MATH 18 Intermediate Calculus, October 5, 2004
EXAM 1 Solutions

(1) Determine whether the following four points are coplanar (with complete justification).

$$P(2,0,0) \qquad Q(1,0,3) \qquad R(3,-1,-1) \qquad S(0,1,2))$$

We check whether the vercors $PQ = \langle -1,0,3\rangle; PR = \langle 1,-1,-1\rangle; PS = \langle -2,1,2\rangle$ are coplanar. The triple product is

$$\begin{vmatrix} -1 & 0 & 3 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{vmatrix} = -2 \neq 0$$

so they are not co-planar.

(2) (a) Find an equation for the plane passing through the point $P = (1,2,2)$ and perpendicular to the line given in symmetric form as

$$\frac{x-2}{3} = \frac{y-3}{4} = z-5.$$

The normal vector is the vector along the line $\vec{\mathbf{N}} = \langle 3,4,1\rangle$. Thus the equation is

$$3(x-1) + 4(y-2) + z - 2 = 0.$$

(b) Find the **cosine of the angle** of intersection of the plane above and the xy–plane.

The angle θ between the planes is the angle between the normal vectors. The normal to the xy-plane is the vector $\vec{\mathbf{k}}$. Thus

$$\cos\theta = \frac{\vec{\mathbf{N}} \cdot \vec{\mathbf{k}}}{|\vec{\mathbf{N}}|\ |\vec{\mathbf{k}}|} = 1/\sqrt{26}.$$

(3) Suppose the vector function $\vec{\mathbf{r}}(t) = \langle t, 2t^2, t^3\rangle$ represents the position of an object as a function of time.

(a) Find the speed of the object at time $t = 1$.

$\vec{\mathbf{r}}'(t) = \langle 1, 4t, 3t^2\rangle$ so $\vec{\mathbf{r}}'(1) = \langle 1,4,3\rangle$ and thus the speed is $|\vec{\mathbf{r}}'(1)| = \sqrt{26}$

(b) Determine the times t (if any) at which the *velocity* and *position* vectors of the object are perpendicular to each other.

we are looking for t such that $\vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{r}}(t) = 0$, which means $t(1+8t^2+3t^4) = 0$. SInce the second term is always positive the only solution is $t = 0$.

(4) (a) Sketch the space curve of the vector function $\vec{\mathbf{r}}(t) = \langle t, \cos t, \sin t\rangle$. Label the point where $t = 0$, and indicate with arrows the direction of increase of t.

You should get a helix spiralling along the x-axis, starting at $(0,1,0)$, going forward along x and upwards along z in the beginning.

(b) Viewing $\vec{\mathbf{r}}(t)$ as the position of a particle at time t, find the velocity and acceleration of the particle as functions of t.

$\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}'(t) = \langle 1, -\sin t, \cos t\rangle$ and $\vec{\mathbf{a}}(t) = \vec{\mathbf{r}}''(t) = \langle 0, -\cos t, -\sin t\rangle$

(c) Write a definite integral describing the arclength of the path traversed by the particle between $t = 1$ and $t = 3$, and evaluate it.

the arc length is $L = \int_1^3 |\vec{\mathbf{r}}'(t)| dt = \int_1^3 \sqrt{2} dt = 2\sqrt{2}$.

(5) Sketch the level curves of $f(x,y) = x^2 + 2y^2$ at height $z = 0$, $z = 1$ and $z = 2$.

for $z = 0$ you get just the origin. for $z = 1$ you get an ellipse centered at the origin aligned to the axes and passing through $(1,0)$ and $(0, 1/\sqrt{2})$. for $z = 2$ you get an ellipse centered at the origin aligned to the axes and passing through $(\sqrt{2}, 0)$ and $(0,1)$.