# THE RIGOROUS DERIVATION OF THE 2D CUBIC FOCUSING NLS FROM QUANTUM MANY-BODY EVOLUTION

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Abstract. We consider a 2D time-dependent quantum system of $N$-bosons with harmonic external confining and *attractive* interparticle interaction in the Gross-Pitaevskii scaling. We derive stability of matter type estimates showing that the $k$-th power of the energy controls the $H^1$ Sobolev norm of the solution over $k$-particles. This estimate is new and more difficult for attractive interactions than repulsive interactions. For the proof, we use a version of the finite-dimensional quantum di Finetti theorem from [49]. A high particle-number averaging effect is at play in the proof, which is not needed for the corresponding estimate in the repulsive case. This a priori bound allows us to prove that the corresponding BBGKY hierarchy converges to the GP limit as was done in many previous works treating the case of repulsive interactions. As a result, we obtain that the *focusing* nonlinear Schrödinger equation is the mean-field limit of the 2D time-dependent quantum many-body system with attractive interatomic interaction and asymptotically factorized initial data. An assumption on the size of the $L^1$-norm of the interatomic interaction potential is needed that corresponds to the sharp constant in the 2D Gagliardo-Nirenberg inequality though the inequality is not directly relevant because we are dealing with a trace instead of a power.

## Contents



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## 1. Introduction

Bose-Einstein condensate (BEC) is a state of matter occurring in a dilute gas of bosons (identical particles with integer spin) at very low tempertures, where all particles fall into the lowest quantum state. This form of matter was predicted in 1924 by Einstein, inspired by calculations for photons by Bose. In 1995, BEC was first produced experimentally by Cornell and Wieman [4] at the University of Colorado at Boulder NIST–JILA lab, in a gas of rubidium cooled to 20 nK. Shortly thereafter, Ketterle [41] at MIT demonstrated important properties of a BEC of sodium atoms. For this work, Cornell, Weiman, and Ketterle received the 2001 Nobel Prize in Physics[1]. Since then, this new state of matter has attracted a lot of attention in physics and mathematics as it can be used to explore fundamental questions in quantum mechanics, such as the emergence of interference, decoherence, superfluidity and quantized vortices.

Let us lay out the quantum mechanical description of the $N$-body problem. Let $t \in \mathbb{R}$ be the time variable and $\mathbf{r}_N = (r_1, \ldots, r_N) \in \mathbb{R}^{nN}$ be the position vector of $N$ particles in $\mathbb{R}^n$. The dynamic of $N$ bosons are described by a symmetric $N$-body wave function $\psi_N(\mathbf{r}_N, t)$ evolving according to the linear $N$-body Schrödinger equation

$$i\partial_t \psi_N = H_N \psi_N$$

with Hamiltonian $H_N$ given by

$$H_N = -\sum_{j=1}^{N} \Delta_{r_j} + \frac{1}{N} \sum_{1 \leq i < j \leq N} N^{n\beta} V(N^\beta(r_i - r_j)) + \sum_{j=1}^{N} W(r_j) \tag{1.1}$$

where $V$ represents the interparticle attraction/repulsion and $W$ represents the external confining potential.

Informally, BEC means that, up to a phase factor depending only on $t$, the $N$-body wave function nearly factorizes

$$\psi_N(\mathbf{r}_N, t) \approx \prod_{j=1}^{N} \varphi(r_j, t) \tag{1.2}$$

In the simplest cases, where is it assumed that interactions between condensate particles are of the contact two-body type and also anomalous contributions to self-energy are neglected, it is widely believed, based upon heuristic and formal calculations, that (1.2) is valid and the one-particle state $\varphi$ evolves according to the nonlinear Schrödinger equation (NLS)

$$i\partial_t \varphi = (-\Delta + W(r))\varphi + 8\pi\mu|\varphi|^2\varphi \tag{1.3}$$

This is one of the main motivations for studying the NLS equation, and there is now a wide body of literature on well-posedness [7, 61], the long-time asymptotics of global-in-time solutions [43], the possibility and structure of finite-time blow-up solutions [60], and the stability and dynamics of coherent solutions called solitary waves [63, 5]. In particular, blow-up and solitary waves only exist in the case of $\mu < 0$, called the *focusing* case.

[1] http://www.nobelprize.org/nobel_prizes/physics/laureates/2001/press.html.

Before proceeding, let us remark on the choice of scaling in the interparticle interaction term. In 2D, it is taken as $N^{-1}V_N(r)$ in (1.1), where $V_N(r) = N^{2\beta}V(N^\beta r)$, for $\beta > 0$.[2] This scaling is intended to capture the so-called *Gross-Pitaevskii limit*, in which the ground-state asymptotics are described by the one-particle Gross-Pitaevskii (GP) energy functional

$$\mathcal{E}(\varphi) = \int (|\nabla\varphi|^2 + W|\varphi|^2 + 4\pi\mu|\varphi|^4)$$

In the case of repulsive interactions $\mu > 0$, in the stationary case, the ground state energy asymptotics in the 2D Gross-Pitaevskii limit from the 2D $N$-body quantum setting, are discussed in [50, Theorem 6.5]. It is found that $a_N$, the 2D scattering length of the microscopic interaction, should scale as $a_N = N^{-1/2}e^{-N/2\mu}$. The *scattering length* associated to a potential is the radius of the hard-sphere potential that gives the same low-wave number phase shifts as the given potential. A precise definition in 2D is given in [50, §9.3]. If we take $V_N(x) = N^{2\beta}V(N^\beta x)$, then by [50, Corollary 9.4] with $\lambda = (\int V)N^{-1}$ and $R = N^{-\beta}$, we have

$$a_N \sim N^{-\beta} \exp\left(-\frac{4\pi N}{\int V}(1+\eta(N))\right) \tag{1.4}$$

where $\eta(N) \to 0$ as $N \to \infty$. Thus $\beta = \frac{1}{2}$ gives the correct $N$-dependence for $a_N$. Other values of $\beta$ could be produced by modifying $N^{2\beta}V(N^\beta r)$ to $(1 + \frac{c \ln N}{N})N^{2\beta}V(N^\beta r)$ for appropriate $c$, and thus changing $\beta$ corresponds to a lower-order correction in the scaling.[3] Moreover, the analysis shows that we have $\mu = \frac{1}{8\pi}\int V$. The corresponding time-dependent problem, for $\mu > 0$, was studied by Kirkpatrick-Schlein-Staffilani [44] in the periodic setting and by X.Chen [15] in the trapping setting($W \neq 0$).

Another way to obtain a 2D limit is to start with a 3D quantum $N$-body system with strong confining in one-dimension (say the $z$-direction). In the stationary repulsive case, this was explored by Schnee-Yngvason [54]. If external confining in the $z$-direction is imposed to give the system an effective width $\omega^{-1/2}$, then one should take the 3D interaction potential to be $(N\sqrt{\omega})^{3\beta-1}V((N\sqrt{\omega})^\beta r)$, where $r = (x, y, z)$, in place of the 2D interaction potential $N^{2\beta-1}V(N^\beta r)$, where $r = (x, y)$, and take $\omega \to \infty$ as $N \to \infty$. In the repulsive case ($\mu > 0$), the corresponding time-dependent problem was studied by X.Chen-Holmer [17]. We will not consider the dimensional reduction problem here.

As indicated earlier, one expects that the nonlinear coefficent $\mu$ in (1.3) is given by $\mu = \frac{1}{8\pi}\int V$, or expressed in terms of the scattering length $a_N$ of $N^{-1}V_N(r)$, the relation is $\mu = -N[\ln(Na_N^2)]^{-1}$.[4] The scattering length can be adjusted experimentally by the method of Feshbach resonance, which exploits the hyperfine structure of the atoms in the condensate. Specifically, we see that the sign of $\mu$ depends on $\int V$, and that $\int V < 0$ leads to *focusing* NLS

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[2]We consider the $\beta > 0$ case solely in this paper. For $\beta = 0$ (Hartree dynamic), see [34, 28, 47, 53, 51, 37, 38, 14, 2, 3].

[3]We note, in particular, that, unlike the 3D case, nothing special happens at $\beta = 1$. Exponential in $N$ scaling would allow one to shift the value of $\mu$, but still would not yield the 2D scattering length of $V$ itself.

[4]Although the definition of scattering length in [50, §9.3] is given in the case of repulsive potentials $V \geq 0$, it can be adjusted to the case of attractive potentials $V \leq 0$, in which the requirement that $\psi(x) = \ln\frac{|x|}{a}$ for $|x| \to \infty$ is replaced with $\psi(x) = -\ln\frac{|x|}{a}$ for $|x| \to \infty$. Then (1.4) is changed by the reciprocal, so we still obtain an exponentially *small* quantity, rather than an unphysical exponentially large quantity in $N$.

with $\mu < 0$. BEC with $\mu < 0$ has been produced in laboratory experiments [24, 26, 59, 42] in different contexts, and solitary waves and blow-up have been observed. Thus there is strong motivation for determining whether the mean-field approximation (1.3) is theoretically valid in different contexts.

We are concerned here with precise conditions under which (1.2) and (1.3) hold, and the rigorous demonstration of this result. For our quantitive formulation of the $N \to \infty$ limit, we use the BBGKY framework. Specifically, let $\gamma_N$ be the projection operator in $L^2(\mathbb{R}^{2N})$ onto the one-dimensional space spanned by $\psi_N$. The kernel is

$$\gamma_N(t, \mathbf{r}_N, \mathbf{r}'_N) = \psi_N(t, \mathbf{r}_N)\overline{\psi_N(t, \mathbf{r}'_N)} \tag{1.5}$$

Let $\gamma_N^{(k)}$ denote the trace of $\gamma_N$ over the last $(N-k)$ particles, called the $k$-th marginal density. Then $\gamma_N^{(k)}$ is a trace-class operator on $L^2(\mathbb{R}^{2k})$ with kernel given by

$$\gamma_N^{(k)}(t, \mathbf{r}_k, \mathbf{r}'_k) = \int_{\mathbf{r}_{N-k}} \gamma_N(t, \mathbf{r}_k, \mathbf{r}_{N-k}; \mathbf{r}'_k, \mathbf{r}_{N-k})\, d\mathbf{r}_{N-k} \tag{1.6}$$

In this language, (1.2) becomes the informal statement

$$\gamma_N^{(k)}(t, \mathbf{r}_k, \mathbf{r}'_k) \approx \prod_{j=1}^{k} \varphi(r_j, t)\overline{\varphi(r'_j, t)}$$

Our main result demonstrates that this holds, in the sense of convergence as $N \to \infty$ in the trace norm. Our result covers the *focusing* case in 2D, also known as the *mass-critical focusing* case in the NLS literature. Previous results either dealt with the defocusing case in dimensions 1,2, or 3, or the focusing case in dimension 1 (obtained either as limit from 1D or 3D quantum many-body dynamics).

**Definition 1.** *We denote $C_{gn}$ the sharp constant of the 2D Gagliardo–Nirenberg estimate:*

$$\|\phi\|_{L^4} \leqslant C_{gn} \|\phi\|_{L^2}^{\frac{1}{2}} \|\nabla\phi\|_{L^2}^{\frac{1}{2}}. \tag{1.7}$$

**Theorem 1.1** (Main Theorem)**.** *Assume that the focusing pair interaction $V$ is an even nonpositive Schwartz class function such that $\|V\|_{L^1} < \frac{2\alpha}{C_{gn}^4}$ for some $\alpha \in (0,1)$. Let $\psi_N(t, \mathbf{x}_N)$ be the $N-$body Hamiltonian evolution $e^{itH_N}\psi_N(0)$, where*

$$H_N = \sum_{j=1}^{k} \left(-\triangle_{x_j} + \omega^2 |x_j|^2\right) + \frac{1}{N}\sum_{i<j} N^{2\beta} V(N^\beta(x_i - x_j)) \tag{1.8}$$

*for some nonzero $\omega \in \mathbb{R}/\{0\}$ and for some $\beta \in \left(0, \frac{1}{6}\right)$, and let $\left\{\gamma_N^{(k)}\right\}$ be the family of marginal densities associated with $\psi_N$. Suppose that the initial datum $\psi_N(0)$ verifies the following conditions:*

*(a) the initial datum is normalized, that is*

$$\|\psi_N(0)\|_{L^2} = 1,$$

*(b) the initial datum is asymptotically factorized, in the sense that,*

$$\lim_{N\to\infty} \operatorname{Tr}\left|\gamma_N^{(1)}(0, x_1; x'_1) - \phi_0(x_1)\overline{\phi_0}(x'_1)\right| = 0, \tag{1.9}$$

*for some one particle wave function $\phi_0$ s.t.* $\left\| \left( -\triangle_x + \omega^2 |x|^2 \right)^{\frac{1}{2}} \phi_0 \right\|_{L^2(\mathbb{R})} < \infty$.

*(c) initially, each particle's energy, though may not be positive, is bounded above*

$$\sup_N \frac{1}{N} \langle \psi_N(0), H_N \psi_N(0) \rangle < \infty. \tag{1.10}$$

*Then $\forall t \geqslant 0$, $\forall k \geqslant 1$, we have the convergence in the trace norm or the propagation of chaos that*

$$\lim_{N\to\infty} \operatorname{Tr} \left| \gamma_N^{(k)}(t, \mathbf{x}_k; \mathbf{x}_k') - \prod_{j=1}^{k} \phi(t, x_j) \overline{\phi}(t, x_j') \right| = 0,$$

*where $\phi(t,x)$ is the solution to the 2D focusing cubic NLS*

$$\begin{aligned} i\partial_t \phi &= \left( -\triangle_x + \omega^2 |x|^2 \right) \phi - b_0 |\phi|^2 \phi \text{ in } \mathbb{R}^{2+1} \\ \phi(0,x) &= \phi_0(x) \end{aligned} \tag{1.11}$$

*and the coupling constant* $b_0 = \left| \int_{\mathbb{R}^2} V(x) dx \right|$.

Theorem 1.1 is equivalent to the following theorem.

**Theorem 1.2** (Main Theorem)**.** *Assume that the focusing pair interaction $V$ is an even nonpositive Schwartz class function such that $\|V\|_{L^1} < \frac{2\alpha}{C_{gn}^4}$ for some $\alpha \in (0,1)$. Let $\psi_N(t, \mathbf{x}_N)$ be the $N-$body Hamiltonian evolution $e^{itH_N}\psi_N(0)$ with $H_N$ given by (1.8) for some nonzero $\omega \in \mathbb{R}/\{0\}$ and for some $\beta \in (0, 1/6)$, and let $\left\{ \gamma_N^{(k)} \right\}$ be the family of marginal densities associated with $\psi_N$. Suppose that the initial datum $\psi_N(0)$ is normalized and asymptotically factorized in the sense of (a) and (b) in Theorem 1.1 and verifies the following energy condition:*

*(c') there is a $C > 0$ independent of $N$ or $k$ such that*

$$\left\langle \psi_N(0), H_N^k \psi_N(0) \right\rangle < C^k N^k, \ \forall k \geqslant 1, \tag{1.12}$$

*though the quantity $\left\langle \psi_N(0), H_N^k \psi_N(0) \right\rangle$ may not be positive.*

*Then $\forall t \geqslant 0$, $\forall k \geqslant 1$, we have the convergence in the trace norm or the propagation of chaos that*

$$\lim_{N\to\infty} \operatorname{Tr} \left| \gamma_N^{(k)}(t, \mathbf{x}_k; \mathbf{x}_k') - \prod_{j=1}^{k} \phi(t, x_j) \overline{\phi}(t, x_j') \right| = 0,$$

*where $\phi(t,x)$ is the solution to the 2D focusing cubic NLS (1.11).*

It follows from the fact that $\psi_N$ evolves according to $i\partial_t \psi_N = H_N \psi_N$ and the definition (1.5), (1.6) of the marginal densities $\gamma_N^{(k)}$ that

$$\begin{aligned} i\partial_t \gamma_N^{(k)} &= \sum_{j=1}^{k} \left[ -\triangle_{x_j} + \omega^2 |x_j|^2, \gamma_N^{(k)} \right] + \frac{1}{N} \sum_{1\leqslant i<j\leqslant k} \left[ V_N(x_i - x_j), \gamma_N^{(k)} \right] \\ &\quad + \frac{N-k}{N} \sum_{j=1}^{k} \operatorname{Tr}_{k+1} \left[ V_N(x_j - x_{k+1}), \gamma_N^{(k+1)} \right], \end{aligned}$$

This coupled sequence of equations is called the BBGKY hierarchy. The use of the BBGKY hierarchy in the quantum setting was suggested by Spohn [58] and has been employed in rigorous work by Adami, Golse, & Teta [1] and Elgart, Erdös, Schlein, & Yau [27, 29, 30, 31, 32]. The latter series of works rigorously derives the 3D cubic defocusing NLS from a 3D time-dependent quantum many-body system with repulsive pair interactions and no trapping ($\omega = 0$). Their program consists of two main steps.[5] First, they derive $H^1$-energy type a priori estimates for the $N$-body Hamiltonian from which a compactness property, for each $k$, of the sequence $\{ \gamma_N^{(k)} \}_{N=1}^{+\infty}$ follows, yielding limit points $\gamma^{(k)}$ solving the 3D Gross-Pitaevskii hierarchy

$$i\partial_t \gamma^{(k)} + \sum_{j=1}^{k} \left[ \triangle_{r_k}, \gamma^{(k)} \right] = b_0 \sum_{j=1}^{k} \operatorname{Tr}_{r_{k+1}} [\delta(r_j - r_{k+1}), \gamma^{(k+1)}], \text{ for all } k \geq 1 . \tag{1.13}$$

Second, they show that (1.13) has a unique solution which satisfies the $H^1$-energy type a priori estimates obtained in the first step. Since a compact sequence with a unique limit point is, in fact, a convergent sequence, it follows that (in an appropriate weak sense) solutions to the BBGKY hierarchy $\gamma_N^{(k)}$ converge to solutions to the GP hierarchy $\gamma^{(k)}$. Moreover, it is easily verified that a tensor product of solutions of NLS (1.3) solves the GP hierarchy, and hence this is the unique solution.

