A SURVEY OF CLUSTER ALGEBRAS

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Abstract. This is a concise expository survey of cluster algebras, introduced by S. Fomin and A. Zelevinsky in their four-part series of foundational papers [1], [2], [3], [4] (the paper [3] is with coauthor A. Berenstein). Our primary focus is on Cluster Algebras IV: Coefficients [4]. We introduce the setting of principal coefficients and define and study F-polynomials and g-vectors. Along the way, we mention the Laurent phenomenon and the classification of cluster algebras of finite type.

1. What is a cluster algebra?

A cluster algebra is a particular kind of commutative subalgebra of a field of rational functions. One specifies a cluster algebra by describing its generators. But instead of providing an a priori infinite list of generators, one gives instead an initial finite set of them, and, in addition, an iterative method of producing new generators from old ones. This method is a combinatorial process which we call mutation.

Cluster algebras were introduced in 2002 by S. Fomin and A. Zelevinsky in their four-part series of foundational papers [1], [2], [3], [4] (the paper [3] is with coauthor A. Berenstein). They were developed as a hopeful tool for studying total positivity and canonical bases of algebraic groups; since then, they have seen connections to many different fields, including tropical geometry and Teichmüller theory, and have generated a phenomenal amount of interest in their own right.

The precise definition of a cluster algebra will take several pages to develop. This will occupy the rest of Section 1. In Sections 2 and 3, we touch on the Laurent phenomenon (Theorem 2.1), the Positivity Conjecture (Conjecture 2.2), and the classification of cluster algebras of finite type (Theorem 3.2). Section 4 contains a survey of the paper [4]; we develop the theory of F-polynomials and g-vectors for cluster algebras with principal coefficients. We conclude in Section 5 with the example of affine Grassmannians $(2, n)$.

Let us work towards the definition of a cluster algebra by recalling the notion of a semifield.

A **semifield** $\mathbb{P}$ is an abelian group, written multiplicatively, which is endowed with a binary operation $\oplus$ called auxiliary addition. This operation is required to be commutative, associative, and distributive under group multiplication. Thus, a semifield has all of the properties of field, except possibly for the existence of an additive identity and additive inverses.

Our primary example of a semifield is the **tropical semifield** of rank n, denoted $\mathrm{Trop}(y_1, \ldots, y_n)$, which consists of the set of Laurent monomials in the variables $y_1, \ldots, y_n$

$$\{y_1^{a_1} \cdots y_n^{a_n} \ : \ a_1, \ldots, a_n \in \mathbb{Z}\}.$$

Multiplication and addition are given by

$$\begin{aligned} y_1^{a_1} \cdots y_n^{a_n} \quad &\cdot \quad y_1^{b_1} \cdots y_n^{b_n} &= \quad & y_1^{a_1+b_1} \cdots y_n^{a_n+b_n}, \\ y_1^{a_1} \cdots y_n^{a_n} \quad &\oplus \quad y_1^{b_1} \cdots y_n^{b_n} &= \quad & y_1^{\min(a_1,b_1)} \cdots y_n^{\min(a_n,b_n)}. \end{aligned}$$

Note that identifying monomials with their exponents vectors identifies $\mathrm{Trop}(y_1, \ldots, y_n)$ with the integer points in the tropical semimodule $(\mathbb{R}^n, \oplus, \odot)$.

Let us fix some notation. If $s = (s_1, \ldots, s_n)$ and $b = (b_1, \ldots, b_n) \in \mathbb{Z}^n$, then we write s^b for $s_1^{b_1} \cdots s_n^{b_n}$. Given an integer x, we let $[x]_+ = \max\{x, 0\}$ and $[x]_- = \max\{-x, 0\}$, and we extend this notation to vectors by applying it coordinatewise. For example,

$$[(3, -4)]_+ = (3, 0) \ \text{ and } \ [(3, -4)]_- = (0, 4).$$

Given an element $t \in \mathbb{P}$, define elements

$$t^+ := t/(t \oplus 1) \ \text{ and } \ t^- := 1/(t \oplus 1). \tag{1}$$

Notice that $t = t^+/t^-$ and $t^+ \oplus t^- = 1$. For example, if $t = (3, -4)$, regarded as an element of the tropical semifield of rank 2 by identifying Laurent monomials with their exponent vectors, then $t^+ = (3, 0)$ and $t^- = (0, 4)$. We can view t^+ and t^- as the positive and negative parts, respectively, of t.

