EXERCISES FOR LECTURE 2

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1. Let k be a field, and let M_k be the log structure on $\mathrm{Spec}(k)$ associated to the map $\mathbb{N} \to k$, $n \mapsto (0)^n$. Let

$$(X, M_X) \to (\mathrm{Spec}(k), M_k)$$

be a morphism, such that étale locally X is isomorphic to

$$\mathrm{Spec}(k[x_1, \ldots, x_n]/(x_1 \cdots x_r)), \quad r \leq n,$$

with the standard log structure. Assume further that X/k is projective. Show that $\Omega^1_{(X,M_X)/(k,M_k)}$ is locally free of finite rank, and that the top exterior power $K_{(X,M_X)/(k,M_k)}$ is the dualizing sheaf on X (in the sense of Serre duality).

2. Let (E, e) be an elliptic curve, and fix two collections of 9 points $\underline{x} = (x_1, \ldots, x_9)$ and $\underline{y} = (y_1, \ldots, y_9)$. Let

$$j_x : E \hookrightarrow P_x, \quad j_y : E \hookrightarrow P_y$$

be the embeddings into rational surfaces obtained as in exercise 5 from lecture 1. Let X be the scheme obtained by gluing P_x and P_y along E. Show that

$$\mathscr{E}xt^1(\Omega^1_{X/k}, \mathscr{O}_X) \simeq \mathscr{O}_E(\sum(x_i - e)) \otimes \mathscr{O}_E(\sum(y_i - e)).$$

In particular, if $\underline{x} = \underline{y} = E[3]$ then $\mathscr{E}xt^1(\Omega^1_{X/k}, \mathscr{O}_X) \simeq \mathscr{O}_E$.

3. Let

$$(X, M_X) \to (\mathrm{Spec}(k), M_k)$$

be the log smooth morphism obtained by taking $\underline{x} = \underline{y} = E[3]$ in problem 2. Show that the Hodge numbers

$$h^i(X, \Omega^j_{(X,M_X)/(k,M_k)})$$

are given by the following table

$$\begin{matrix} 1 & 0 & 1 \\ 0 & 20 & 0 \\ 1 & 0 & 1 \end{matrix}$$

and that $K_{(X,M_X)/(k,M_k)} \simeq \mathscr{O}_X$.

4. Let k be a field and let $\theta : Q \to P$ be a morphism of fine monoids such that the orders of

$$\mathrm{Ker}(\theta^{gp}), \quad \mathrm{Coker}(\theta^{gp})_{\mathrm{tors}}$$

are invertible in k. Show that

$$\mathrm{Spec}(P \to k[P]) \to \mathrm{Spec}(Q \to k[Q])$$

is log smooth.