In the defocusing literature, a major difficulty is that the uniqueness theory for the hierarchy (1.13) is surprisingly delicate due to the fact that it is a system of infinitely many coupled equations over an unbounded number of variables and there has been much work on it. Klainerman & Machedon [45] gave a Strichartz type uniqueness theorem using a collapsing estimate originating from the multilinear Strichartz estimates and a board game argument inspired by the Feynman graph argument in [30]. The method by Klainerman & Machedon [45] was taken up by Kirkpatrick, Schlein, & Staffilani [44], who derived the 2D cubic defocusing NLS from the 2D time-dependent quantum many-body system; by T. Chen & Pavlović [10], who considered the 1D and 2D 3-body repelling interaction problem; by X. Chen [15, 16], who investigated the defocusing problem with trapping in 2D and 3D; by X. Chen & Holmer [17], who proved the effectiveness of the defocusing 3D to 2D reduction problem, and by T.Chen & Pavlović [11] and X.Chen & Holmer [16, 18, 21], who proved the Strichartz type bound conjectured by Klainerman & Machedon. Such a method has also inspired the study of the general existence theory of hierarchy (1.13), see [12, 9, 35, 57]. Recently, using a version of the quantum de Finetti theorem from [48][6], T.Chen, Hainzl, Pavlović, & Seiringer [8] provided an alternative proof to the uniqueness theorem in [30] and showed that it is an unconditional uniqueness result in the sense of NLS theory. With this method, Sohinger derived the 3D defocusing cubic NLS in the periodic case [56]. See also [22, 40].

However, for the focusing case, things are different. How to obtain the needed $H^1$-energy type a priori estimates is the central question. To be precise, without such a priori estimates, one *cannot* check the requirements of the various uniqueness theorems

[5] See [6, 36, 52, 46] for different approaches.

[6] See also [3, 2].

[30, 45, 44, 15, 16, 56, 8, 22, 40] at all.[7] It is already highly nontrivial and may not be possible to even prove the weaker type II stability of matter estimate

$$\langle \psi_N, H_N \psi_N \rangle \geqslant -CN \text{ for all } \psi_N \in L^2_s(\mathbb{R}^{nN}) \tag{1.14}$$

when $H_N$ is given by (1.1) with $V < 0$ while it is obviously true when $V \geqslant 0$. The first complete work on the focusing problem was done by X.Chen and Holmer [19, 20] for the time-dependent 1D problem. The key is to explore the structure of the 2-body operator

$$H_{+ij} = -\Delta_{r_i} + W(r_i) - \Delta_{r_j} + W(r_j) + \frac{N-1}{N} N^{n\beta} V(N^\beta (r_i - r_j)) \tag{1.15}$$

generated in the decomposition of $H_N$. Such a technique was later used independently by Lewin, Nam, & Rougerie in [49], where they investigated the ground state problem in the focusing setting. The main portion of this paper is devoted to this problem in 2D. In particular, we prove

**Theorem 1.3.** *Consider the focusing many-body Hamiltonian*

$$H_N = \sum_{j=1}^{k} \left( -\triangle_{x_j} + \omega^2 |x_j|^2 \right) + \frac{1}{N} \sum_{i<j} N^{2\beta} V(N^\beta (x_i - x_j)),$$

*in 2D. Assume $\omega > 0$, $\beta < \frac{1}{6}$, and $\|V\|_{L^1} < \frac{2\alpha}{C_{gn}^4}$ for some $\alpha \in (0,1)$, then let $c_0 = \min(\frac{1-\alpha}{\sqrt{2}}, \frac{1}{2})$, we have $\forall k = 0, 1, ...$, there is an $N_0(k) > 0$ such that*

$$\left\langle \psi_N, \left( N^{-1} H_N + 1 \right)^k \psi_N \right\rangle \geqslant c_0^k \left\| S^{(k)} \psi_N \right\|_{L^2}^2, \tag{1.16}$$

*for all $N > N_0(k)$ and for all $\psi_N \in L^2_s(\mathbb{R}^{2N})$. Here*

$$S^{(k)} = \prod_{j=1}^{k} S_j$$

*and $S_j^2$ is the Hermite operator $-\triangle_{x_j} + \omega^2 |x_j|^2$.*

The difficulty of proving Theorem 1.3 is self-evident. In the 2D setting in which the kinetic energy, effectively the $H^1$ norm, cannot control $V_N$, effectively a Dirac $\delta$-mass[8], not only Theorem 1.3 provides stability of matter, it also proves regularity. The key to the proof, as we will explain later, is to make use of a large $N$ averaging effect which is revealed via a clever application of a finite dimensional quantum de Finette theorem in [49].

[7] In fact, one of the authors of [27, 29, 30, 31, 32] remarked the a priori bound was the most delicate part in the defocusing case as well when the results were revisited in [6].

[8] Different from the limit NLS in which the $L^4$ norm is easily controlled in $H^1$, in the $N$-body setting, one has to control a trace with $H^1$.

1.1. **Organization of the paper.** As mentioned before, the main portion of this paper is devoted to proving Theorem 1.3. We do so in §2. We will first prove the $k=1$ case:

$$\left\langle \psi_N, \left(N^{-1}H_N+1\right)\psi_N\right\rangle \geqslant (1-\alpha)\left\|S_1\psi_N\right\|_{L^2}^2 \tag{1.17}$$

which is Theorem 2.1 in §2.1.

We remark that not only the proof of Theorem 2.1 departs totally from its analogues in the previous work, its underlying machinery is also significantly different. Theorem 2.1 works because of a large $N$ averaging effect not observed before. To explain this fact, consider the general Hamiltonian (1.1) and let $H_{+ij}$ be defined as in (1.15), then by symmetry,

$$\left\langle \psi_N, \left(N^{-1}H_N+1\right)\psi_N\right\rangle_{\mathbf{x}_N} = \left\langle \psi_N, \left(2+H_{+12}\right)\psi_N\right\rangle_{\mathbf{x}_N},$$

that is, (1.17) is equivalent to

$$\left\langle \psi_N, \left(2+H_{+12}\right)\psi_N\right\rangle \geqslant C\left\|\left(-\Delta_{r_1}+W(r_2)\right)^{\frac{1}{2}}\psi_N\right\|_{L^2}^2. \tag{1.18}$$

In all the defocusing work [1, 10, 15, 16, 17, 27, 29, 30, 32, 31, 44, 56], estimates like (1.18) are automatically true because $V\geqslant 0$. In the previous focusing work [19, 20], it takes substantial work to prove the similar estimates but they actually do not rely on the fact that $\psi_N$ is a $N$-body bosonic wave function in the sense that they hold even if one replaces $\psi_N$ by some $f(x_1,x_2)$ in (1.18). However, Theorem 2.1 requires that $\psi_N$ is a $N$-body bosonic wave function. In fact, when $V<0$, in 2D, the quantity $\left\langle f, \left(2+H_{+12}\right)f\right\rangle$ is not even bounded below, because of the $\delta$-function emerging from $V_N$. Hence, we are observing a large $N$ averaging effect, or more precisely, "though $V_N$ gets more singular as $N\to\infty$, larger $N$ beats it.", as we will see in the proof.[9] Moreover, this is the only energy estimate in the "$n$D to $n$D" [10] literature which requires the trapping $\omega\neq 0$ at the moment.

Based on the $k=1$ case, we then prove the $k>1$ case in §2.2 with a delicate computation using the 2-body operator. In §2.3, by giving a counter example, we show that with the current technique, one can not reach a higher $\beta$.

With Theorem 1.3 established, we prove Theorems 1.1 and 1.2 in §3. Though the technique in §3 is standard by now, this is the first time the derivation of the trapping case is written down without using the lens transform in [15, 16, 19] and it simplifies the argument.

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## 2. Stability of Matter / Energy Estimates for Focusing Quantum Many-body System

In this section, we prove stability of matter / energy estimate (1.16).[11]

[9] See Remark 3.

[10] Here, "$n$D to $n$D" means "deriving $n$D NLS from $n$D $N$-body dynamic".

[11] For the defocusing case ($V\geqslant 0$) in which there is no need to worry about particles focusing to a point, it certainly makes sense to only call estimates like (1.16) "energy estimates". However, that is obviously not the case when $V<0$. Moreover, (1.16) does have a similar form with the stability of matter estimates like (1.14). Hence we use the word "stability of matter / energy estimates" here.

### 2.1. Stability of Matter when $k = 1$.

**Theorem 2.1** (Stability of Matter). *Assume* $\omega > 0$, $\beta < \frac{1}{6}$, *and* $\|V\|_{L^1} < \frac{2\alpha}{C_{gn}^4}$ *for some* $\alpha \in (0,1)$, *then* $\forall C_0 > 0$, *there exists an* $N_0 > 0$ *such that*

$$\left\langle \psi_N, \left(N^{-1}H_N + C_0\right)\psi_N\right\rangle \geqslant (1-\alpha)\left\|S_1\psi_N\right\|_{L^2}^2,$$

*for all* $N > N_0$ *and for all* $\psi_N \in L_s^2(\mathbb{R}^{2N})$. *Here,* $N_0$ *grows to infinity as* $C_0$ *approaches* 0. *In particular, the* $N$*-body system is stable provided* $N$ *is larger than a threshold.*

**Remark 1.** *In the previous focusing work* [19, 20], *there is a positive lower bound for the* $C_0$ *while there is no such requirement in Theorem 2.1 as long as* $C_0 > 0$.

To prove Theorem 2.1, we adopt the notation that: for any function $f$, write

$$f_{Nij} = N^{2\beta} f(N^\beta(x_i - x_j)).$$

The key of the proof of Theorem 2.1 is the following theorem.

**Theorem 2.2.** *Define*

$$H_{ij} = S_i^2 + S_j^2 + \frac{N-1}{N}V_{Nij}. \tag{2.1}$$

*Assume* $\omega > 0$, $\beta < \frac{1}{6}$, *and* $\|V\|_{L^1} < \frac{2\alpha}{C_{gn}^4}$ *for some* $\alpha \in (0,1)$, *then* $\forall C_0 > 0$, *there exists an* $N_0 > 0$ *such that*

$$\left\langle \psi_N, \left(2C_0 + H_{12}\right)\psi_N\right\rangle \geqslant 2(1-\alpha)\left\|S_1\psi_N\right\|_{L^2}^2,$$

*for all* $N > N_0$ *and for all* $\psi_N \in L_s^2(\mathbb{R}^{2N})$. *Here,* $N_0$ *grows to infinity as* $C_0$ *approaches* 0.

*Proof of Theorem 2.1 assuming Theorem 2.2.* We decompose the Hamiltonian $H_N$ into

$$N^{-1}H_N + C_0 = \frac{1}{2N(N-1)} \sum_{\substack{i,j=1,\ldots,N \\ i\neq j}} \left(2C_0 + H_{ij}\right). \tag{2.2}$$

Hence

$$\begin{aligned}
\left\langle \psi_N, \left(N^{-1}H_N + C_0\right)\psi_N\right\rangle &= \frac{1}{2N(N-1)} \sum_{\substack{i,j=1,\ldots,N \\ i\neq j}} \left\langle \psi_N, \left(2C_0 + H_{ij}\right)\psi_N\right\rangle \\
&= \frac{1}{2N(N-1)} \sum_{\substack{i,j=1,\ldots,N \\ i\neq j}} \left\langle \psi_N, \left(2C_0 + H_{12}\right)\psi_N\right\rangle \\
&\geqslant (1-\alpha)\left\|S_1\psi_N\right\|_{L^2}^2.
\end{aligned}$$

□

We then turn our attention onto the proof of Theorem 2.2. We will prove the following proposition.

**Proposition 2.1.** *Assume $\omega > 0$, $\beta < \frac{1}{6}$, and $\|V\|_{L^1} < \frac{2\alpha}{C_{gn}^4}$ for some $\alpha \in (0,1)$, define the operator*

$$H_{ij,\alpha} = \alpha S_i^2 + \alpha S_j^2 + \frac{N-1}{N} V_{Nij}.$$

*Then $\forall C_0 > 0$, there exists an $N_0 > 0$ such that*

$$2C_0 + H_{ij,\alpha} \geqslant 0,\ \forall N > N_0.$$

*Here, $N_0$ grows to infinity as $C_0$ approaches $0$.*

*Proof.* See §2.1.1. □

In fact, assuming Proposition 2.1, then

$$\begin{aligned} \langle \psi_N, (2C_0 + H_{12})\, \psi_N \rangle &= (1-\alpha) \left\langle \psi_N, \left(S_1^2 + S_2^2\right) \psi_N \right\rangle + \langle \psi_N, (2C_0 + H_{12,\alpha})\, \psi_N \rangle \\ &\geqslant 2(1-\alpha) \|S_1 \psi_N\|_{L^2}^2 . \end{aligned}$$

Hence we are left with the proof of Proposition 2.1.

2.1.1. *Proof of Proposition 2.1.* Define the Littlewood-Paley projectors (eigenspace projectors) by

$$\begin{aligned} P_{\leqslant M}^j &= \chi_{(0,M]}(S_j), \\ P_{>M}^j &= \chi_{(M,\infty)}(S_j), \\ P_{\leqslant M}^{(k)} &= \prod_{j=1}^k P_{\leqslant M}^j,\ P_{>M}^{(k)} = \prod_{j=1}^k P_{>M}^j \end{aligned}$$

We will need the following lemmas.

