Solutions to Selected Homework Problems
Math 9 — Fall 2005

§6.2 Volume

§6.2, #2.

$$V = \int_0^2 \pi (e^x)^2\, dx = \pi \int_0^2 e^{2x}\, dx = \pi \left[\frac{1}{2}e^{2x}\right]_0^2 = \frac{1}{2}\pi(e^4 - 1).$$

§6.2, #3.

$$V = \int_1^2 \pi(1/x)^2\, dx = \pi \int_1^2 x^{-2}\, dx = \pi \left[-x^{-1}\right]_1^2 = \pi\left(-\frac{1}{2} + 1\right) = \frac{\pi}{2}.$$

§6.2, #11.

$$\begin{aligned} V &= \int_0^1 \pi(1-x)^2 - \pi(1-\sqrt{x}\,)^2\, dx \\ &= \pi \int_0^1 (1 - 2x + x^2) - (1 - 2\sqrt{x} + x)\, dx \\ &= \pi \int_0^1 x^2 - 3x + 2\sqrt{x}\, dx \\ &= \pi \left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{4}{3}x^{3/2}\right]_0^1 \\ &= \pi\left(\frac{1}{3} - \frac{3}{2} + \frac{4}{3}\right) = \frac{\pi}{6}. \end{aligned}$$

§6.2, #51. The horizontal slices are rectangles. We let the top be $y = 0$ and we let y increase as we move down until we get to $y = h$ at the base of the pyramid. Then the rectangular slice at y is y/h as large as the base, so the area of the rectangular slice at y is

$$A(y) = \frac{2by}{h} \times \frac{by}{h} = \frac{2b^2y^2}{h^2}.$$

To get the volume, we integrate with respect to y from $y = 0$ to $y = h$, which gives (remember that b and h are constants)

$$V = \int_0^h A(y)\,dy = \int_0^h \frac{2b^2y^2}{h^2}\,dy$$
$$= \frac{2b^2}{h^2}\int_0^h y^2\,dy = \frac{2b^2}{h^2}\left[\frac{1}{3}y^3\right]_0^h = \frac{2b^2}{h^2}\cdot\frac{1}{3}h^3 = \frac{2b^2h}{3}.$$

§6.2, #54. We did this problem in class for a circle of radius 1. Put the circle in the xy-plane. Then x ranges from $-r$ to r, and for any given value of x, the slice (cross section) is a square whose side has length $2\sqrt{r^2 - x^2}$. (This is because the circle is given by the equation $x^2 + y^2 = 1$, so this is the distance from the top of the circle to the bottom of the circle.) The area of the square slice at x is thus

$$A(x) = \left(2\sqrt{r^2 - x^2}\right)^2 = 4(r^2 - x^2).$$

Then the volume of the solid is

$$V = \int_{-r}^r A(x)\,dx = \int_{-r}^r 4(r^2 - x^2) = 4\left[r^2x - \frac{1}{3}x^3\right]_{-r}^r = \frac{16}{3}r^3.$$

§6.4 Work

§6.4, #1. The force is constant, so

$$\text{Work} = \text{Force} \times \text{Distance} = (900\text{ N}) \times (8\text{ m}) = 7200\text{ J}.$$

§6.4, #3.

$$W = \int_0^9 \frac{10}{(1+x)^2}\,dx = \left[\frac{-10}{1+x}\right]_0^9 = -1 - (-10) = 9\text{ ft-lb}.$$

§6.4, #5.

$$W = \int_0^8 (\text{Force at } x)\,dx.$$

This is the area under the pictured curve. You can write down equations and integrate if you want, but it's easier to just observe that the region consists of two parts: (1) a right triangle whose base is 4 and height is 3, so it has area $\frac{1}{2} \cdot 4 \cdot 3 = 6$; (2) a rectangle that is 4-by-3, so it has area 12. Hence the total work is $W = 18$ J.