Lemma 1.1. *Let $\mathbb{P}$ be a semifield, and let $\mathbb{ZP}$ denote the group ring associated to the multiplicative group $\mathbb{P}$ and the ring of integers $\mathbb{Z}$. Then $\mathbb{ZP}$ is an integral domain.*

Proof. Notice that the multiplicative group $\mathbb{P}$ must be torsion-free [1, Section 5]: indeed, if $p^m = 1$, then

$$p \cdot (p^{m-1} \oplus \cdots \oplus 1) = 1 \oplus p^{m-1} \oplus \cdots \oplus p \ \text{ implies } p = 1.$$

Now, we argue, following [6, Theorems 6.29, 6.31], that $\mathbb{ZP}$ has no zero divisors. First, we note that the group ring $\mathbb{Z}G$ associated to any ordered group $(G, <)$ has no zero divisors: if $a = n_1 g_1 + \cdots + n_k g_k$ and

$a' = n'_1 g'_1 + \cdots + n'_l g'_l$ are two nonzero elements of $\mathbb{Z}G$, where n_i, n'_j are nonzero integers and $g_1 < \cdots < g_k$ and $g'_1 < \cdots < g'_l$ are elements of G, then the coefficient of $g_1 g'_1$ in the product aa' is $n_1 n'_1 \neq 0$. So $aa' \neq 0$.

Then it suffices to show that $\mathbb{P}$ can be ordered. Indeed, we now show that any torsion-free abelian group A can be ordered. Note that A embeds into a $\mathbb{Q}$-vector space via $A \hookrightarrow A \otimes_{\mathbb{Z}} \mathbb{Q}$, so we may assume A is a $\mathbb{Q}$-vector space. Choose and linearly order a basis of A over $\mathbb{Q}$. Given an element

$$a = \lambda_1 v_1 + \cdots + \lambda_n v_n \in A$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{Q}$, and $v_1 < \cdots < v_n$ are basis elements, say that a is *positive* if $\lambda_1 > 0$. Then the relation on A given by $a < b$ whenever ba^{-1} is positive is a linear order on A. □

Now, fix a semifield $\mathbb{P}$. Lemma 1.1 implies that the group ring $\mathbb{Z}\mathbb{P}$ is a domain, so we write K for the field of fractions of $\mathbb{Z}\mathbb{P}$. (We can now safely forget about Lemma 1.1.) Let $\mathcal{F}$ denote the field of rational functions in n indeterminates $x_1, \ldots, x_n$, with coefficients in $\mathbb{Z}\mathbb{P}$, thus

$$\mathcal{F} = K(x_1, \ldots, x_n).$$

For example, if $\mathbb{P} = \mathrm{Trop}(y_1, \ldots, y_n)$, then $\mathbb{Z}\mathbb{P} = \mathbb{Z}[y_1^{\pm}, \ldots, y_n^{\pm}]$ and $K = \mathbb{Q}(y_1, \ldots, y_n)$. Note that the addition operation $\oplus$ does not play a role in the definition of $\mathcal{F}$, but it will appear in the definition of a cluster algebra, Definition 1.4.

A **labeled seed** is an ordered triple

$$((s_1, \ldots, s_n), (t_1, \ldots, t_n), B),$$

satisfying the following conditions:

- $s_1, \ldots, s_n \in \mathcal{F}$ are algebraically independent over K,
- $t_1, \ldots, t_n \in \mathbb{P}$, and
- B is an $n \times n$ matrix of integers that is *skew-symmetrizable*; that is, one may scale the rows of B by positive numbers to obtain a skew-symmetric matrix.

We call the s_i variables **cluster variables**, the t_j variables **coefficient variables**, and B the **exchange matrix** of the seed (s, t, B).

Now, suppose we are given a labeled seed $(s = (s_1, \ldots, s_n), t = (t_1, \ldots, t_n), B)$ and an integer $k \in \{1, \ldots, n\}$. Then the **seed obtained by mutation** of (s, t, B) in direction k is the ordered triple $(s' = (s'_1, \ldots, s'_n), t' = (t'_1, \ldots, t'_n), B')$ defined as follows.

- For $j \in \{1, \ldots, n\} \setminus k$, we let $s'_j = s_j$, and we let

$$s'_k s_k = t_k^+ s^{[B_k]_+} + t_k^- s^{[B_k]_-}, \tag{2}$$

where B_k denotes the k^{th} column vector of the matrix B.