**Lemma 2.1.** [12]*Let $H_{ij,\alpha}$ be defined as in Proposition 2.1, then, for all $\varepsilon \in (0,1)$, as long as $M \geqslant \sqrt{\frac{3\|V\|_\infty}{2\alpha} \frac{N^\beta}{\varepsilon}}$, we have*

$$H_{12,\alpha} \geqslant P_{\leqslant M}^{(2)} H_{12,\alpha} P_{\leqslant M}^{(2)} - 2\varepsilon^2 P_{\leqslant M}^{(2)} |V_{N12}|\, P_{\leqslant M}^{(2)}.$$

*Proof.* We write

$$S_j^2 = \left(P_{\leqslant M}^{(2)} + P_{>M}^{(2)}\right) S_j^2 \left(P_{\leqslant M}^{(2)} + P_{>M}^{(2)}\right) = P_{\leqslant M}^{(2)} S_j^2 P_{\leqslant M}^{(2)} + P_{>M}^{(2)} S_j^2 P_{>M}^{(2)}$$

because

$$P_{>M}^{(2)} S_j^2 P_{\leqslant M}^{(2)} = P_{\leqslant M}^{(2)} S_j^2 P_{>M}^{(2)} = 0.$$

We then write

$$\begin{aligned} V_{N12} &= \left(P_{\leqslant M}^{(2)} + P_{>M}^{(2)}\right) V_{N12} \left(P_{\leqslant M}^{(2)} + P_{>M}^{(2)}\right) \\ &= P_{\leqslant M}^{(2)} V_{N12} P_{\leqslant M}^{(2)} + P_{>M}^{(2)} V_{N12} P_{>M}^{(2)} + P_{>M}^{(2)} V_{N12} P_{\leqslant M}^{(2)} + P_{\leqslant M}^{(2)} V_{N12} P_{>M}^{(2)} \end{aligned}$$

We estimate the high-high terms by

$$\left\langle \psi_N, P_{>M}^{(2)} V_{N12} P_{>M}^{(2)} \psi_N \right\rangle \geqslant -N^{2\beta} \|V\|_\infty \left\| P_{>M}^{(2)} \psi_N \right\|_{L^2}^2 .$$

[12]This lemma is essentially [49, Lemma 3.6].

and the high-low and the low-high terms by Cauchy-Schwarz,

$$\begin{aligned}\left\langle \psi_N, P_{>M}^{(2)} V_{N12} P_{\leqslant M}^{(2)} \psi_N \right\rangle &\geqslant -\frac{1}{\varepsilon^2} \left\langle P_{>M}^{(2)} \psi_N, |V_{N12}| P_{>M}^{(2)} \psi_N \right\rangle - \varepsilon^2 \left\langle P_{\leqslant M}^{(2)} \psi_N, |V_{N12}| P_{\leqslant M}^{(2)} \psi_N \right\rangle \\ &\geqslant -\frac{N^{2\beta} \|V\|_\infty}{\varepsilon^2} \left\| P_{>M}^{(2)} \psi_N \right\|_{L^2}^2 - \varepsilon^2 \left\langle P_{\leqslant M}^{(2)} \psi_N, |V_{N12}| P_{\leqslant M}^{(2)} \psi_N \right\rangle .\end{aligned}$$

Hence

$$\begin{aligned}H_{12,\alpha} &\geqslant P_{\leqslant M}^{(2)} H_{12,\alpha} P_{\leqslant M}^{(2)} + P_{>M}^{(2)} C_0 P_{>M}^{(2)} + \alpha P_{>M}^{(2)} S_1^2 P_{>M}^{(2)} + \alpha P_{>M}^{(2)} S_2^2 P_{>M}^{(2)} \\ &\quad - P_{>M}^{(2)} \|V\|_\infty (1 + \frac{2}{\varepsilon^2}) N^{2\beta} P_{>M}^{(2)} - 2\varepsilon^2 P_{\leqslant M}^{(2)} |V_{N12}| P_{\leqslant M}^{(2)}.\end{aligned}$$

Whenever $M \geqslant \sqrt{\frac{3\|V\|_\infty}{2\alpha}} \frac{N^\beta}{\varepsilon}$, we have

$$\begin{aligned}&\alpha P_{>M}^{(2)} S_1^2 P_{>M}^{(2)} + \alpha P_{>M}^{(2)} S_2^2 P_{>M}^{(2)} - P_{>M}^{(2)} \|V\|_\infty (1 + \frac{2}{\varepsilon^2}) N^{2\beta} P_{>M}^{(2)} \\ \geqslant\ & P_{>M}^{(2)} 2\alpha M^2 P_{>M}^{(2)} - P_{>M}^{(2)} \|V\|_\infty (1 + \frac{2}{\varepsilon^2}) N^{2\beta} P_{>M}^{(2)} \geqslant 0.\end{aligned}$$

Hence

$$H_{12,\alpha} \geqslant P_{\leqslant M}^{(2)} H_{12,\alpha} P_{\leqslant M}^{(2)} - 2\varepsilon^2 P_{\leqslant M}^{(2)} |V_{N12}| P_{\leqslant M}^{(2)}$$

as claimed. □

**Lemma 2.2** (Finite dimensional quantum de Finetti [23, Theorem II.8] or [49, Lemma 3.4])**.** [13]*Assume* $\left\{ \gamma_N^{(k)} \right\}_{k=1}^N$ *is the marginal density generated by a $N$-body wave function $\psi_N \in L_s^2(\mathbb{R}^{2N})$. Then there is a probability measure $d\mu_N$ supported on the unit sphere of $P_{\leqslant M}\left(L_s^2(\mathbb{R}^2)\right)$ such that*

$$\operatorname{Tr} \left| P_{\leqslant M}^{(2)} \gamma_N^{(2)} P_{\leqslant M}^{(2)} - \int_{S(P_{\leqslant M}(L_s^2(\mathbb{R}^2)))} \left| \phi^{\otimes 2} \right\rangle \left\langle \phi^{\otimes 2} \right| d\mu_N(\phi) \right| \leqslant \frac{8 D_M}{N}$$

*where $D_M$ is the dimension of $P_{\leqslant M}\left(L_s^2(\mathbb{R}^2)\right)$.*

**Remark 2.** *Lemma 2.2 is the only place in which this paper needs $\omega > 0$. It is a major open problem to prove Lemma 2.2 without assuming a finite dimensional Hilbert space.*

**Lemma 2.3.** *If $\|V\|_{L^1} < \frac{2\alpha}{C_{gn}^4}$, then there exists $\varepsilon$ which depends solely on $\|V\|_{L^1}$ such that, for all $\phi \in L^2(\mathbb{R}^2)$ with $\|\phi\|_{L^2} = 1$, we have*

$$E_\varepsilon(\phi) = \left\langle \phi(x_1)\phi(x_2), H_{12,\alpha}^\varepsilon \phi(x_1)\phi(x_2) \right\rangle \geqslant 0$$

*where*

$$H_{12,\alpha}^\varepsilon = \alpha S_1^2 + \alpha S_2^2 + \frac{N-1}{N} V_{N12} - 2\varepsilon^2 |V_{N12}| \tag{2.3}$$

[13]To be precise, this version we are using is [49, Lemma 3.4]. If one uses [23, Theorem II.8] to prove it, one will have a 16 instead of a 8. The optimal coefficient is important in the literature of de Finetti theorems, but it does not matter for our application here.

*Proof.* We first compute directly that

$$\begin{aligned} E_{\varepsilon}(\phi) &= 2\alpha\int|S\phi|^{2}\,dx+\frac{N-1}{N}\int V_{N12}\,|\phi(x_{1})\phi(x_{2})|^{2}\,dx_{1}dx_{2} \\ &\quad -2\varepsilon^{2}\int|V_{N12}|\,|\phi(x_{1})\phi(x_{2})|^{2}\,dx_{1}dx_{2}. \end{aligned}$$

Apply Cauchy-Schwarz,

$$\geqslant 2\alpha\int|S\phi|^{2}\,dx-(1+2\varepsilon^{2})\left\||\phi|^{2}\right\|_{L^{2}}\left\|V_{N}*|\phi|^{2}\right\|_{L^{2}}.$$

Use Young's convolution inequality,

$$\begin{aligned} &\geqslant 2\alpha\int|S\phi|^{2}\,dx-(1+2\varepsilon^{2})\,\|V_{N}\|_{L^{1}}\left\||\phi|^{2}\right\|_{L^{2}}\left\||\phi|^{2}\right\|_{L^{2}} \\ &= 2\alpha\int|S\phi|^{2}\,dx-(1+2\varepsilon^{2})\,\|V\|_{L^{1}}\,\|\phi\|_{L^{4}}^{4}. \end{aligned}$$

With estimate (1.7), we get to

$$E_{\varepsilon}(\phi)\geqslant 2\alpha\int|S\phi|^{2}\,dx-(1+2\varepsilon^{2})C_{gn}^{4}\,\|V\|_{L^{1}}\,\|\nabla\phi\|_{L^{2}}^{2}.$$

Hence, when $\|V\|_{L^{1}}<\frac{2\alpha}{C_{gn}^{4}}$, we can select $\varepsilon$ small enough so that

$$E_{\varepsilon}(\phi)\geqslant 0.$$

□

With Lemmas 2.1 to 2.3, we now prove Proposition 2.1.

*Proof of Proposition 2.1.* The trick is to notice the equaltiy

$$\left\langle P_{\leqslant M}^{(2)}\psi_{N},H_{12,\alpha}^{\varepsilon}P_{\leqslant M}^{(2)}\psi_{N}\right\rangle=\operatorname{Tr}H_{12,\alpha}^{\varepsilon}P_{\leqslant M}^{(2)}\gamma_{N}^{(2)}P_{\leqslant M}^{(2)}$$

where $H_{12,\alpha}^{\varepsilon}$ is defined in (2.3). It helps because

$$\begin{aligned} &\left\langle\psi_{N},\left(2C_{0}+H_{12,\alpha}\right)\psi_{N}\right\rangle \\ \geqslant\;& 2C_{0}+\left\langle P_{\leqslant M}^{(2)}\psi_{N},H_{12,\alpha}^{\varepsilon}P_{\leqslant M}^{(2)}\psi_{N}\right\rangle \\ =\;& 2C_{0}+\operatorname{Tr}H_{12,\alpha}^{\varepsilon}P_{\leqslant M}^{\otimes 2}\gamma_{N}^{(2)}P_{\leqslant M}^{(2)} \end{aligned}$$

provided that $M\geqslant\sqrt{\frac{3\|V\|_{\infty}}{2\alpha}\frac{N^{\beta}}{\varepsilon}}$, by Lemma 2.1.

Rewrite

$$\begin{aligned} &\operatorname{Tr}H_{12,\alpha}^{\varepsilon}P_{\leqslant M}^{(2)}\gamma_{N}^{(2)}P_{\leqslant M}^{(2)} \\ =\;&\operatorname{Tr}\int_{S(P_{\leqslant M}(L_{s}^{2}(\mathbb{R}^{2})))}H_{12,\alpha}^{\varepsilon}\left|\phi^{\otimes 2}\right\rangle\left\langle\phi^{\otimes 2}\right|d\mu_{N}(\phi) \\ &+\left[\operatorname{Tr}H_{12,\alpha}^{\varepsilon}P_{\leqslant M}^{(2)}\gamma_{N}^{(2)}P_{\leqslant M}^{(2)}-\int_{S(P_{\leqslant M}(L_{s}^{2}(\mathbb{R}^{2})))}H_{12,\alpha}^{\varepsilon}\left|\phi^{\otimes 2}\right\rangle\left\langle\phi^{\otimes 2}\right|d\mu_{N}(\phi)\right] \end{aligned}$$

We can use the inequality $\operatorname{Tr} AB \leqslant \|A\|_{op} \operatorname{Tr}|B|$ to get to

$$\begin{aligned} & \langle \psi_N, (2C_0 + H_{12,\alpha})\, \psi_N \rangle \\ \geqslant\ & 2C_0 + \int_{S(P_{\leqslant M}(L^2_s(\mathbb{R}^2)))} E_\varepsilon(\phi) d\mu_N(\phi) \\ & - \left\| H^\varepsilon_{12,\alpha} \right\|_{op} \operatorname{Tr} \left| P^{(2)}_{\leqslant M} \gamma^{(2)}_N P^{(2)}_{\leqslant M} - \int_{S(P_{\leqslant M}(L^2_s(\mathbb{R}^2)))} \left|\phi^{\otimes 2}\right\rangle \left\langle \phi^{\otimes 2}\right| d\mu_N(\phi) \right| \end{aligned}$$

Now fix $\varepsilon$ as in Lemma 2.3, apply Lemma 2.3 on the second term and Lemma 2.2 on the third term, it becomes

$$\langle \psi_N, (2C_0 + H_{12,\alpha})\, \psi_N \rangle \geqslant 2C_0 - \left\| H^\varepsilon_{12,\alpha} \right\|_{op} \frac{8D_M}{N}.$$

On the one hand, with frequency smaller than $M$, the Hermite operator in 2D has at most $M^4$ eigenfunctions, that is

$$D_M \leqslant \left(M^2\right)^2 \leqslant \frac{CN^{4\beta}}{\varepsilon^4}.$$

On the other hand,

$$\left\| H^\varepsilon_{12,\alpha} \right\|_{op} \leqslant 2\alpha M^2 + (1+2\varepsilon^2) \|V\|_{L^\infty} N^{2\beta} \leqslant \frac{CN^{2\beta}}{\varepsilon^2}.$$

Thus we conclude that

$$\langle \psi_N, (2C_0 + H_{12,\alpha})\, \psi_N \rangle \geqslant 2C_0 - \frac{CN^{6\beta}}{N} \geqslant 0$$

provided that $N$ is large enough and $\beta < \frac{1}{6}$. Thence we have completed the proof of Proposition 2.1, concluded Theorem 2.2, and obtained Theorem 2.1. $\square$

**Remark 3.** *The above proof is exactly what we meant by saying "though $V_N$ gets more singular as $N \to \infty$, but larger $N$ beats it." in the introduction.*

2.2. **High Energy Estimates when $k > 1$.** Assuming (1.16) holds for $k$, we now prove it for $k+2$. Using the induction hypothesis, we arrive at

$$\frac{1}{c_0^{k+2}} \langle \psi_N, (N^{-1}H_N + 1)^{k+2} \psi_N \rangle \geqslant \frac{1}{c_0^2} \langle S^{(k)}(N^{-1}H_N+1)\psi_N, S^{(k)}(N^{-1}H_N+1)\psi_N \rangle. \tag{2.4}$$

We decompose $N^{-1}H_N + 1$ like in (2.2), but this time we separate the sum as

$$N^{-1}H_N + 1 = \frac{1}{N(N-1)} \sum_{\substack{1 \le i < j \le N \\ i \le k}} (2 + H_{ij}) + \frac{1}{N(N-1)} \sum_{\substack{1 \le i < j \le N \\ i > k}} (2 + H_{ij}).$$

Then (2.4) unfold into three terms if we combine the two crossing terms, namely

$$\frac{1}{c_0^{k+2}} \left\langle \psi_N, (N^{-1}H_N + 1)^{k+2} \psi_N \right\rangle \geqslant M + E_C + E_P$$

where the main term $M$ is

$$M=\frac{1}{c_0^2N^2(N-1)^2}\sum_{\substack{1\leq i_1<j_1\leq N\\ 1\leq i_2<j_2\leq N\\ \text{such that } i_1>k,\, i_2>k}}\left\langle S^{(k)}\left(2+H_{i_1j_1}\right)\psi_N, S^{(k)}\left(2+H_{i_2j_2}\right)\psi_N\right\rangle,$$

the cross error term $E$ is

$$E_C=\frac{1}{c_0^2N^2(N-1)^2}\sum_{\substack{1\leq i_1<j_1\leq N\\ 1\leq i_2<j_2\leq N\\ \text{such that } i_1\leq k,\, i_2>k}}2\operatorname{Re}\left\langle S^{(k)}\left(2+H_{i_1j_1}\right)\psi_N, S^{(k)}\left(2+H_{i_2j_2}\right)\psi_N\right\rangle,$$

and the nonnegative error term $E_P$ is

$$\begin{aligned}E_P &= \frac{1}{c_0^2N^2(N-1)^2}\sum_{\substack{1\leq i_1<j_1\leq N\\ 1\leq i_2<j_2\leq N\\ \text{such that } i_1\leq k,\, i_2\leq k}}\left\langle S^{(k)}\left(2+H_{i_1j_1}\right)\psi_N,\ S^{(k)}\left(2+H_{i_2j_2}\right)\psi_N\right\rangle \\ &= \frac{1}{c_0^2N^2(N-1)^2}\left\langle \sum_{\substack{1\leq i<j\leq N\\ i\leqslant k}} S^{(k)}\left(2+H_{i_1j_1}\right)\psi_N, \sum_{\substack{1\leq i<j\leq N\\ i\leqslant k}} S^{(k)}\left(2+H_{i_2j_2}\right)\psi_N\right\rangle \geqslant 0.\end{aligned}$$

Here, we distinguish the terms by the cardinality of the sums. Implicitly, we always have $N>>k$, hence the main contribution comes from the sum $\sum_{k<i<N}$. In fact, $M$ has $\sim N^4$ summands inside while the cross error term $E_C$ has $\sim N^3$ summands.

