WP Metric Geometry Quick View SAW May 3, 2007

Let $\mathcal{T}$ be the Teichmüller space for marked genus g, n punctured Riemann surfaces R of negative Euler characteristic. By Uniformization a conformal structure determines a unique complete compatible hyperbolic metric ds^2. $\mathcal{T}$ is a complex manifold of $\dim_{\mathbb{C}} = 3g-3+n$ with the cotangent space at R represented by $Q(R)$, the space of holomorphic quadratic differentials on R with at most simple poles at the punctures. The Weil-Petersson cometric is

$$\langle \varphi, \psi \rangle = \int_R \varphi \overline{\psi} (ds^2)^{-1}.$$

The WP metric is invariant under the action of the *mapping class group* MCG, [Iva02]. The metric is Kähler, non complete with negative sectional curvature with $\sup_{\mathcal{T}} = 0$ (except for $\dim \mathcal{T} = 1$) and $\inf_{\mathcal{T}} = -\infty$. *In practice and experience the WP geometry of $\mathcal{T}$ corresponds to the hyperbolic geometry of surfaces.*

The augmented Teichmüller space, [Abi77, Ber74]. The augmented Teichmüller space $\overline{\mathcal{T}}$ is a MCG invariant bordification (a partial compactification) following the approach for Baily-Borel bordifications. ($\overline{\mathcal{T}}$ is an analog of the $SL(2;\mathbb{Z})$ invariant bordification $\mathbb{Q}$ for $\mathbb{H}$ provided with the Satake horocycle topology.) The bordification is also described as the Chabauty topology closure of the faithful cofinite representations of $\pi_1(R)$ into $PSL(2;\mathbb{R})$. $\overline{\mathcal{T}}/MCG$ is homeomorphic to the Deligne-Mumford stable curve compactification of the moduli space of Riemann surfaces. The elements of $\overline{\mathcal{T}} - \mathcal{T}$ are marked degenerate hyperbolic structures for which certain simple non peripheral closed curves are *represented* by pairs of cusps.

CAT(0) geometry. Important is that $\overline{\mathcal{T}}$ is WP complete with $(\overline{\mathcal{T}}, d_{WP})$ a $CAT(0)$ metric space (a generalized complete, simply connected, non positively curved space.) The large-scale geometry is described by $\overline{\mathcal{T}}$ being quasi isometric to $\mathcal{PG}(R)$ the *pants graph* of R with unit-length edge metric, [Bro03]. $\overline{\mathcal{T}}$ is a stratified space with each open strata characterized as the union of all geodesics (distance realizing paths) containing a given point as an interior point. $\overline{\mathcal{T}}$ itself is characterized as the closed WP convex hull of the maximally degenerate hyperbolic structures (the unions of thrice punctured spheres.) An application of the $CAT(0)$ geometry and the important rigidity of the *complex of curves* $\mathcal{C}(R)$, [Iva02, Chap. 3] is that the isometry group $Isom_{WP}$ coincides with MCG.

Geodesic-length functions. Associated to each non trivial, non peripheral free homotopy class on R is the length of the unique ds^2 geodesic in the homotopy class. Geodesic-lengths $\ell_\alpha(R)$ on $\mathcal{T}$ are elements of the WP geometry. The Fenchel-Nielsen twist (right earthquake) deformation about α is described by $2t_\alpha = J \operatorname{grad} \ell_\alpha$ for J the Teichmüller almost complex structure. The WP Hermitian pairing is described in terms of the hyperbolic trigonometry for $R = \mathbb{H}/\Gamma$ with

$$\langle \operatorname{grad} \ell_\alpha, J \operatorname{grad} \ell_\beta \rangle = 4\, \omega_{WP}(t_\alpha, t_\beta) = -2 \sum_{p \in \alpha \cap \beta} \cos \theta_p$$

for the geodesics α, β and intersection angles θ_* on R and in upper half plane

$$\langle \operatorname{grad} \ell_\alpha, \operatorname{grad} \ell_\beta \rangle = \frac{2}{\pi}\big(\ell_\alpha \delta_{\alpha\beta} \ + \ \sum\nolimits'_{\langle A\rangle\backslash\Gamma/\langle B\rangle} (u \log \frac{u+1}{u-1} - 2)\big)$$

for the Kronecker delta δ_*, where for $C \in \langle A\rangle\backslash\Gamma/\langle B\rangle$ then $u = u(\tilde{\alpha}, C(\tilde{\beta}))$ is the cosine of the intersection angle if the lifts $\tilde{\alpha}$ and $C(\tilde{\beta})$ intersect and is otherwise $\cosh d(\tilde{\alpha}, C(\tilde{\beta}))$.