- For $j \in \{1, \ldots, n\} \setminus k$, let

$$t'_j = t_j \cdot \frac{(t_k^+)^{[b_{kj}]_+}}{(t_k^-)^{[b_{kj}]_-}}, \tag{3}$$

and let $t'_k = t_k^{-1}$.

- For $i, j \in \{1, \ldots, n\}$, if $i = k$ or $j = k$, let $b'_{ij} = -b_{ij}$. Otherwise, let

$$b'_{ij} = b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_-. \tag{4}$$

Remark 1.2. One may check that the new triple (s', t', B') really is another labeled seed, and furthermore, that mutation in direction k of (s', t', B') yields the original seed (s, t, B).

Remark 1.3. The notation above loosely follows the notation of [2, Section 1.2], and has the advantage that it makes the symmetry in Remark 1.2 apparent. We can, however, use the definitions of t^+ and t^- in (1) to rewrite the relations (2) and (3). The cluster exchange relation (2) becomes

$$s'_k s_k = \frac{t_k s^{[B_k]_+} + s^{[B_k]_-}}{t_k \oplus 1}, \tag{5}$$

and the coefficient exchange relation (3) becomes

$$t'_j = t_j \cdot t_k^{[b_{kj}]_+} (1 \oplus t_k)^{-b_{kj}}. \tag{6}$$

This presentation follows [4] and will be the starting point of the discussion of F-polynomials and g-vectors in Section 4.

Now, let $\mathbb{T}_n$ denote the **infinite n-ary tree**, that is, a connected acyclic simple n-regular graph. Equip $\mathbb{T}_n$ with an n-edge coloring. For example, $\mathbb{T}_2$ is an infinite two-way path, and we give its edges alternating colors. A **cluster pattern** is an assignment of labeled seeds to the vertices of $\mathbb{T}_n$ such that for any edge, say between vertices v and v' and of color $k \in \{1, \ldots, n\}$, the seeds associated to v and v' are obtained from each other by mutation in direction k.

We are finally equipped to define a cluster algebra.

Definition 1.4. A **cluster algebra** $\mathcal{A}$ of rank n is the K-subalgebra of $\mathcal{F} = K(x_1, \ldots, x_n)$ which is generated by the set of all cluster variables in some cluster pattern on $\mathbb{T}_n$.

We say that $\mathcal{A}$ is of **geometric type** if the semifield $\mathbb{P}$ is a tropical semifield. In Section 4, we will discuss a very interesting situation that arises among cluster algebras of geometric type: the case of principal coefficients.

These definitions are quite opaque at first glance, so in Section 5, we will provide a family of familiar examples. We will prove that the coordinate ring of the affine Grassmannian $\mathbb{C}[\mathrm{Gr}(2,n)]$ carries the structure of a cluster algebra, with 3-term Plücker relations playing the role of cluster exchange relations.

2. The Laurent phenomenon

Theorem 2.1. [1, Theorem 3.1, Laurent Phenomenon] *Let $\mathcal{A}$ be a cluster algebra over semifield $\mathbb{P}$, let $(s_1, \ldots, s_n)$ be any cluster, and let x be any cluster variable. Then x can be expressed as a Laurent polynomial in $s_1, \ldots, s_n$ with coefficients in $\mathbb{ZP}$.*

In [1], Fomin and Zelevinsky conjectured that the coefficients of the Laurent polynomials in Theorem 2.1 are always positive combinations of semifield elements.

Conjecture 2.2. [1, Positivity Conjecture] *The Laurent polynomials in Theorem 2.1 have coefficients of the form $n_1p_1 + \cdots + n_kp_k$, where $n_1, \ldots, n_k$ are positive integers and $p_1, \ldots, p_k \in \mathbb{P}$.*

The positivity conjecture has been studied extensively; many authors have proved special cases. For example, it has been proved for cluster algebras of finite type [4, Corollary 11.7], and for cluster algebras of geometric type arising from surfaces [7, Theorem 1.1].

3. Cluster algebras of finite type

Definition 3.1. A cluster algebra is of **finite type** if the set of distinct seeds in the associated cluster pattern is finite.

This section summarizes several important theorems from the paper [2]. We need some definitions to make sense of the main theorem, Theorem 3.2, which characterizes finite type cluster algebras.