Since the nonnegative error term $E_P\geqslant 0$, we drop it and (2.4) becomes

$$\frac{1}{c_0^{k+2}}\left\langle\psi_N,(N^{-1}H_N+1)^{k+2}\psi_N\right\rangle\geqslant M+E_C. \tag{2.5}$$

The strategy is to first extract the desired kinetic energy part from the main term $M$ in §2.2.1 then prove that the cross error term $E_C$ can be absorbed into $M$ for large $N$ in §2.2.2. During the course of the proof, we will need the following lemma.

**Lemma 2.4** ([19, Lemma A.2]). *If $A_1\geq A_2\geq 0$, $B_1\geq B_2\geq 0$ and $A_iB_j=B_jA_i$ for all $1\leq i,j\leq 2$, then $A_1B_1\geq A_2B_2$.*

2.2.1. *Handling the Main Term.* Commute $(1+H_{i_1j_1})$ and $(1+H_{i_2j_2})$ with $S^{(k)}$ in $M$,

$$M=\frac{1}{c_0^2N^2(N-1)^2}\sum_{\substack{1\leq i_1<j_1\leq N\\ 1\leq i_2<j_2\leq N\\ \text{such that } i_1>k,\, i_2>k}}\langle S^{(k)}\psi_N,\ \left(2+H_{i_1j_1}\right)\left(2+H_{i_2j_2}\right)S^{(k)}\psi_N\rangle \tag{2.6}$$

We decompose the sum into three pieces

$$M=M_1+M_2+M_3$$

where $M_1$ consists of the terms with

$$\{i_1,j_1\}\cap\{i_2,j_2\}=\varnothing,$$

$M_2$ consists of the terms with

$$|\{i_1,j_1\}\cap\{i_2,j_2\}|=1,$$

and $M_3$ consists of the terms with

$$|\{i_1, j_1\} \cap \{i_2, j_2\}| = 2.$$

By symmetry of $\psi_N$, we have

$$\begin{aligned}
M_1 &= \frac{1}{4c_0^2}\langle \left(2 + H_{(k+1)(k+2)}\right) S^{(k)}\psi_N, \left(2 + H_{(k+3)(k+4)}\right) S^{(k)}\psi_N\rangle \\
M_2 &= \frac{1}{2c_0^2}N^{-1}\langle \left(2 + H_{(k+1)(k+2)}\right) S^{(k)}\psi_N, \left(2 + H_{(k+2)(k+3)}\right) S^{(k)}\psi_N\rangle \\
M_3 &= \frac{1}{2c_0^2}N^{-2}\langle \left(2 + H_{(k+1)(k+2)}\right) S^{(k)}\psi_N, \left(2 + H_{(k+1)(k+2)}\right) S^{(k)}\psi_N\rangle
\end{aligned}$$

We drop $M_3$ since it is nonnegative. Thus (2.6) becomes

$$M \geqslant M_1 + M_2.$$

By the fact that

$$\left[2 + H_{(k+1)(k+2)}, 2 + H_{(k+3)(k+4)}\right] = 0,$$

we deduce

$$M_1 \geqslant \frac{4(1-\alpha)^2}{4c_0^2}\langle S^{(k)}\psi_N, S_{k+1}^2 S_{k+2}^2 S^{(k)}\psi_N\rangle$$

using Theorem 2.2 and Lemma 2.4. Recall $c_0 = \min(\frac{1-\alpha}{\sqrt{2}}, \frac{1}{2})$, hence

$$M_1 \geqslant 2\langle S^{(k+2)}\psi_N, S^{(k+2)}\psi_N\rangle = 2\left\|S^{(k+2)}\psi_N\right\|_{L^2}^2. \tag{2.7}$$

We now deal with $M_2$. We expand

$$M_2 = M_{21} + M_{22} + M_{23}$$

where

$$\begin{aligned}
M_{21} &= \frac{N^{-1}}{2c_0^2}\langle \left(2 + S_{k+1}^2 + S_{k+2}^2\right) S^{(k)}\psi_N, \left(2 + S_{k+2}^2 + S_{k+3}^2\right) S^{(k)}\psi_N\rangle, \\
M_{22} &= \frac{N^{-1}}{c_0^2}\operatorname{Re}\langle \left(2 + S_{k+1}^2 + S_{k+2}^2\right) S^{(k)}\psi_N, V_{N(k+2)(k+3)}S^{(k)}\psi_N\rangle, \\
M_{23} &= \frac{N^{-1}}{2c_0^2}\langle V_{N(k+1)(k+2)}S^{(k)}\psi_N, V_{N(k+2)(k+3)}S^{(k)}\psi_N\rangle.
\end{aligned}$$

We keep only the $S_{k+2}^4$ terms inside $M_{21}$, which carries as many derivatives as in (2.7) and hence is the second main contribution. That is

$$M_{21} \geqslant \frac{N^{-1}}{2c_0^2}\langle S_{k+2}^2 S_{k+2}^2 S^{(k)}\psi_N, S^{(k)}\psi_N\rangle \geqslant 2N^{-1}\langle S_{k+1}^4 S^{(k)}\psi_N, S^{(k)}\psi_N\rangle = 2N^{-1}\|S_1 S^{(k+1)}\psi_N\|_{L^2}^2. \tag{2.8}$$

For $M_{22}$, we first rearrange the derivatives

$$\begin{aligned} M_{22} &= \frac{2N^{-1}}{c_0^2}\langle S^{(k)}\psi_N, V_{N(k+2)(k+3)}S^{(k)}\psi_N\rangle \\ &+\frac{N^{-1}}{c_0^2}\langle S^{(k+1)}\psi_N, V_{N(k+2)(k+3)}S^{(k+1)}\psi_N\rangle \\ &+\frac{N^{-1}}{c_0^2}\langle S_{k+2}S^{(k)}\psi_N, V_{N(k+2)(k+3)}S_{k+2}S^{(k)}\psi_N\rangle \\ &+\frac{N^{\beta-1}}{c_0^2}\operatorname{Re}\langle S_{k+2}S^{(k)}\psi_N, (\nabla V)_{N(k+2)(k+3)}\, S^{(k)}\psi_N\rangle \end{aligned}$$

Notice that, in the above, we have used the fact that $\nabla$ is the only thing inside $S_j$ that needs the Leibniz's rule.[14] Do Hölder,

$$\begin{aligned} |M_{22}| &\lesssim N^{-1}\left\|V_{N(k+2)(k+3)}\right\|_{L^1_{x_{k+3}}}\left\|S^{(k)}\psi_N\right\|^2_{L^2L^\infty_{x_{k+3}}} \\ &+N^{-1}\left\|V_{N(k+2)(k+3)}\right\|_{L^{1+}_{x_{k+3}}}\left\|S^{(k+1)}\psi_N\right\|^2_{L^2L^{\infty-}_{x_{k+3}}} \\ &+N^{-1}\left\|V_{N(k+2)(k+3)}\right\|_{L^{1+}_{x_{k+3}}}\left\|S_{k+2}S^{(k)}\psi_N\right\|^2_{L^2L^{\infty-}_{x_{k+3}}} \\ &+N^{\beta-1}\left\|(\nabla V)_{N(k+2)(k+3)}\right\|_{L^{1+}_{x_{k+3}}}\left\|S_{k+2}S^{(k)}\psi_N\right\|_{L^2L^{\infty-}_{x_{k+3}}}\left\|S^{(k)}\psi_N\right\|_{L^2L^{\infty-}_{x_{k+3}}}, \end{aligned}$$

Apply Sobolev,

$$\begin{aligned} |M_{22}| &\lesssim N^{-1}\left\|S^{(k+2)}\psi_N\right\|^2_{L^2}+N^{-1+}\left\|S^{(k+2)}\psi_N\right\|^2_{L^2}+N^{-1+}\left\|S^{(k+2)}\psi_N\right\|^2_{L^2} \\ &\quad +N^{\beta-1+}\left\|S^{(k+2)}\psi_N\right\|^2_{L^2} \\ &\leqslant CN^{\beta-1+}\left\|S^{(k+2)}\psi_N\right\|^2_{L^2}, \end{aligned} \tag{2.9}$$

which is easily absorbed into the positive contributions. Alert reader should notice the loss due to the failure of the 2D endpoint Sobolev: $\frac{1}{2}-\frac{1}{\infty}=\frac{1}{2}$.

Do the same thing for $M_{23}$,

$$\begin{aligned} |M_{23}| &\lesssim N^{-1}\left\|V_{N(k+1)(k+2)}\right\|_{L^{1+}_{x_{k+1}}}\left\|V_{N(k+2)(k+3)}\right\|_{L^{1+}_{x_{k+3}}}\left\|S^{(k)}\psi_N\right\|_{L^2L^{\infty-}_{x_{k+1}}L^{\infty-}_{x_{k+3}}} \\ &\leqslant CN^{-1+}\left\|S^{(k+2)}\psi_N\right\|^2_{L^2}. \end{aligned} \tag{2.10}$$

Collecting (2.7)-(2.10), we arrive at the following estimate for $M$:

$$M \geqslant \left(2-CN^{\beta-1+}\right)\left(\|S^{(k+2)}\psi\|^2_{L^2}+N^{-1}\|S_1S^{(k+1)}\psi\|^2_{L^2}\right). \tag{2.11}$$

2.2.2. *Handling the Cross Error Term.* Next we turn our attention to estimating $E_C$. We will prove that

$$E_C \geqslant -C\max(N^{2\beta-\frac{3}{2}+}, N^{\beta-1+})\left(\|S^{(k+2)}\psi_N\|^2_{L^2}+N^{-1}\|S_1S^{(k+1)}\psi_N\|^2_{L^2}\right). \tag{2.12}$$

That is, $E_C$ is an absorbable error if added into (2.11).

[14] This is a fact proved and used by many authors. See, for example, [62].

We assume $k \geqslant 1$, since $E_C = 0$ when $k = 0$. We decompose the sum into three parts

$$E_C = E_1 + E_2 + E_3 \tag{2.13}$$

where $E_1$ contains the terms with $j_1 \leqslant k$, $E_2$ contains the terms with $j_1 > k$ and $j_1 \in \{i_2, j_2\}$, and $E_3$ contains those terms with $j_1 > k$, $j_1 \neq i_2$ and $j_1 \neq j_2$.

Since $H_{ij} = H_{ji}$, by symmetry of $\psi_N$, we have

$$\begin{aligned} E_1 &= k^2 N^{-2} \langle S^{(k)} (2 + H_{12}) \psi_N, S^{(k)} \left(2 + H_{(k+1)(k+2)}\right) \psi_N \rangle \\ E_2 &= k N^{-2} \langle S^{(k)} \left(2 + H_{1(k+1)}\right) \psi_N, S^{(k)} \left(2 + H_{(k+1)(k+2)}\right) \psi_N \rangle \\ E_3 &= N^{-1} \langle S^{(k)} \left(2 + H_{1(k+1)}\right) \psi_N, S^{(k)} \left(2 + H_{(k+2)(k+3)}\right) \psi_N \rangle \end{aligned}$$

We first address $E_1$. We commute $(2 + H_{12})$ with $S^{(k)}$ and obtain

$$E_1 = E_{11} + E_{12} + E_{13},$$

where

$$\begin{aligned} E_{11} &= N^{-2} \langle (2 + H_{12}) S^{(k)} \psi_N, \left(2 + H_{(k+1)(k+2)}\right) S^{(k)} \psi_N \rangle \\ E_{12} &= N^{-2} \langle S_1 [S_2, H_{12}] \frac{S^{(k)}}{S_1 S_2} \psi_N, \left(2 + H_{(k+1)(k+2)}\right) S^{(k)} \psi_N \rangle \\ E_{13} &= N^{-2} \langle [S_1, H_{12}] \frac{S^{(k)}}{S_1} \psi_N, \left(2 + H_{(k+1)(k+2)}\right) S^{(k)} \psi_N \rangle \end{aligned}$$

By Theorem 2.2 and Lemma 2.4, $E_{11} \geqslant 0$ and we drop it. For $E_{12}$, since $[S_2, H_{12}] = -N^{\beta} (\nabla V)_{N12}$, expanding $\left(2 + H_{(k+1)(k+2)}\right)$ gives

$$\begin{aligned} E_{12} = \; & -2N^{\beta-2} \langle (\nabla V)_{N12} \frac{S^{(k)}}{S_1 S_2} \psi_N, S_1 S^{(k)} \psi_N \rangle \\ & -N^{\beta-2} \langle (\nabla V)_{N12} \frac{S^{(k)}}{S_1 S_2} \psi_N, (S_{k+1}^2 + S_{k+2}^2) S_1 S^{(k)} \psi_N \rangle \\ & -N^{\beta-2} \langle (\nabla V)_{N12} \frac{S^{(k)}}{S_1 S_2} \psi_N, V_{N(k+1)(k+2)} S_1 S^{(k)} \psi_N \rangle \end{aligned}$$

Use Holder,

$$\begin{aligned} & |E_{12}| \\ \lesssim \; & N^{\beta - \frac{3}{2}} \left\| (\nabla V)_{N12} \right\|_{L^{2+}_{x_1}} \left\| \frac{S^{(k)}}{S_1 S_2} \psi_N \right\|_{L^2 L^{\infty-}_{x_1}} N^{-\frac{1}{2}} \left\| S_1 S^{(k)} \psi_N \right\|_{L^2} \\ & + N^{\beta - \frac{3}{2}} \left\| (\nabla V)_{N12} \right\|_{L^{2+}_{x_1}} \left\| \frac{S^{(k+1)}}{S_1 S_2} \psi_N \right\|_{L^2 L^{\infty-}_{x_1}} N^{-\frac{1}{2}} \left\| S_1 S^{(k+1)} \psi_N \right\|_{L^2} \\ & + N^{\beta - \frac{3}{2}} \left\| (\nabla V)_{N12} \right\|_{L^{2+}_{x_1}} \left\| V_{N(k+1)(k+2)} \right\|_{L^{1+}_{x_{k+1}}} \left\| \frac{S^{(k)}}{S_1 S_2} \psi_N \right\|_{L^2 L^{\infty-}_{x_1} L^{\infty-}_{x_{k+1}}} N^{-\frac{1}{2}} \left\| S_1 S^{(k)} \psi_N \right\|_{L^2 L^{\infty-}_{x_{k+1}}} \end{aligned}$$

Use Sobolev and notice that $\|(\nabla V)_{N12}\|_{L^{2+}_{x_1}} \sim N^{\beta+}$ in 2D, we have

$$
\begin{aligned}
|E_{12}| &\lesssim N^{2\beta-\frac{3}{2}+}\left\|S^{(k-1)}\psi_N\right\|_{L^2} N^{-\frac{1}{2}}\left\|S_1 S^{(k)}\psi_N\right\|_{L^2} \\
&\quad +N^{2\beta-\frac{3}{2}+}\left\|S^{(k)}\psi_N\right\|_{L^2} N^{-\frac{1}{2}}\left\|S_1 S^{(k)}\psi_N\right\|_{L^2} \\
&\quad +N^{2\beta-\frac{3}{2}+}\left\|S^{(k)}\psi_N\right\|_{L^2} N^{-\frac{1}{2}}\left\|S_1 S^{(k+1)}\psi_N\right\|_{L^2} \\
&\lesssim N^{2\beta-\frac{3}{2}+}\left(\left\|S^{(k)}\psi_N\right\|_{L^2}^2 + N^{-1}\left\|S_1 S^{(k+1)}\psi_N\right\|_{L^2}^2\right).
\end{aligned} \tag{2.14}
$$

