EXERCISES FOR LECTURE 1

MARTIN OLSSON

1. Show that the inclusions

$$\text{(integral monoids)} \subset \text{(monoids)}$$

and

$$\text{(integral and saturated monoids)} \subset \text{(integral monoids)}$$

have left adjoints.

2. Let $f : X \to Y$ be a morphism of schemes, P a monoid, and M the log structure on Y associated to a morphism $\beta : P \to \Gamma(Y, \mathscr{O}_Y)$. Show that f^*M is the log structure associated to the composite map

$$P \to \Gamma(Y, \mathscr{O}_Y) \to \Gamma(X, \mathscr{O}_X).$$

3. Let $f : X \to Y$ be a morphism of schemes. Show that the functor

$$f^* : (\text{log structures on } Y) \to (\text{log structures on } X)$$

has a right adjoint $f_*^{\log}$. Give an example to show that $f_*^{\log}$ does not in general take fine log structures to fine log structures.

4. Let P be a fs monoid, k a field, and let X be the scheme $\mathrm{Spec}(k[P])$. Let $j : U \hookrightarrow X$ be the open subset

$$\mathrm{Spec}(k[P^{gp}]) \hookrightarrow \mathrm{Spec}(k[P]).$$

Show that $j_*^{\log}\mathscr{O}_U^*$ is the log structure associated to the natural map $P \to k[P]$.

5. Let (E, e) be an elliptic curve over an algebraically closed field k of characteristic $\neq 3$. Let $j : E \hookrightarrow \mathbb{P}^2$ be an embedding given by choosing an isomorphism $\Gamma(E, \mathscr{O}_E(3e)) \simeq k^3$. Let $x_1, \ldots, x_9 \in E(k)$ be 9 points, and let $\pi : P \to \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at the nine points $j(x_i)$. Show the following:

(i) The strict transform of E in P projects isomorphically onto E, so we get a lifting $\tilde{j} : E \hookrightarrow P$ of j.

(ii) $E \in |-K_P|$.

(iii) Let I be the ideal of E in $\mathscr{O}_P$. Then

$$\tilde{j}^* I \simeq \mathscr{O}_E\left(\sum_{i=1}^{9}(e - x_i)\right).$$