WP convexity and curvature. Information on geodesics is provided by the strict convexity of geodesic-lengths (more generally convexity of the total length of a *measured geodesic lamination*) along WP geodesics (square roots are also convex.) Geodesic-length sublevel sets are convex. On $\mathcal{T}$ the WP Levi-Civita connection D is described for small geodesic-length, $\lambda = \operatorname{grad} \ell_\alpha^{1/2}$ and tangent U by

$$D_U \lambda_\alpha \ = \ 3\ell_\alpha^{-1/2} \langle J\lambda_\alpha, U\rangle J\lambda_\alpha \ + \ O(\ell_\alpha^{3/2}\|U\|).$$

The remainder term constant is uniform for bounded geodesic-length. Bounds for the gradient and Hessian of geodesic-length give rise to applications. The WP curvature of the span $\{t_\alpha, Jt_\alpha\}$ is $O(-\ell_\alpha^{-1})$. Similarly for a pair of deformations supported on different components of $R - \{\textit{short geodesics}\}$ the curvature of the corresponding 2-plane is $O(-\textit{short lengths})$.

Fenchel-Nielsen coordinates. Marked hyperbolic pairs of pants are determined by their three boundary geodesic-lengths in $\mathbb{R}_+$. The FN twist-length coordinates $(\ell_j, \tau_j)_{j=1}^{3g-3+n}$ for assembling hyperbolic pants provide global coordinates for $\mathcal{T}$ with expansions

$$\omega_{WP} = \frac{1}{2} \sum d\ell_j \wedge d\tau_j$$

and on the Bers region $\{\ell_j < c_0\}$ the expansions

$$\langle\ ,\ \rangle \ \asymp \ \sum (d\ell_j^{1/2})^2 + (d\ell_j^{1/2} \circ J)^2 \ \asymp \ \sum \operatorname{Hess} \ell_j$$

with comparability uniform in terms of c_0. At the maximally degenerate structure the WP metric has the expansions

$$\begin{aligned}\langle\ ,\ \rangle \ &= 2\pi \sum (d\ell_j^{1/2})^2 + (d\ell_j^{1/2} \circ J)^2 \ + \ O(\sum \ell_j^3 \langle\ ,\ \rangle) \\ &= \frac{\pi}{6} \sum \frac{\operatorname{Hess} \ell_j^2}{\ell_j} \ + \ O(\sum \ell_j^2 \langle\ ,\ \rangle).\end{aligned}$$

Alexandrov tangent cone. For a pair of geodesics (distance realizing curves) from a common point in a $CAT(0)$ metric space there is a well-defined initial angle, [BH99, Chap. II.3]. A tangent cone is defined in terms of initial angle. At a point p of a strata $\mathcal{T}(\sigma) \subset \overline{\mathcal{T}} - \mathcal{T}$, $\sigma \in \mathcal{C}(R)$, the WP Alexandrov tangent cone AC_p is isometric to a product of a Euclidean orthant and the tangent space $\mathbf{T}\mathcal{T}(\sigma)$ with WP pairing. The isometry is given in terms of initial derivatives of geodesic-length functions. The dimension of the Euclidean orthant is the count of geodesic-lengths trivial on $\mathcal{T}(\sigma)$ (the count of nodes.) A property of $\overline{\mathcal{T}}$ is that each WP geodesic tangential to $\mathcal{T}(\sigma)$ at p actually lies in $\mathcal{T}(\sigma)$. Applications include that geodesics do not *refract* at the bordification and that the angle of incidence and reflection coincide for certain limits of degenerating geodesics. A further application is for combinatorial harmonic maps. Certain groups acting on Euclidean buildings and group extensions acting on Cayley graphs satisfying a Poincaré type inequality for links of points will have a global fixed point for an action on $\overline{\mathcal{T}}$.

All important attributions and references are provided in the introductions of [Wol03, Wol07].

References

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