Now, for $E_{13}$, notice that $[S_1, H_{12}] = N^{\beta}(\nabla V)_{N12}$, writting out $\left(2+H_{(k+1)(k+2)}\right)$ gives,

$$
\begin{aligned}
E_{13} &= 2N^{\beta-2}\langle(\nabla V)_{N12}\frac{S^{(k)}}{S_1}\psi_N, S^{(k)}\psi_N\rangle \\
&\quad +N^{\beta-2}\langle(\nabla V)_{N12}\frac{S^{(k)}}{S_1}\psi_N, (S_{k+1}^2+S_{k+2}^2)S^{(k)}\psi_N\rangle \\
&\quad +N^{\beta-2}\langle(\nabla V)_{N12}\frac{S^{(k)}}{S_1}\psi_N, V_{N(k+1)(k+2)}S^{(k)}\psi_N\rangle.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&|E_{13}| \\
\lesssim\ & N^{\beta-2}\|(\nabla V)_{N12}\|_{L^{2+}_{x_1}}\left\|\frac{S^{(k)}}{S_1}\psi_N\right\|_{L^2 L^{\infty-}_{x_1}}\left\|S^{(k)}\psi_N\right\|_{L^2} \\
&+N^{\beta-2}\|(\nabla V)_{N12}\|_{L^{2+}_{x_1}}\left\|\frac{S^{(k+1)}}{S_1}\psi_N\right\|_{L^2 L^{\infty-}_{x_1}}\left\|S^{(k+1)}\psi_N\right\|_{L^2} \\
&+N^{\beta-2}\|(\nabla V)_{N12}\|_{L^{2+}_{x_1}}\left\|V_{N(k+1)(k+2)}\right\|_{L^{1+}_{x_{k+1}}}\left\|\frac{S^{(k)}}{S_1}\psi_N\right\|_{L^2 L^{\infty-}_{x_1} L^{\infty-}_{x_{k+1}}}\left\|S^{(k)}\psi_N\right\|_{L^2 L^{\infty-}_{x_{k+1}}}.
\end{aligned}
$$

Hence, with the Sobolev estimates,

$$
\begin{aligned}
&|E_{13}| \\
\lesssim\ & N^{2\beta-2+}\left\|S^{(k)}\psi_N\right\|_{L^2}\left\|S^{(k)}\psi_N\right\|_{L^2} + N^{2\beta-2+}\left\|S^{(k+1)}\psi_N\right\|_{L^2}\left\|S^{(k+1)}\psi_N\right\|_{L^2} \\
&+N^{2\beta-2+}\left\|S^{(k+1)}\psi_N\right\|_{L^2}\left\|S^{(k+1)}\psi_N\right\|_{L^2} \\
\lesssim\ & N^{2\beta-2+}\left\|S^{(k+1)}\psi_N\right\|_{L^2}^2.
\end{aligned} \tag{2.15}
$$

Hence, combining with (2.14), we have acquired

$$
E_1 \geqslant -CN^{2\beta-\frac{3}{2}+}\left(\left\|S^{(k+1)}\psi_N\right\|_{L^2}^2 + N^{-1}\left\|S_1 S^{(k+1)}\psi_N\right\|_{L^2}^2\right) \tag{2.16}
$$

since $E_{11} \geqslant 0$.

Next, we deal with $E_2$. We remind the readers that

$$
E_2 = kN^{-2}\langle S^{(k)}\left(2+H_{1(k+1)}\right)\psi_N, S^{(k)}\left(2+H_{(k+1)(k+2)}\right)\psi_N\rangle.
$$

Commuting $\left(2+H_{1(k+1)}\right)$ to the front, we write

$$
E_2 = E_{21} + E_{22}
$$

where

$$\begin{aligned}
E_{21} &= N^{-2}\langle\left(2+H_{1(k+1)}\right)S^{(k)}\psi_N, S^{(k)}\left(2+H_{(k+1)(k+2)}\right)\psi_N\rangle,\\
E_{22} &= N^{-2}\langle[S_1,H_{1(k+1)}]\frac{S^{(k)}}{S_1}\psi_N, S^{(k)}\left(2+H_{(k+1)(k+2)}\right)\psi_N\rangle.
\end{aligned}$$

For $E_{21}$, expanding $2+H_{ij}$ yields

$$E_{21}=E_{211}+E_{212}+E_{213}+E_{214}$$

where

$$\begin{aligned}
E_{211} &= N^{-2}\langle\left(2+S_1^2+S_{k+1}^2\right)S^{(k)}\psi_N, S^{(k)}\left(2+S_{k+1}^2+S_{k+2}^2\right)\psi_N\rangle,\\
E_{212} &= N^{-2}\langle\left(2+S_1^2+S_{k+1}^2\right)S^{(k)}\psi_N, V_{N(k+1)(k+2)}S^{(k)}\psi_N\rangle,\\
E_{213} &= N^{-2}\langle V_{N1(k+1)}S^{(k)}\psi_N, S^{(k)}\left(2+S_{k+1}^2+S_{k+2}^2\right)\psi_N\rangle,\\
E_{214} &= N^{-2}\langle V_{N1(k+1)}S^{(k)}\psi_N, V_{N(k+1)(k+2)}S^{(k)}\psi_N\rangle.
\end{aligned}$$

Note that $E_{211}\geqslant 0$, so we can discard it. Expand $E_{212}$,

$$\begin{aligned}
E_{212} =\ & 2N^{-2}\langle S^{(k)}\psi_N, V_{N(k+1)(k+2)}S^{(k)}\psi_N\rangle\\
&+N^{-2}\langle S_1S^{(k)}\psi_N, V_{N(k+1)(k+2)}S_1S^{(k)}\psi_N\rangle\\
&+N^{\beta-2}\langle S^{(k+1)}\psi_N, (\nabla V)_{N(k+1)(k+2)}S^{(k)}\psi_N\rangle\\
&+N^{-2}\langle S^{(k+1)}\psi_N, V_{N(k+1)(k+2)}S^{(k+1)}\psi_N\rangle
\end{aligned}$$

Apply Hölder,

$$\begin{aligned}
&|E_{212}|\\
\lesssim\ & N^{-2}\left\|V_{N(k+1)(k+2)}\right\|_{L^1_{x_{k+1}}}\left\|S^{(k)}\psi_N\right\|^2_{L^2L^\infty_{x_{k+1}}}\\
&+N^{-1}\left\|V_{N(k+1)(k+2)}\right\|_{L^{1+}_{x_{k+1}}}N^{-1}\left\|S_1S^{(k)}\psi_N\right\|^2_{L^2L^{\infty-}_{x_{k+1}}}\\
&+N^{\beta-2}\left\|(\nabla V)_{N(k+1)(k+2)}\right\|_{L^{1+}_{x_{k+2}}}\left\|S^{(k+1)}\psi_N\right\|_{L^2L^{\infty-}_{x_{k+2}}}\left\|S^{(k)}\psi_N\right\|_{L^2L^{\infty-}_{x_{k+2}}}\\
&+N^{-2}\left\|V_{N(k+1)(k+2)}\right\|_{L^{1+}_{x_{k+2}}}\left\|S^{(k+1)}\psi_N\right\|^2_{L^2L^{\infty-}_{x_{k+2}}}
\end{aligned}$$

With Sobolev, we see

$$\begin{aligned}
|E_{212}| \lesssim\ & N^{-2}\left\|S^{(k+2)}\psi_N\right\|^2_{L^2}+N^{-1+}N^{-1}\left\|S_1S^{(k+1)}\psi_N\right\|^2_{L^2}\\
&+N^{\beta-2+}\left\|S^{(k+2)}\psi_N\right\|_{L^2}\left\|S^{(k+1)}\psi_N\right\|_{L^2}\\
&+N^{-2+}\left\|S^{(k+2)}\psi_N\right\|^2_{L^2}\\
\lesssim\ & N^{-1+}\left(\left\|S^{(k+2)}\psi_N\right\|^2_{L^2}+N^{-1}\left\|S_1S^{(k+1)}\psi_N\right\|^2_{L^2}\right)
\end{aligned}\tag{2.17}$$

where we used $\max(N^{\beta-2+},N^{-1+})=N^{-1+}$ for our problem in which $\beta<1$.

For $E_{213}$,

$$\begin{aligned}
E_{213} &= N^{-2}\langle V_{N1(k+1)}S^{(k)}\psi_N, S^{(k)}\left(2+S_{k+1}^2+S_{k+2}^2\right)\psi_N\rangle \\
&= 2N^{-2}\langle V_{N1(k+1)}S^{(k)}\psi_N, S^{(k)}\psi_N\rangle + N^{-2}\langle V_{N1(k+1)}S^{(k)}S_{k+2}\psi_N, S^{(k)}S_{k+2}\psi_N\rangle \\
&\quad + N^{\beta-2}\langle (\nabla V)_{N1(k+1)}S^{(k)}\psi_N, S^{(k+1)}\psi_N\rangle + N^{-2}\langle V_{N1(k+1)}S^{(k+1)}\psi_N, S^{(k+1)}\psi_N\rangle
\end{aligned}$$

Apply Hölder,

$$\begin{aligned}
|E_{213}| &\lesssim N^{-2}\left\|V_{N1(k+1)}\right\|_{L^{1+}_{x_{k+1}}}\left\|S^{(k)}\psi_N\right\|^2_{L^2L^{\infty-}_{x_{k+1}}} \\
&\quad + N^{-2}\left\|V_{N1(k+1)}\right\|_{L^{1+}_{x_{k+1}}}\left\|S_{k+2}S^{(k)}\psi_N\right\|^2_{L^2L^{\infty-}_{x_{k+1}}} \\
&\quad + N^{\beta-1}\left\|(\nabla V)_{N1(k+1)}\right\|_{L^{1+}_{x_1}}N^{-1}\left\|S^{(k)}\psi_N\right\|_{L^2L^{\infty-}_{x_1}}\left\|S^{(k+1)}\psi_N\right\|_{L^2L^{\infty-}_{x_1}} \\
&\quad + N^{-2}\left\|(\nabla V)_{N1(k+1)}\right\|_{L^{\infty}_{x_1}}\left\|S^{(k+1)}\psi_N\right\|^2_{L^2}.
\end{aligned}$$

Utilize Sobolev,

$$\begin{aligned}
|E_{213}| &\lesssim N^{-2+}\left\|S^{(k+1)}\psi_N\right\|^2_{L^2} + N^{-2+}\left\|S^{(k+2)}\psi_N\right\|^2_{L^2} \\
&\quad + N^{\beta-1+}N^{-1}\left\|S_1S^{(k)}\psi_N\right\|_{L^2}\left\|S_1S^{(k+1)}\psi_N\right\|_{L^2} \\
&\quad + N^{2\beta-2}\left\|S^{(k+1)}\psi_N\right\|^2_{L^2}. \\
&\lesssim N^{\beta-1+}\left(N^{-1}\left\|S_1S^{(k+1)}\psi_N\right\|^2_{L^2} + \left\|S^{(k+2)}\psi_N\right\|^2_{L^2}\right)
\end{aligned} \tag{2.18}$$

Then, for $E_{214}$

$$\begin{aligned}
|E_{214}| &= \left|N^{-2}\langle V_{N1(k+1)}S^{(k)}\psi_N, V_{N(k+1)(k+2)}S^{(k)}\psi_N\rangle\right| \\
&\lesssim N^{-2}\left\|V_{N1(k+1)}\right\|_{L^{1+}_{x_1}}\left\|V_{N(k+1)(k+2)}\right\|_{L^{1+}_{x_{k+2}}}\left\|S^{(k)}\psi_N\right\|^2_{L^2L^{\infty-}_{x_1}L^{\infty-}_{x_{k+2}}} \\
&\lesssim N^{-1+}N^{-1}\left\|S_1S^{(k+1)}\psi_N\right\|^2_{L^2}.
\end{aligned}$$

Together with (2.17)-(2.18), we have the estimate for $E_{21}$,

$$E_{21} \geqslant -CN^{\beta-1+}\left(N^{-1}\left\|S_1S^{(k+1)}\psi_N\right\|^2_{L^2} + \left\|S^{(k+2)}\psi_N\right\|^2_{L^2}\right), \tag{2.19}$$

because $E_{211} \geqslant 0$.

We now turn to $E_{22}$ which is

$$E_{22} = N^{-2}\langle [S_1, H_{1(k+1)}]\frac{S^{(k)}}{S_1}\psi_N, S^{(k)}\left(2+H_{(k+1)(k+2)}\right)\psi_N\rangle.$$

Substitute $[S_1, H_{1(k+1)}] = N^{\beta}(\nabla V)_{N1(k+1)}$ and expand $2+H_{(k+1)(k+2)}$ to obtain

$$E_{22} = E_{221} + E_{221} + E_{223}$$

where

$$\begin{aligned}
E_{221} &= N^{\beta-2}\langle(\nabla V)_{N1(k+1)}\frac{S^{(k)}}{S_1}\psi_N, S^{(k)}S_{k+1}^2\psi_N\rangle \\
E_{222} &= N^{\beta-2}\langle(\nabla V)_{N1(k+1)}\frac{S^{(k)}}{S_1}\psi_N, S^{(k)}S_{k+2}^2\psi_N\rangle \\
E_{223} &= N^{\beta-2}\langle(\nabla V)_{N1(k+1)}\frac{S^{(k)}}{S_1}\psi_N, S^{(k)}V_{N(k+1)(k+2)}\psi_N\rangle
\end{aligned}$$

For $E_{221}$, we first Hölder at $x_1$ as follows:

$$|E_{221}| \lesssim N^{\beta-2}\|(\nabla V)_{N1(k+1)}\|_{L^{2+}_{x_1}}\|\frac{S^{(k)}}{S_1}\psi_N\|_{L^2L^{\infty-}_{x_1}}\|S^{(k+1)}S_{k+1}\psi_N\|_{L^2},$$

then Soblev to obtain

$$\begin{aligned}
|E_{221}| &\lesssim N^{2\beta-\frac{3}{2}+}\|S^{(k)}\psi_N\|_{L^2}N^{-\frac{1}{2}}\|S_1S^{(k+1)}\psi_N\|_{L^2} \\
&\lesssim N^{2\beta-\frac{3}{2}+}\left(\|S^{(k)}\psi_N\|_{L^2}^2+N^{-1}\|S_1S^{(k+1)}\psi_N\|_{L^2}^2\right).
\end{aligned} \tag{2.20}$$

Use Hölder in $x_{k+1}$ for $E_{222}$, we get

$$\begin{aligned}
|E_{222}| &\lesssim N^{\beta-2}\|(\nabla V)_{N1(k+1)}\|_{L^{1+}_{x_{k+1}}}\|\frac{S^{(k)}}{S_1}S_{k+2}\psi_N\|_{L^2L^{\infty-}_{x_{k+1}}}\|S^{(k)}S_{k+2}\psi_N\|_{L^2L^{\infty-}_{x_{k+1}}} \\
&\lesssim N^{\beta-2+}\|S^{(k+2)}\psi_N\|_{L^2}^2.
\end{aligned} \tag{2.21}$$

Then, argue in the same way for $E_{223}$,

$$\begin{aligned}
|E_{223}| &\lesssim N^{\beta-2}\|(\nabla V)_{N1(k+1)}\|_{L^{1+}_{x_1}}\left\|V_{N(k+1)(k+2)}\right\|_{L^{1+}_{x_{k+2}}} \\
&\quad\times\|\frac{S^{(k)}}{S_1}\psi_N\|_{L^2L^{\infty-}_{x_1}L^{\infty-}_{x_{k+2}}}\|S^{(k)}\psi_N\|_{L^2L^{\infty-}_{x_1}L^{\infty-}_{x_{k+2}}} \\
&\lesssim N^{\beta-\frac{3}{2}+}\|S^{(k+1)}\psi_N\|_{L^2}N^{-\frac{1}{2}}\|S_1S^{(k+1)}\psi_N\|_{L^2} \\
&\lesssim N^{\beta-\frac{3}{2}+}\left(\|S^{(k+1)}\psi_N\|_{L^2}^2+N^{-1}\|S_1S^{(k+1)}\psi_N\|_{L^2}^2\right)
\end{aligned}$$