The **Cartan counterpart** of an integer square matrix is the matrix obtained by replacing each off-diagonal entry with the negative of its modulus and setting all diagonal entries equal to 2. A matrix $B \in \mathbb{Z}^{n\times n}$ is **sign-skew-symmetric** if for all $i, j \in \{1, \ldots, n\}$, $b_{ij} = b_{ji} = 0$ or $b_{ij}b_{ji} < 0$. Given a sign-skew-symmetric matrix $B \in \mathbb{Z}^{n\times n}$, the **diagram associated to** B is the edge-weighted directed graph on vertices $\{1, \ldots, n\}$ with a directed edge from i to j, of weight $|b_{ij}b_{ji}|$, whenever $b_{ij} > 0$.

Theorem 3.2. [2, Theorems 1.5, 1.6, 1.8] *Let $\mathcal{A}$ be a cluster algebra. The following are equivalent:*

(1) *$\mathcal{A}$ has finite type.*
(2) *For some exchange matrix B of a seed (s, t, B) in the cluster pattern, the diagram associated to B is an orientation of a Dynkin diagram.*
(3) *For some exchange matrix B of a seed (s, t, B) in the cluster pattern, the Cartan counterpart of B is a Cartan matrix.*
(4) *For all seeds (s, t, B) of $\mathcal{A}$, the matrix B satisfies $|b_{ij}b_{ji}| \leq 3$ for all $i, j \in \{1, \dots, n\}$.*

Thus, if $\mathcal{A}$ is a finite type cluster algebra, then we say that it has **type** $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$, or G_2 according to the Cartan-Killing classification of its corresponding Dynkin diagram, or equivalently of its corresponding Cartan matrix. That this notion of type is well-defined is a consequence of [2, Theorem 1.4], which we omit from this survey.

4. F-polynomials and g-vectors

Let $\mathcal{A}$ be a cluster algebra arising from a cluster pattern on the infinite n-ary tree $\mathbb{T}_n$, and let $\mathbb{P}$ be the underlying semifield. We say that $\mathcal{A}$ has **principal coefficients** at a vertex $v_0 \in \mathbb{T}_n$ if $\mathbb{P} = \text{Trop}(t_1, \dots, t_n)$, where $(t_1, \dots, t_n)$ is the n-tuple of coefficient variables in the labeled seed at v_0.

The case of principal coefficients is very interesting and we will summarize some of the main theorems, following the treatment in [4, Sections 4-6]. We will see that a cluster algebra with principal coefficients has a natural $\mathbb{Z}^n$-grading; this leads to the notion of a g-vector. We will also see that the Laurent polynomials expressing a given cluster variable in terms of an initial cluster in a cluster algebra with principal coefficients are sufficient to recover the corresponding Laurent polynomials for an arbitrary cluster algebra. We will do this by studying F-polynomials.

Suppose a cluster algebra $\mathcal{A}$ has principal coefficients at vertex v_0; let

$$((s_1, \dots, s_n), (t_1, \dots, t_n), B)$$

be the labeled seed at v_0, so $\mathbb{P} = \text{Trop}(t_1, \dots, t_n)$. Let $s_{l,v}$ be the cluster variable in the l^{th} coordinate of the labeled seed at vertex $v \in \mathbb{T}_n$. We would like to write $s_{l,v}$ in terms of $s_1, \dots, s_n$. We can do so by applying the mutations (2), (3), and (4) along the unique path from v_0 to v in $\mathbb{T}_n$. Now, since every tropical sum of elements of $\mathbb{P}$ is just a monomial in $t_1, \ldots, t_n$, we see inductively that $s_{l,v}$ can be written as a *subtraction-free rational expression* in $s_1, \ldots, s_n, t_1, \ldots, t_n$. For clarity:

Definition 4.1. The semifield $\mathbb{Q}_{sf}(x_1, \ldots, x_n)$ of **subtraction-free rational expressions** in n variables consists of those rational functions $f \in \mathbb{Q}(x_1, \ldots, x_n)$ such that $f \cdot q = p$ for some polynomials p, q with all coefficients positive. (It is clear that these elements form a semifield.)