§6.4 Work

§6.4, #7. The force is given by $f(x) = kx$. We are given that

$$f(4 \text{ inches}) = 10 \text{ pounds},$$

so $k = 5/2$ inch-pounds. So the work to stretch from $x = 0$ to $x = 6$ inches is

$$W = \int_0^6 \frac{5}{2}x\,dx = \frac{5}{4}x^2\Big|_0^6 = 45 \text{ inch-pounds} = \frac{15}{4} \text{ foot-pounds}.$$

§6.4, #10. The force is given by $f(x) = kx$. We are given that the work to stretch 1 foot is 12 ft-lb. So

$$12 \text{ ft-lb} = \int_0^1 kx\,dx = \frac{1}{2}kx^2\Big|_0^1 = \frac{1}{2}k.$$

This tells us that $k = 24$ ft-lb. So the work done in stretching 9 inches beyond the spring's natural length is (note that 9 inches is $\frac{3}{4}$ foot)

$$W = \int_0^{3/4} 24x\,dx = 12x^2\Big|_0^{3/4} = \frac{27}{4} \text{ ft-lb}.$$

§6.4, #13. (a) Break the rope into n pieces and label starting at the top and going downward. Each little piece has length $50/n$ feet, so we let $\Delta x = \frac{50}{n}$ feet. The force on this little piece is $\frac{1}{2}\Delta x$ lb, since we're told that the rope weighs $\frac{1}{2}$ lb/ft.

The i^{th} little piece is x_i^* feet below the top of the building (where we can take $x_i^* = 50i/n$). So the work to move the i^{th} little piece to the top is

$$\begin{aligned} i^{\text{th}} \text{ piece of work} &= \text{Distance Moved} \times \text{Force} \\ &= (x_i^* \text{ feet}) \times (\frac{1}{2}\Delta x \text{ lbs}) \\ &= \frac{1}{2}x_i^*\Delta x \text{ ft-lb}. \end{aligned}$$

Adding up the little pieces of work and taking the limit as $n \to \infty$, the total work is

$$W = \int_0^{50} \frac{1}{2} x \, dx = \frac{1}{4} x^2 \Big|_0^{50} = 625 \text{ ft-lb.}$$

(b) The work done on the top half of the rope is computed in the same way as in (a) (just pretend the rope is only 25 feet long),

$$W^{\text{top half}} = \int_0^2 5\frac{1}{2} x \, dx = \frac{1}{4} x^2 \Big|_0^2 5 = \frac{625}{4} \text{ ft-lb.}$$

For the bottom half of the rope, each little piece is being raised exactly 25 feet (since none of it actually reaches the top of the building). That 25 feet of rope weighs $\frac{25}{2}$ lbs, so the work done on the bottom half of the rope is

$$W^{\text{bottom half}} = \left(\frac{25}{2} \text{ lbs} \right) \times (25 \text{ feet}) = \frac{625}{2} \text{ ft-lb.}$$

Hence the total work done is

$$W = W^{\text{top half}} + W^{\text{bottom half}} = \frac{625}{4} \text{ ft-lb} + \frac{625}{2} \text{ ft-lb} = \frac{1875}{4} \text{ ft-lb.}$$

§6.5 Average Value of a Function

§6.5, #1.

$$f_{\text{avg}} = \frac{1}{1-(-1)} \int_{-1}^{1} x^2 \, dx = \frac{1}{2} \left[\frac{1}{3} x^3 \right]_{-1}^{1} = \frac{1}{3}.$$

§6.5, #2.

$$f_{\text{avg}} = \frac{1}{4-1} \int_1^4 \frac{dx}{x} = \frac{1}{3} \ln(x) \Big|_1^4 = \frac{1}{3} \ln(4) = \frac{2}{3} \ln(2).$$

§6.5, #5.

$$f_{\text{avg}} = \frac{1}{5-0} \int_0^5 t e^{-t^2} \, dt = \frac{1}{5} \left[-\frac{1}{2} e^{-t^2} \right]_0^5$$
$$= \frac{1}{5} \left(-\frac{1}{2} e^{-25} + \frac{1}{2} \right) = \frac{1}{10} \left(1 - e^{-25} \right).$$

§6.5, #9. (a)

$$f_{\text{avg}} = \frac{1}{5-2}\int_2^5 (x-3)^2\,dx = \frac{1}{3}\left[\frac{1}{3}(x-3)^3\right]_2^5 = \frac{1}{9}(2^3-(-1)^3) = 1$$

(b) We need to solve $f(c) = f_{\text{avg}}$. So solve

$$(c-3)^2 = 1.$$

Thus $c - 3 = \pm 1$, so $c = 2$ or $c = 4$.