Together with (2.20) and (2.21), we have the estimate for $E_{22}$,

$$|E_{22}| \lesssim N^{2\beta-\frac{3}{2}+}\left(\|S^{(k)}\psi_N\|_{L^2}^2+N^{-1}\|S_1S^{(k+1)}\psi_N\|_{L^2}^2\right). \tag{2.22}$$

This completes the treatment of $E_2$. Specifically, (2.19) and (2.22) give

$$E_2 \geqslant -C\max(N^{2\beta-\frac{3}{2}+},N^{\beta-1+})\left(\|S^{(k)}\psi_N\|_{L^2}^2+N^{-1}\|S_1S^{(k+1)}\psi_N\|_{L^2}^2\right). \tag{2.23}$$

Finally, we treat $E_3$ which is

$$E_3 = N^{-1}\langle S^{(k)}\left(2+H_{1(k+1)}\right)\psi_N, S^{(k)}\left(2+H_{(k+2)(k+3)}\right)\psi_N\rangle.$$

Commute $\left(2+H_{1(k+1)}\right)$ and $S^{(k)}$,

$$E_3 = E_{31}+E_{32},$$

where

$$\begin{aligned} E_{31} &= N^{-1}\langle\left(2+H_{1(k+1)}\right)S^{(k)}\psi_N, S^{(k)}\left(2+H_{(k+2)(k+3)}\right)\psi_N\rangle, \\ E_{32} &= N^{-1}\langle\left[S_1, H_{1(k+1)}\right]\frac{S^{(k)}}{S_1}\psi_N, S^{(k)}\left(2+H_{(k+2)(k+3)}\right)\psi_N\rangle. \end{aligned}$$

We first discard $E_{31}$ because $E_{31} \geqslant 0$ by Theorem 2.2 and Lemma 2.4. For $E_{32}$, we plug in $\left[S_1, H_{1(k+1)}\right] = N^{\beta}(\nabla V)_{N1(k+1)}$ and expand $\left(2+H_{(k+2)(k+3)}\right)$ to obtain

$$\begin{aligned} E_{32} &= N^{\beta-1}\langle(\nabla V)_{N1(k+1)}\frac{S^{(k)}}{S_1}\psi_N, S^{(k)}\left(2+S_{k+2}^2+S_{k+3}^2\right)\psi_N\rangle \\ &\quad +N^{\beta-1}\langle(\nabla V)_{N1(k+1)}\frac{S^{(k)}}{S_1}\psi_N, S^{(k)}V_{N(k+2)(k+3)}\psi_N\rangle \\ &= 2N^{\beta-1}\langle(\nabla V)_{N1(k+1)}\frac{S^{(k)}}{S_1}\psi_N, S^{(k)}\psi_N\rangle \\ &\quad +2N^{\beta-1}\langle(\nabla V)_{N1(k+1)}\frac{S^{(k)}}{S_1}S_{k+2}\psi_N, S^{(k)}S_{k+2}\psi_N\rangle \\ &\quad +N^{\beta-1}\langle(\nabla V)_{N1(k+1)}\frac{S^{(k)}}{S_1}\psi_N, S^{(k)}V_{N(k+2)(k+3)}\psi_N\rangle. \end{aligned}$$

First Hölder again

$$\begin{aligned} &|E_{32}| \\ \lesssim\ & N^{\beta-1}\left\|(\nabla V)_{N1(k+1)}\right\|_{L^{1+}_{x_{k+1}}}\left\|\frac{S^{(k)}}{S_1}\psi_N\right\|_{L^2L^{\infty-}_{x_{k+1}}}\left\|S^{(k)}\psi_N\right\|_{L^2L^{\infty-}_{x_{k+1}}} \\ &+N^{\beta-1}\left\|(\nabla V)_{N1(k+1)}\right\|_{L^{1+}_{x_{k+1}}}\left\|\frac{S^{(k)}}{S_1}S_{k+2}\psi_N\right\|_{L^2L^{\infty-}_{x_{k+1}}}\left\|S^{(k)}S_{k+2}\psi_N\right\|_{L^2L^{\infty-}_{x_{k+1}}} \\ &+N^{\beta-1}\left\|(\nabla V)_{N1(k+1)}\right\|_{L^{1+}_{x_{k+1}}}\left\|V_{N(k+2)(k+3)}\right\|_{L^{1+}_{x_{k+2}}}\left\|\frac{S^{(k)}}{S_1}\psi_N\right\|_{L^2L^{\infty-}_{x_{k+1}}L^{\infty-}_{x_{k+2}}}\left\|S^{(k)}\psi_N\right\|_{L^2L^{\infty-}_{x_{k+1}}L^{\infty-}_{x_{k+2}}}, \end{aligned}$$

then Sobolev gives

$$\begin{aligned} |E_{32}| &\lesssim N^{\beta-1+}\|S^{(k)}\psi_N\|_{L^2}\|S^{(k+1)}\psi_N\|_{L^2}+N^{\beta-1+}\|S^{(k+1)}\psi_N\|_{L^2}\|S^{(k+2)}\psi_N\|_{L^2} \\ &\quad +N^{\beta-1+}\|S^{(k+1)}\psi_N\|_{L^2}\|S^{(k+2)}\psi_N\|_{L^2} \\ &\lesssim N^{\beta-1+}\|S^{(k+2)}\psi_N\|_{L^2}^2. \end{aligned}$$

That is

$$E_3 \geqslant -CN^{\beta-1+}\|S^{(k+2)}\psi_N\|_{L^2}^2. \tag{2.24}$$

Putting (2.16), (2.23) and (2.24) in one line, we obtain the estimate for the cross error term

$$E_C \geqslant -C\max(N^{2\beta-\frac{3}{2}+}, N^{\beta-1+})\left(\|S^{(k+2)}\psi_N\|_{L^2}^2+N^{-1}\|S_1S^{(k+1)}\psi_N\|_{L^2}^2\right),$$

which is exactly (2.12).

Finally, combining (2.11)and (2.12), we have

$$\begin{aligned}
&\frac{1}{c_0^{k+2}}\langle\psi_N,(N^{-1}H_N+1)^{k+2}\psi_N\rangle\\
\geqslant\;&\left(2-C\max(N^{2\beta-\frac{3}{2}+},N^{\beta-1+})\right)\left(\|S^{(k+2)}\psi\|_{L^2}^2+N^{-1}\|S_1S^{(k+1)}\psi\|_{L^2}^2\right)\\
\geqslant\;&\|S^{(k+2)}\psi\|_{L^2}^2+N^{-1}\|S_1S^{(k+1)}\psi\|_{L^2}^2
\end{aligned}$$

for $N$ larger than some threshold, as originally claimed. Whence, we have proved (1.16) for all $k$ and established Theorem 1.3.

2.3. **Remark on higher $\beta$.** It is easy to see from §2.2 that Theorem 1.3 will hold up to $\beta<3/4$ as long as Theorem 2.1 works for higher $\beta$. It is certainly of mathematical and physical interest to push for a higher $\beta$ in Theroem 1.3. On the one hand, higher $\beta$ makes the convergence $V_N\to -b_0\delta$ as $N\to\infty$ faster and hence is more singular, difficult, and interesting to deal with. On the other hand, larger $\beta$ means stronger and more localized interaction.

Examing the proof of Theorem 2.1, one immediately notice the obstacles lie in Lemmas 2.1 and 2.2. While it is extremely difficult to improve Lemma 2.2, one would certainly wonder how to improve the crude estimate, Lemma 2.1. However, it turns out that the crude estimate is actually optimal in the sense that it fails if $M\leqslant C\frac{N^{\beta-\delta}}{\varepsilon}$ for some $\delta>0$. (See Lemma 2.5 below.) Thus, there is no obvious way to improve the current result and reach a higher $\beta$.

**Lemma 2.5.** *Suppose that $V\in\mathcal{S}(\mathbb{R}^2)$ with $\hat{V}(\xi)=1$ for $|\xi|\leq 4$. Suppose that $M_j=M_j(N)$, $j=1,2$ are dyads with $0\leq M_1\leq M_2\leq N^\beta$ and $\lim_{N\to\infty}\frac{M_2}{M_1}=\infty$. There does not exist a constant $C$ independent of $N$ such that the following estimate holds: for all symmetric $\psi(x_1,x_2)$,*

$$\int |V_N(x_1-x_2)||P^{(2)}_{M_1\leq\bullet\leq M_2}\psi(x_1,x_2)|^2\,dx_1\,dx_2\leq C\|\nabla_1\psi\|_{L^2}^2 \tag{2.25}$$

Before proceeding with the proof, we make a few remarks. First, the assumption $\hat{V}(\xi)=1$ for $|\xi|\leq 4$ can be eliminated, but we add it since it simplifies the proof and still covers a wide class of Schwartz class potentials. Second, we note the estimate (2.25) is in fact true when $M_2/M_1$ remains bounded as $N\to\infty$. This follows readily from scaling and the Bernstein inequality: if $M$ is a single dyadic interval, then $\|P_M\phi\|_{L^\infty}\leq M\|P_M\phi\|_{L^2}$. Moreover, the core of Lemma 2.1 is effectively the estimate

$$\int |V_N(x_1-x_2)||P^{(2)}_{>M_1}\psi(x_1,x_2)|^2\,dx_1\,dx_2\leq C\|\nabla_1\psi\|_{L^2}^2 \text{ for } M_1\geqslant N^\beta. \tag{2.26}$$

Lemma 2.5 shows that Lemma 2.1 cannot be improved in the sense that one cannot select $M_2=N^\beta$ and $M_1\ll N^\beta$ (for example $M_1=N^{\beta-\delta}$ for any $\delta>0$) and expect (2.26) to hold.

*Proof.* Replacing $x_j$ by $\frac{x_j}{M_1^{1/2}M_2^{1/2}}$ and $\tilde{N}=\frac{N}{M_1^{1/2\beta}M_2^{1/2\beta}}$, we obtain that the estimate (2.25) is equivalent to

$$\int |V_{\tilde{N}}(x_1-x_2)|P^{(2)}_{\left(\frac{M_1}{M_2}\right)^{1/2}\leq\bullet\leq\left(\frac{M_2}{M_1}\right)^{1/2}}\psi(x_1,x_2)|^2\,dx_1\,dx_2\leq C\|\nabla_1\psi\|_{L^2}^2$$

Notice that $M_2 \leq N^\beta$ implies $\left(\frac{M_2}{M_1}\right)^{1/2} \leq \tilde{N}^\beta$ and $\lim_{N\to\infty} \frac{M_2}{M_1} = \infty$ implies that $\lim_{N\to\infty} \left(\frac{M_2}{M_1}\right)^{1/2} = \infty$ and hence $\lim_{N\to\infty} \tilde{N} = \infty$.

Thus, it suffices to assume that in (2.25), we in fact have $\lim_{N\to\infty} M_1 = 0$ and $\lim_{N\to\infty} M_2 = \infty$.

For any functions $W$, $\psi_1$, $\psi_2$, consider

$$\begin{aligned}
I &\stackrel{\text{def}}{=} \int_{x_1,x_2} W(x_1 - x_2)\psi_1(x_1, x_2)\overline{\psi_2(x_1, x_2)}\, dx_1\, dx_2 \\
&= \int_{x_1,x_2,\eta,\xi_1,\xi_2} e^{i(x_1-x_2)\eta} e^{ix_1\xi_1} e^{ix_2\xi_2} \hat{W}(\eta)\hat{\psi}_1(\xi_1, \xi_2)\overline{\psi_2(x_1, x_2)}\, dx_1\, dx_2 \\
&= \int_{\eta,\xi_1,\xi_2} \hat{W}(\eta)\psi_1(\xi_1, \xi_2)\overline{\int_{x_1,x_2} e^{-ix_1(\xi_1+\eta)} e^{-ix_2(\xi_2-\eta)}\psi_2(x_1, x_2)\, dx_1\, dx_2\, d\eta\, d\xi_1\, dx_2} \\
&= \int_{\eta,\xi_1,\xi_2} \hat{W}(\eta)\psi_1(\xi_1, \xi_2)\overline{\psi_2(\xi_1 + \eta, \xi_2 - \eta)}\, d\eta\, d\xi_1\, d\xi_2 \\
&= \int_{\eta,\xi_1,\xi_2} \hat{W}(\eta)\hat{\psi}_1(\xi_1 - \frac{\eta}{2}, \xi_2 + \frac{\eta}{2})\overline{\hat{\psi}_2(\xi_1 + \frac{\eta}{2}, \xi_2 - \frac{\eta}{2})}\, d\xi_1 d\xi_2 d\eta \\
&= \int_{\eta,\xi_1,\xi_2} \hat{W}(2\eta)\hat{\psi}_1(\xi_1 - \eta, \xi_2 + \eta)\overline{\hat{\psi}_2(\xi_1 + \eta, \xi_2 - \eta)}\, d\xi_1 d\xi_2 d\eta
\end{aligned}$$

Let

$$J_V \stackrel{\text{def}}{=} \int |V_N(x_1 - x_2)||P_{M_1\leq\bullet\leq M_2}\psi(x_1, x_2)|^2\, dx_1\, dx_2$$

and

$$\begin{aligned}
J_\delta &\stackrel{\text{def}}{=} \int \delta(x_1 - x_2)|P_{M_1\leq\bullet\leq M_2}\psi(x_1, x_2)|^2\, dx_1\, dx_2 \\
&= \int |P_{M_1\leq\bullet\leq M_2}\psi(x, x)|^2\, dx
\end{aligned}$$

We show that $J_V = J_\delta$. To obtain $I = J_V - J_\delta$, in the expression for $I$, we take $W = V_N - \delta$ and $\psi_j = P_{M_1\leq\bullet\leq M_2}\psi$. Then

$$\hat{W}(2\eta) = \hat{V}(\frac{2\eta}{N^\beta}) - 1$$

so $\hat{W}(2\eta) = 0$ for $|\eta| \leq 2N^\beta$. On the other hand, the frequency restrictions on $\psi_j$ imply that $|\xi_1 - \eta| \leq M_2 \leq N^\beta$ and $|\xi_1 + \eta| \leq M_2 \leq N^\beta$. It follows that

$$|2\eta| = |(\xi_1 + \eta) - (\xi_1 - \eta)| \leq |\xi_1 + \eta| + |\xi_1 - \eta| \leq 2N^\beta$$

Consequently $I = 0$, completing the proof of the claim.

We argue by contradiction assuming that (2.25) holds with $C$ independent of $N$. Since $J_V = J_\delta$,

$$J_\delta = \int |P_{M_1\leq\bullet\leq M_2}\psi(x, x)|^2\, dx \leq C\|\nabla_{x_1}\psi\|_{L^2}^2$$

with a constant $C$ independent of $N$, where $M_1 \to 0$ and $M_2 \to \infty$ as $N \to \infty$. By Fatou's lemma,

$$\int |\psi(x,x)|^2 \, dx \leq C \|\nabla_{x_1} \psi\|_{L^2}^2 \tag{2.27}$$

which is the (false) 2D endpoint trace estimate. A counterexample can be constructed as follows. Let $\chi$ be a smooth function with $\chi(-x) = \chi(x)$, $\chi(x) = 1$ for $|x| \leq \frac{1}{4}$ and $\chi(x) = 0$ for $|x| \geq \frac{1}{2}$. Then

$$\psi(x_1, x_2) = \chi(x_1 - x_2)\chi(x_1)\chi(x_2) \ln(-\ln|x_1 - x_2|)$$

is a symmetric function for which the left side of (2.27) is infinite but the right side is finite.