Definition 4.2. Let $\mathcal{A}$ be a cluster algebra with principal coefficients at vertex v_0, and let (s, t, B) be the labeled seed at v_0. Let v be any vertex of $\mathbb{T}_n$, and let $l \in \{1, \ldots, n\}$. Then define

$$X_{l,v} \in \mathbb{Q}_{sf}(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

to be the subtraction-free rational expression such that

$$s_{l,v} = X_{l,v}(s_1, \ldots, s_n, t_1, \ldots, t_n).$$

Note that $X_{l,v}$ is uniquely defined since $s_1, \ldots, s_n$ are algebraically independent.

Now we can define the F-polynomial.

Definition 4.3. With $\mathcal{A}, v_0, v$, and l as in Definition 4.2, define the **F-polynomial** $F_{l,v}$ by setting all x variables in $X_{l,v}$ to 1:

$$F_{l,v}(y_1, \ldots, y_n) := X_{l,v}(1, \ldots, 1, y_1, \ldots, y_n) \in \mathbb{Q}_{sf}(y_1, \ldots, y_n).$$

Proposition 4.4. [4, Proposition 3.6] *Each $X_{l,v}$ is an element of $\mathbb{Z}[x_1^{\pm}, \ldots, x_n^{\pm}, y_1, \ldots, y_n]$, and each $F_{l,v}$ is an element of $\mathbb{Z}[x_1, \ldots, x_n]$.*

In fact, it is conjectured that the coefficients of $X_{l,v}$ and of $F_{l,v}$ are always positive integers [4, Section 3]. This is proved for cluster algebras of finite type [4, Corollary 11.7]. In view of Theorem 4.7, the aforementioned conjecture is a reformulation of Conjecture 2.2.

Next, we note that a cluster algebra $\mathcal{A}$ of rank n with principal coefficients has a natural $\mathbb{Z}^n$-grading.

Proposition 4.5. [4, Proposition 6.1, Corollary 6.2] *Let $\mathcal{A}$ be a cluster algebra with principal coefficients at v_0; let (s, t, B) be the labeled seed at v_0. Then every polynomial $X_{l,v}$, as given in Definition 4.2, is homogeneous with respect to the multigrading*

$$\deg : \mathbb{Z}[x_1^{\pm}, \ldots, x_n^{\pm}, y_1, \ldots, y_n] \to \mathbb{Z}^n$$

sending $x_i \mapsto e_i$ and $y_i \mapsto -B_i$, where B_i denotes the i^{th} column of the matrix B.

Thus, $\mathcal{A}$ is a $\mathbb{Z}^n$-graded $\mathbb{Z}$-subalgebra of $\mathbb{Z}[x_1^{\pm}, \ldots, x_n^{\pm}, y_1, \ldots, y_n]$.

Definition 4.6. We let $g_{l,v} \in \mathbb{Z}^n$ be the degree of the cluster variable $s_{l,v}$ according to the grading in Proposition 4.5. These vectors are called **g-vectors**.

In fact, we will see in Theorem 4.7 that the F-polynomials and g-vectors of a cluster algebra $\mathcal{A}$ with principal coefficients completely determine formulas for the cluster variables of an arbitrary cluster algebra.

To state Theorem 4.7, we need some notation: if F is a subtraction-free rational expression and $\mathbb{P}$ is a semifield, let $F|_{\mathbb{P}}$ denote the semifield polynomial that is obtained by writing F using only addition, multiplication, and division, then replacing usual operations with operations from the semifield $\mathbb{P}$.

Theorem 4.7. [4, Corollary 6.3] *Let $\mathcal{A}'$ be any cluster algebra, and let (x, y, B) denote the seed at vertex v_0. Let $\mathcal{A}$ be the unique (up to isomorphism) cluster algebra with principal coefficients at v_0 whose labeled seed at v_0 has exchange matrix equal to B.*

Given a vertex v of $\mathbb{T}_n$ and $l \in \{1, \dots, n\}$, associate to $\mathcal{A}$ the F-polynomial $F_{l,v}$ and the g-vector $g = g_{l,v}$ as in Definitions 4.3 and 4.6. Then the cluster variable of $\mathcal{A}$ at the l^{th} coordinate of vertex v, denoted $x_{l,v}$, is given by

$$x_{l,v} = \frac{F_{l,v}(\hat{y}_1, \dots, \hat{y}_n)}{F_{l,v}|_{\mathbb{P}}(y_1, \dots, y_n)} x_1^{g_1} \cdots x_n^{g_n}, \tag{7}$$

where $\hat{y}_i$ denotes the degree-0 element $y_i \cdot x_1^{B_{1,i}} \cdots x_n^{B_{n,i}}$, for each $i = 1, \dots, n$.