More properly written, we can introduce a smooth function

$$\psi_\epsilon(x_1, x_2) = \chi(x_1 - x_2)\chi(x_1)\chi(x_2) \ln(-\ln(|x_1 - x_2| + \epsilon))$$

Then

$$\int |\psi_\epsilon(x,x)|^2 \, dx \sim \ln \ln \epsilon^{-1}$$

while $\|\nabla_{x_1} \psi_\epsilon\|_{L^2}$ is bounded independently of $\epsilon$ as $\epsilon \to 0$. Sending $\epsilon \to 0$ shows that any choice of $C$ in (2.27) can be beat, giving us the contradiction. $\square$

## 3. Derivation of the 2D Focusing NLS

3.1. **Proof of Theorem 1.2.** We start by introducing an appropriate topology on the density matrices as was previously done in [27, 28, 29, 30, 31, 32, 44, 10, 15, 16, 17, 18]. Denote the spaces of compact operators and trace class operators on $L^2\left(\mathbb{R}^{2k}\right)$ as $\mathcal{K}_k$ and $\mathcal{L}_k^1$, respectively. Then $(\mathcal{K}_k)' = \mathcal{L}_k^1$. By the fact that $\mathcal{K}_k$ is separable, we select a dense countable subset $\{J_i^{(k)}\}_{i\geqslant 1} \subset \mathcal{K}_k$ in the unit ball of $\mathcal{K}_k$ (so $\|J_i^{(k)}\|_{\text{op}} \leqslant 1$ where $\|\cdot\|_{\text{op}}$ is the operator norm). For $\gamma^{(k)}, \tilde{\gamma}^{(k)} \in \mathcal{L}_k^1$, we then define a metric $d_k$ on $\mathcal{L}_k^1$ by

$$d_k(\gamma^{(k)}, \tilde{\gamma}^{(k)}) = \sum_{i=1}^{\infty} 2^{-i} \left| \operatorname{Tr} J_i^{(k)} \left(\gamma^{(k)} - \tilde{\gamma}^{(k)}\right) \right|.$$

A uniformly bounded sequence $\gamma_N^{(k)} \in \mathcal{L}_k^1$ converges to $\gamma^{(k)} \in \mathcal{L}_k^1$ with respect to the weak* topology if and only if

$$\lim_{N\to\infty} d_k(\gamma_N^{(k)}, \gamma^{(k)}) = 0.$$

For fixed $T > 0$, let $C\left([0,T], \mathcal{L}_k^1\right)$ be the space of functions of $t \in [0,T]$ with values in $\mathcal{L}_k^1$ which are continuous with respect to the metric $d_k$. On $C\left([0,T], \mathcal{L}_k^1\right)$, we define the metric

$$\hat{d}_k(\gamma^{(k)}(\cdot), \tilde{\gamma}^{(k)}(\cdot)) = \sup_{t\in[0,T]} d_k(\gamma^{(k)}(t), \tilde{\gamma}^{(k)}(t)),$$

and denote by $\tau_{prod}$ the topology on the space $\oplus_{k\geqslant 1} C\left([0,T], \mathcal{L}_k^1\right)$ given by the product of topologies generated by the metrics $\hat{d}_k$ on $C\left([0,T], \mathcal{L}_k^1\right)$.

By Theorem 1.3, we have, $\forall k=0,1,...,$

$$\begin{aligned}
\operatorname{Tr} S^{(k)}\gamma_N^{(k)}(t)S^{(k)} &= \left\|S^{(k)}\psi_N(t)\right\|_{L^2}^2 \\
&\leqslant C^k\left\langle \psi_N(t),\left(N^{-1}H_N+1\right)^k\psi_N(t)\right\rangle \\
&= C^k\left\langle \psi_N(0),\left(N^{-1}H_N+1\right)^k\psi_N(0)\right\rangle \\
&= \sum_{j=0}^{k}\binom{k}{j}\langle\psi_N(0),\frac{1}{N^{k-j}}H_N^{k-j}\psi_N(0)\rangle \\
&\leqslant \sum_{j=0}^{k}\binom{k}{j}1^jC^{k-j} \\
&\leqslant C^k
\end{aligned}$$

provided that $N\geqslant N_0(k)$. That is the energy estimate:

$$\sup_t \operatorname{Tr} S^{(k)}\gamma_N^{(k)}(t)S^{(k)}\leqslant C^k. \tag{3.1}$$

With estimate (3.1), one can go through Lemmas 3.1, 3.2, and 3.3 to conclude that, as trace class operators:

$$\gamma_N^{(k)}(t)\rightharpoonup\left|\phi(t)\right\rangle\left\langle\phi(t)\right|^{\otimes k}\ \text{ weak*}.$$

By the argument on [16, p.398-399][15], we can upgrade the above weak* convergence to strong and hence finish the proof of Theorem 1.2.

**Lemma 3.1** (Compactness)**.** *For all finite $T>0$, the sequence*

$$\left\{\Gamma_N(t)=\left\{\gamma_N^{(k)}\right\}_{k=1}^{N}\right\}\subset\bigoplus_{k\geqslant1}C\left(\left[0,T\right],\mathcal{L}_k^1\right),$$

*which satisfies the 2D focusing BBGKY hierarchy*

$$\begin{aligned}
i\partial_t\gamma_N^{(k)} &= \sum_{j=1}^{k}\left[-\triangle_{x_j}+\omega^2\left|x_j\right|^2,\gamma_N^{(k)}\right]+\frac{1}{N}\sum_{1\leqslant i<j\leqslant k}\left[V_N(x_i-x_j),\gamma_N^{(k)}\right] \\
&\quad+\frac{N-k}{N}\sum_{j=1}^{k}\operatorname{Tr}_{k+1}\left[V_N(x_j-x_{k+1}),\gamma_N^{(k+1)}\right],
\end{aligned}\tag{3.2}$$

*where $V<0$, subject to energy condition (3.1) is compact with respect to the product topology $\tau_{prod}$. For any limit point $\Gamma(t)=\left\{\gamma^{(k)}\right\}_{k=1}^{N}$, $\gamma^{(k)}$ is a symmetric nonnegative trace class operator with trace bounded by 1, and it verifies the energy bound*

$$\sup_{t\in[0,T]}\operatorname{Tr} S^{(k)}\gamma^{(k)}S^{(k)}\leqslant C^k. \tag{3.3}$$

[15]The proof [16, p.398-399] is actually for more general datum.

**Lemma 3.2** (Convergence)**.** *Let $\Gamma(t)=\left\{\gamma^{(k)}\right\}_{k=1}^{\infty}$ be a limit point of $\left\{\Gamma_N(t)=\left\{\gamma_N^{(k)}\right\}_{k=1}^{N}\right\}$, the sequence in Theorem 3.1, with respect to the product topology $\tau_{prod}$, then $\Gamma(t)$ is a solution to the focusing GP hierarchy*

$$i\partial_t\gamma^{(k)}=\sum_{j=1}^{k}\left[-\triangle_{x_j}+\omega^2\left|x_j\right|^2,\gamma^{(k)}\right]-b_0\sum_{j=1}^{k}\operatorname{Tr}_{k+1}\left[\delta(x_j-x_{k+1}),\gamma^{(k+1)}\right], \tag{3.4}$$

*subject to initial data $\gamma^{(k)}(0)=\left|\phi_0\right\rangle\left\langle\phi_0\right|^{\otimes k}$ with coupling constant $b_0=\int\left|V(x)\right|dx$. which, written in integral form, is*

$$\gamma^{(k)}(t)=U^{(k)}(t)\gamma^{(k)}(0)+ib_0\sum_{j=1}^{k}\int_0^t U^{(k)}(t-s)\operatorname{Tr}_{k+1}\left[\delta\left(x_j-x_{k+1}\right),\gamma^{(k+1)}(s)\right]ds. \tag{3.5}$$

*where*

$$U^{(k)}(t)=e^{it\left(-\triangle_{x_j}+\omega^2|x_j|^2\right)}e^{-it\left(-\triangle_{x_j'}+\omega^2\left|x_j'\right|^2\right)}.$$

**Lemma 3.3.** [16]*If $\Gamma(t)=\left\{\gamma^{(k)}\right\}_{k=1}^{\infty}$ is a solution to (3.4) subject to the following two conditions:*

*(a) $\Gamma(t)$ is sequence of normalized symmetry nonnegative trace class opertors which is a limit point of some $N$-body marginals with respect to the product topology $\tau_{prod}$ or satisifes $\operatorname{Tr}_{k+1}\gamma^{(k+1)}=\gamma^{(k)}$.*

*(b) For some $\alpha\geqslant\frac{2}{3}$, we have the regularity estimate*

$$\sup_{t\in[0,T]}\operatorname{Tr}\left(S^{(k)}\right)^{\alpha}\gamma^{(k)}\left(S^{(k)}\right)^{\alpha}\leqslant C^k,$$

*then $\Gamma(t)$ is also the only solution of (3.4) subject to (a) and (b).*

*In particular, if $\Gamma(t)$ checks (a) and (b) of this lemma and $\gamma^{(k)}(0)=\left|\phi_0\right\rangle\left\langle\phi_0\right|^{\otimes k}$ where $\phi_0$ satisfies $\left\|\left(-\triangle_x+\omega^2\left|x\right|^2\right)^{\frac{1}{2}}\phi_0\right\|_{L^2(\mathbb{R})}<\infty$, then*

$$\gamma^{(k)}(t)=\left|\phi(t)\right\rangle\left\langle\phi(t)\right|^{\otimes k}$$

*where $\phi(t)$ solves the 2D focusing cubic NLS (1.11). This is because $\left|\phi(t)\right\rangle\left\langle\phi(t)\right|^{\otimes k}$ is a solution to (3.4) subject to (a) and (b) of this lemma.*

To prove Lemma 3.1 and 3.2, we need the following lemma.

**Lemma 3.4** ([44, Lemma A.2])**.** *Let $f\in L^1\left(\mathbb{R}^2\right)$ such that $\int_{\mathbb{R}^2}\langle r\rangle\left|f(r)\right|dr<\infty$ and $\int_{\mathbb{R}^2}f(r)\,dr=1$ but we allow that $f$ not be nonnegative everywhere. Define $f_\alpha(r)=\alpha^{-2}f\left(\frac{r}{\alpha}\right)$.*

[16]One can also use the Strichartz type uniqueness theorems [15, Theorem 3] or [44, Theorem 7.1] here.

*Then, for every $\kappa \in (0,1)$ , there exists $C_\kappa > 0$ s.t.*

$$\begin{aligned}
& \left|\operatorname{Tr} J^{(k)}\left(f_\alpha\left(r_j-r_{k+1}\right)-\delta\left(r_j-r_{k+1}\right)\right)\gamma^{(k+1)}\right| \\
\leqslant\; & C_\kappa\left(\int |f(r)|\,|r|^\kappa\, dr\right)\alpha^\kappa \\
& \times\left(\left\|(1-\triangle_{x_j})^{\frac{1}{2}}J^{(k)}(1-\triangle_{x_j})^{-\frac{1}{2}}\right\|_{\text{op}}+\left\|(1-\triangle_{x_j})^{-\frac{1}{2}}J^{(k)}(1-\triangle_{x_j})^{\frac{1}{2}}\right\|_{\text{op}}\right) \\
& \times \operatorname{Tr}\left(1-\triangle_{x_j}\right)\left(1-\triangle_{x_{k+1}}\right)\gamma^{(k+1)} \\
\leqslant\; & C_\kappa\left(\int |f(r)|\,|r|^\kappa\, dr\right)\alpha^\kappa\left(\left\|S_jJ^{(k)}S_j^{-1}\right\|_{\text{op}}+\left\|S_j^{-1}J^{(k)}S_j\right\|_{\text{op}}\right)\operatorname{Tr} S_jS_{k+1}\gamma^{(k+1)}S_jS_{k+1}
\end{aligned}$$

*for all nonnegative $\gamma^{(k+1)} \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^{2k+2}\right)\right)$.*

*Proof of Compactness.* By [32, Lemma 6.2], this is equivalent to the statement that for every test function $f^{(k)}$ from a dense subset of $\mathcal{K}_k$ and for every $\varepsilon > 0$, there exists $\delta(f^{(k)},\varepsilon)$ such that for all $t_1, t_2 \in [0,T]$ with $|t_1 - t_2| \leqslant \delta$, we have

$$\sup_N \left|\operatorname{Tr} f^{(k)}\gamma_N^{(k)}(t_1)-\operatorname{Tr} f^{(k)}\gamma_N^{(k)}(t_2)\right| \leqslant \varepsilon\,.$$

We select the test functions $f^{(k)} \in \mathcal{K}_k$ which satisfy

$$\left\|S_iS_jf^{(k)}S_i^{-1}S_j^{-1}\right\|_{\text{op}}+\left\|S_i^{-1}S_j^{-1}f^{(k)}S_iS_j\right\|_{\text{op}}<\infty,$$

Let $0 \leqslant t_1 \leqslant t_2 \leqslant T$, we take advantage of the $\partial_t\gamma_N^{(k)}$ in the hierarchy (3.2) and use the fundamental theorem of calculus to get to

$$\begin{aligned}
& \left|\operatorname{Tr} f^{(k)}\gamma_N^{(k)}(t_2)-\operatorname{Tr} f^{(k)}\gamma_N^{(k)}(t_1)\right| \\
\leqslant\; & \sum_{j=1}^k\int_{t_1}^{t_2}\left|\operatorname{Tr} f^{(k)}\left[S_j^2,\gamma_N^{(k)}(s)\right]\right|ds \\
& +\frac{1}{N}\sum_{1\leqslant i<j\leqslant k}\int_{t_1}^{t_2}\left|\operatorname{Tr} f^{(k)}\left[V_N\left(x_i-x_j\right),\gamma_N^{(k)}(s)\right]\right|ds \\
& +\frac{N-k}{N}\sum_{j=1}^k\int_{t_1}^{t_2}\left|\operatorname{Tr} f^{(k)}\left[V_N\left(x_j-x_{k+1}\right),\gamma_N^{(k+1)}(s)\right]\right|ds.
\end{aligned}$$

We estimate each term as follow. The first term can be easily estimated

$$\begin{aligned}
& \int_{t_1}^{t_2}\left|\operatorname{Tr} f^{(k)}\left[S_j^2,\gamma_N^{(k)}(s)\right]\right|ds \\
=\; & \int_{t_1}^{t_2}\left|\operatorname{Tr} S_j^{-1}f^{(k)}S_jS_j\gamma_N^{(k)}(s)\,S_j-\operatorname{Tr} S_jf^{(k)}S_j^{-1}S_j\gamma_N^{(k)}(s)\,S_j\right|ds \\
\leqslant\; & \int_{t_1}^{t_2}\left(\left\|S_j^{-1}f^{(k)}S_j\right\|_{op}+\left\|S_jf^{(k)}S_j^{-1}\right\|_{op}\right)\left(\operatorname{Tr} S_j\gamma_N^{(k)}(s)\,S_j\right)ds \\
\leqslant\; & C_fC\left|t_2-t_1\right|.
\end{aligned}$$

For the second and the third terms, we use the fact that conjugation preserves traces and the Sobolev inequality

$$\left\|S_{ij}^{-1}S_{k+1}^{-1}V_N\left(x_i-x_j\right)S_j^{-1}S_{k+1}^{-1}\right\|_{op}\leqslant C\left\|V_N\right\|_{L^1}=C\left\|V\right\|_{L^1} \tag{3.6}$$

to deduce

$$\begin{aligned}
&\frac{1}{N}\sum_{1\leqslant i<j\leqslant k}\int_{t_1}^{t_2}\left|\operatorname{Tr}f^{(k)}\left[V_N\left(x_i-x_j\right),\gamma_N^{(k)}\left(s\right)\right]\right|ds\\
\leqslant\;&\frac{k^2}{N}\int_{t_1}^{t_2}|\operatorname{Tr}S_i^{-1}S_j^{-1}f^{(k)}S_iS_jS_i^{-1}S_j^{-1}V_N\left(x_i-x_j\right)S_i^{-1}S_j^{-1}S_iS_j\gamma_N^{(k)}\left(s\right)S_iS_j|ds\\
&+\frac{k^2}{N}\int_{t_1}^{t_2}|\operatorname{Tr}S_iS_jf^{(k)}S_i^{-1}S_j^{-1}S_iS_j\gamma_N^{(k)}\left(s\right)S_iS_jS_i^{-1}S_j^{-1}V_N\left(x_i-x_j\right)S_i^{-1}S_j^{-1}|ds\\
\leqslant\;&\frac{Ck^2}{N}\left(\left\|S_i^{-1}S_j^{-1}f^{(k)}S_iS_j\right\|_{op}+\left\|S_iS_jf^{(k)}S_i^{-1}S_j^{-1}\right\|_{op}\right)\left\|S_i^{-1}S_j^{-1}V_N\left(x_i-x_j\right)S_i^{-1}S_j^{-1}\right\|\\
&\int_{t_1}^{t_2}\operatorname{Tr}S_iS_j\gamma_N^{(k)}\left(s\right)S_iS_jds\\
\leqslant\;&\frac{k^2}{N}C_fC^2\left|t_2-t_1\right|,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{N-k}{N}\sum_{j=1}^{k}\int_{t_1}^{t_2}\left|\operatorname{Tr}f^{(k)}\left[V_N\left(x_j-x_{k+1}\right),\gamma_N^{(k+1)}\left(s\right)\right]\right|ds\\
\leqslant\;&k\int_{t_1}^{t_2}|\operatorname{Tr}S_j^{-1}S_{k+1}^{-1}f^{(k)}S_jS_{k+1}S_j^{-1}S_{k+1}^{-1}V_N\left(x_j-x_{k+1}\right)S_j^{-1}S_{k+1}^{-1}S_jS_{k+1}\gamma_N^{(k+1)}\left(s\right)S_jS_{k+1}|ds\\
&+k\int_{t_1}^{t_2}|\operatorname{Tr}S_jS_{k+1}f^{(k)}S_j^{-1}S_{k+1}^{-1}S_jS_{k+1}\gamma_N^{(k+1)}\left(s\right)S_jS_{k+1}S_j^{-1}S_{k+1}^{-1}V_N\left(x_j-x_{k+1}\right)S_j^{-1}S_{k+1}^{-1}|ds\\
\leqslant\;&Ck\left(\left\|S_j^{-1}f^{(k)}S_j\right\|_{op}+\left\|S_jf^{(k)}S_j^{-1}\right\|_{op}\right)\left\|S_{ij}^{-1}S_{k+1}^{-1}V_N\left(x_i-x_j\right)S_j^{-1}S_{k+1}^{-1}\right\|\\
&\int_{t_1}^{t_2}\operatorname{Tr}S_jS_{k+1}\gamma_N^{(k+1)}\left(s\right)S_jS_{k+1}ds\\
\leqslant\;&kC_fC^2\left|t_2-t_1\right|.
\end{aligned}$$

That is

$$\left|\operatorname{Tr}f^{(k)}\gamma_N^{(k)}(t_2)-\operatorname{Tr}f^{(k)}\gamma_N^{(k)}(t_1)\right|\leqslant C_{f,k}\left|t_2-t_1\right|,$$

which is enough to end the proof of Theorem 3.1. □

*Proof of Convergence.* By Theorem 3.1, passing to subsequences if necessary, we have

$$\lim_{N\to\infty}\sup_{t\in[0,T]}\operatorname{Tr}f^{(k)}\left(\gamma_N^{(k)}-\gamma^{(k)}\right)=0,\ \forall f^{(k)}\in\mathcal{K}_k. \tag{3.7}$$

We test (3.5) against the test functions $f^{(k)}$ in Theorem 3.1. We prove that the limit point verifies

$$\operatorname{Tr}f^{(k)}\gamma^{(k)}\left(0\right)=\operatorname{Tr}f^{(k)}\left|\phi_0\right\rangle\left\langle\phi_0\right|^{\otimes k}, \tag{3.8}$$

and

$$\begin{aligned}\operatorname{Tr} f^{(k)}\gamma^{(k)} &= \operatorname{Tr} f^{(k)}U^{(k)}(t)\gamma^{(k)}(0) \\ &\quad +ib_0\sum_{j=1}^{k}\int_0^t \operatorname{Tr} f^{(k)}U^{(k)}(t-s)\left[\delta\left(x_j-x_{k+1}\right),\gamma^{(k+1)}(s)\right]ds. \end{aligned}\tag{3.9}$$

Rewrite the BBGKY hierarchy (3.2) as the following

$$\begin{aligned}\operatorname{Tr} f^{(k)}\gamma_N^{(k)} &= \operatorname{Tr} f^{(k)}U^{(k)}(t)\gamma_N^{(k)}(0) \\ &\quad +\frac{i}{N}\sum_{1\leqslant i<j\leqslant k}\int_0^t \operatorname{Tr} f^{(k)}U^{(k)}(t-s)\left[-V_N(x_i-x_j),\gamma_N^{(k)}(s)\right]ds \\ &\quad +i\frac{N-k}{N}\sum_{j=1}^{k}\int_0^t \operatorname{Tr} f^{(k)}U^{(k)}(t-s)\left[-V_N(x_j-x_{k+1}),\gamma_N^{(k+1)}(s)\right]ds \\ &= I+\frac{i}{N}\sum_{1\leqslant i<j\leqslant k} II+i\left(1-\frac{k}{N}\right)\sum_{j=1}^{k} III.\end{aligned}$$

Notice that $b_0=-\int V_N(x)dx$, we have put a minus sign in front of $V_N$ to match 3.9.

Immediately following (3.7), we have

$$\begin{aligned}\lim_{N\to\infty}\operatorname{Tr} f^{(k)}u_N^{(k)} &= \operatorname{Tr} f^{(k)}u^{(k)}, \\ \lim_{N\to\infty}\operatorname{Tr} f^{(k)}U^{(k)}(t)\gamma_N^{(k)}(0) &= \operatorname{Tr} f^{(k)}U^{(k)}(t)f^{(k)}(0).\end{aligned}$$

By the well-known argument on [50, p.64], we know $\gamma_N^{(k)}(0)\to\left|\phi_0\right\rangle\left\langle\phi_0\right|^{\otimes k}$ strongly as trace operators because $\gamma_N^{(1)}(0)\to\left|\phi_0\right\rangle\left\langle\phi_0\right|$ strongly as trace operators. So we have checked relation (3.8) and the left hand side and the first term on the right hand side of (3.9) for $\Gamma(t)$.

We now prove

$$\lim_{N\to\infty}\frac{II}{N}=\lim_{N\to\infty}\frac{k}{N}III=0,\tag{3.10}$$

and

$$\lim_{N\to\infty}III=\int_0^t \operatorname{Tr} J^{(k)}U^{(k)}(t-s)\left[\delta\left(x_j-x_{k+1}\right),\gamma^{(k+1)}(s)\right]ds.\tag{3.11}$$

In the proof of Theorem 3.1, we have already shown that $|II|$ and $|III|$ are uniformly bounded for every finite time, thus (3.10) has been checked. So we are left to prove 3.11. To use Lemma 3.4, we take a probability measure $\rho\in L^1\left(\mathbb{R}^2\right)$, define $\rho_\alpha(y)=\frac{1}{\alpha^2}\rho\left(\frac{y}{\alpha}\right)$. Adopt the notation $f_{s-t}^{(k)}=f^{(k)}U^{(k)}(t-s)$, we have

$$\begin{aligned}
&\left|\operatorname{Tr} f^{(k)}U^{(k)}(t-s)\left(-V_N(x_j-x_{k+1})\gamma_N^{(k+1)}(s)-b_0\delta(x_j-x_{k+1})\gamma^{(k+1)}(s)\right)\right|\\
&\quad\leqslant\left|\operatorname{Tr} f_{s-t}^{(k)}\left(-V_N(x_j-x_{k+1})-b_0\delta(x_j-x_{k+1})\right)\gamma_N^{(k+1)}(s)\right|\\
&\qquad+b_0\left|\operatorname{Tr} f_{s-t}^{(k)}\left(\delta(x_j-x_{k+1})-\rho_\alpha(x_j-x_{k+1})\right)\gamma_N^{(k+1)}(s)\right|\\
&\qquad+b_0\left|\operatorname{Tr} f_{s-t}^{(k)}\rho_\alpha(x_j-x_{k+1})\left(\gamma_N^{(k+1)}(s)-\gamma^{(k+1)}(s)\right)\right|\\
&\qquad+b_0\left|\operatorname{Tr} f_{s-t}^{(k)}\left(\rho_\alpha(x_j-x_{k+1})-\delta(x_j-x_{k+1})\right)\gamma^{(k+1)}(s)\right|\\
&\quad=IV+V+VI+VII.
\end{aligned}$$

Lemma 3.4 and the energy condition (3.1) gives

$$\begin{aligned}
IV &\leqslant \frac{C}{N^{\kappa\beta}}\left(\left\|S_j^{-1}f^{(k)}S_j\right\|_{op}+\left\|S_jf^{(k)}S_j^{-1}\right\|_{op}\right)\operatorname{Tr} S_jS_{k+1}\gamma_N^{(k+1)}S_jS_{k+1}\\
&\leqslant \frac{C_f}{N^{\kappa\beta}}\to 0 \text{ as } N\to\infty, \text{ uniformly for } s\in[0,T] \text{ with } T<\infty.
\end{aligned}$$

Similarly, we obtain $V,VII\leqslant C_f\alpha^\kappa\to 0$ as $\alpha\to 0$. For VI,

$$\begin{aligned}
G &\leqslant b_0\left|\operatorname{Tr} f_{s-t}^{(k)}\rho_\alpha(x_j-x_{k+1})\frac{1}{1+\varepsilon S_{k+1}}\left(\gamma_N^{(k+1)}(s)-\gamma^{(k+1)}(s)\right)\right|\\
&\quad+b_0\left|\operatorname{Tr} f_{s-t}^{(k)}\rho_\alpha(x_j-x_{k+1})\frac{\varepsilon S_{k+1}}{1+\varepsilon S_{k+1}}\left(\gamma_N^{(k+1)}(s)-\gamma^{(k+1)}(s)\right)\right|.
\end{aligned}$$

The first term in the above inequality tends to zero as $N\to\infty$ for every $\varepsilon>0$, since we have assumed (3.7) and $f_{s-t}^{(k)}\rho_\alpha(x_j-x_{k+1})\frac{1}{1+\varepsilon S_{k+1}}$ is a compact operator. Due to the energy bounds (3.1) and (3.3), the second term tends to zero as $\varepsilon\to 0$, uniformly in $N$.

Combining the estimates for $IV-VII$, we have justified limit (3.11) and thus limit (3.9). Hence, we have proved Theorem 3.2. □

*Proof of Uniqueness.* The proof is essentially already in [8] and [22]. One merely needs to set $\mathcal{A}=0$, switch the Strichartz estimate for $e^{it\triangle}$ to the ones for $e^{it\left(\triangle-\omega^2|x|^2\right)}$ in [22] and notice that $\|f\|_{H^\alpha}\lesssim\|S^\alpha f\|_{L^2}$ for $\alpha\geqslant 0$. We skip the details. □

3.2. **Proof of Theorem 1.1.** Assuming Theorem 1.2, we now prove Theorem 1.1. If $\psi_N(0)$ satisfies (a), (b), and (c) in Theorem 1.1, then $\psi_N(0)$ checks the requirements of the following lemma.

**Lemma 3.5.** *Assume $\psi_N(0)$ satisfies (a), (b), and (c) in Theorem 1.1. Let $\chi\in C_0^\infty(\mathbb{R})$ be a cut-off such that $0\leqslant\chi\leqslant 1$, $\chi(s)=1$ for $0\leqslant s\leqslant 1$ and $\chi(s)=0$ for $s\geqslant 2$. For $\kappa>0$, we define an approximation $\psi_N^\kappa(0)$ of $\psi_N(0)$ by*

$$\psi_N^\kappa(0)=\frac{\chi(\kappa H_N/N)\psi_N(0)}{\|\chi(\kappa H_N/N)\psi_N(0)\|}. \tag{3.12}$$

*This approximation has the following properties:*

*(i)* $\psi_N^\kappa(0)$ *verifies the energy condition*

$$\langle \psi_N^\kappa(0), H_N^k \psi_N^\kappa(0)\rangle \leqslant \frac{2^k N^k}{\kappa^k}.$$

*(ii)*

$$\sup_N \|\psi_N^\kappa(0) - \psi_N(0)\|_{L^2} \leqslant C\kappa^{\frac{1}{2}}.$$

*(iii) For small enough* $\kappa > 0$, $\psi_N^\kappa(0)$ *is asymptotically factorized as well*

$$\lim_{N\to\infty} \operatorname{Tr}\left|\gamma_N^{\kappa,(1)}(0, x_1; x_1') - \phi_0(x_1)\overline{\phi_0}(x_1')\right| = 0,$$

*where* $\gamma_N^{\kappa,(1)}(0)$ *is the marginal density associated with* $\psi_N^\kappa(0)$ *and* $\phi_0$ *is the same as in assumption (b) in Theorem 1.1.*

*Proof.* (i) and (ii) follows from [19, Lemma B.1] and [20, Lemma B.1]. (iii) follows from the proof of [30, Proposition 5.1 (iii)]. Notice that for two dimension, we get a $N^\beta$ instead of a $N^{\frac{3\beta}{2}}$ in [30, (5.20)] and hence we get a $N^{\beta-1}$ in the estimate of [30, (5.18)] which goes to zero for $\beta \in (0,1)$. □

Thus we can define an approximation $\psi_N^\kappa(0)$ of $\psi_N(0)$ as in (3.12). Via (i) and (iii) of Lemma 3.5, $\psi_N^\kappa(0)$ verifies the requirements of Theorem 1.2 for small enough $\kappa > 0$. Therefore, for $\gamma_N^{\kappa,(k)}(t)$, the marginal density associated with $e^{itH_N}\psi_N^\kappa(0)$, Theorem 1.2 gives the convergence:

$$\gamma_N^{(k)}(t) \to |\phi(t)\rangle\langle\phi(t)|^{\otimes k} \text{ strongly, } \forall k, t \tag{3.13}$$

as trace class operators, for all small enough $\kappa > 0$.

For $\gamma_N^{(k)}(t)$ in Theorem 1.2, we notice that, for any test function $f^{(k)} \in \mathcal{K}_k$ and any $t \in \mathbb{R}$, we have

$$\begin{aligned} & \left|\operatorname{Tr} f^{(k)}\left(\gamma_N^{(k)}(t) - |\phi(t)\rangle\langle\phi(t)|^{\otimes k}\right)\right| \\ \leqslant & \left|\operatorname{Tr} f^{(k)}\left(\gamma_N^{(k)}(t) - \gamma_N^{\kappa,(k)}(t)\right)\right| \\ & + \left|\operatorname{Tr} f^{(k)}\left(\gamma_N^{\kappa,(k)}(t) - |\phi(t)\rangle\langle\phi(t)|^{\otimes k}\right)\right| \\ = & \ \mathrm{A} + \mathrm{B}. \end{aligned}$$

Convergence (3.13) then takes care of B. To handle A, part (ii) of Lemma 3.5 yields

$$\left\|e^{itH_N}\psi_N^\kappa(0) - e^{itH_N}\psi_N(0)\right\|_{L^2} = \|\psi_N^\kappa(0) - \psi_N(0)\|_{L^2} \leqslant C\kappa^{\frac{1}{2}}$$

which implies

$$A = \left|\operatorname{Tr} f^{(k)}\left(\gamma_N^{(k)}(t) - \gamma_N^{\kappa,(k)}(t)\right)\right| \leqslant C\left\|f^{(k)}\right\|_{op}\kappa^{\frac{1}{2}}.$$

Since $\kappa > 0$ is arbitrary, we deduce that

$$\lim_{N\to\infty}\left|\operatorname{Tr} f^{(k)}\left(\gamma_N^{(k)}(t) - |\phi(t)\rangle\langle\phi(t)|^{\otimes k}\right)\right| = 0,$$

i.e.

$$\gamma_N^{(k)}(t) \rightharpoonup |\phi(t)\rangle\langle\phi(t)|^{\otimes k} \text{ weak*}$$

as trace class operators. Notice that the limit has the same trace norm as $\gamma_N^{(k)}(t)$ for every $N$, the Grümm's convergence theorem then upgrades the above weak* convergence to strong. Thence, we have concluded Theorem 1.1 via Theorem 1.2.

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