Remark 4.8. Equation 7 demonstrates the *separation of addition* phenomenon: notice that the numerator contains all instances of group ring addition in $\mathbb{ZP}$, while the denominator contains all instances of the semifield addition in $\mathbb{P}$.

Another sense in which the case of principal coefficients is universal is with regard to exchange graphs. First, let us define an equivalence relation on labeled seeds, where $(s, t, B) \sim (s', t', B')$ if (s', t', B') can be obtained from (s, t, B) by a simultaneous permutation of the entries of s, the entries of t, and the rows and columns of B. Then the **exchange graph** of a cluster algebra $\mathcal{A}$ is the graph on equivalence classes of labeled seeds of $\mathcal{A}$ with an edge between two classes if and only if they have representatives which are obtained from each other by a single mutation.

Note that the exchange graph $G(\mathcal{A})$ is therefore equipped with a canonical surjective graph homomorphism $p_{\mathcal{A}} : \mathbb{T}_n \twoheadrightarrow G(\mathcal{A})$ sending a vertex of $\mathbb{T}_n$ to the equivalence class of its labeled seed. (A homomorphism $G \to H$ of graphs is a map $V(G) \to V(H)$ of sets that sends adjacent vertices in G to adjacent vertices in H.)

Now, given cluster algebras $\mathcal{A}$ and $\mathcal{A}'$ of rank n, we say that the graph $G(\mathcal{A})$ **covers** the graph $G(\mathcal{A}')$ if there exists a graph homomorphism ϕ such that the diagram

$$\begin{array}{ccc} & & \mathbb{T}_n \\ & \swarrow^{p_{\mathcal{A}}} & \downarrow^{p_{\mathcal{A}'}} \\ G(\mathcal{A}) & \xrightarrow{\phi} & G(\mathcal{A}') \end{array}$$

commutes.

Theorem 4.9. *Let $\mathcal{A}'$ be any cluster algebra of rank n, and let B be the exchange matrix of $\mathcal{A}'$ at the vertex $v \in \mathbb{T}_n$. Let $\mathcal{A}$ be the unique rank n cluster algebra with exchange matrix B at v and principal coefficients at v. Then $G(\mathcal{A})$ covers $G(\mathcal{A}')$.*

In fact, the following much stronger statement is conjectured to be true:

Conjecture 4.10. [4, Conjecture 4.3] *The exchange graph $G(\mathcal{A})$ of a cluster algebra $\mathcal{A}$ does not depend on the choice of coefficients. More precisely, if $\mathcal{A}$ and $\mathcal{A}'$ are two cluster algebras such that the matrix B occurs as an exchange matrix for each, then $G(\mathcal{A}) = G(\mathcal{A}')$.*

Conjecture 4.10 is known to hold for cluster algebras of finite type [2, Theorem 1.13] and for cluster algebras of rank 2 [2, Example 7.6].

5. Example: Affine Grassmannians of 2-planes

In this section, we will sketch the proof that for each $n \geq 4$, the affine cone over the complex Grassmannian $\mathrm{Gr}(2, n)$ is a cluster algebra of classical type A_{n-3}. Our reference throughout is Section 12.2 of [2].

Fix $n \geq 4$, let S denote the polynomial ring $\mathbb{C}[z_{12}, z_{13}, \ldots, z_{n-1,n}]$ in $\binom{n}{2}$ variables, and let $I \subseteq S$ be the ideal of 3-term *Plücker relations*

$$I = \langle z_{ij}z_{kl} - z_{ik}z_{jl} + z_{il}z_{ik} \;:\; i, j, k, l \in [n] \text{ distinct}\rangle.$$

By the affine cone over the Grassmannian $\mathrm{Gr}(2, n)$, we mean $\mathrm{Spec}(S/I)$.

The **cluster complex** $\Delta(A)$ of a cluster algebra $\mathcal{A}$ is defined to be the simplicial complex whose ground set is the set of cluster variables of $\mathcal{A}$ and whose maximal cells are the clusters. If $\mathcal{A}$ is of type A_{n-3}, then $\Delta(A)$ is known to be the dual complex of the rank n *associahedron*, a polytope whose vertices correspond to triangulations of the plane polygon P_n with n vertices. The exchange graph $G(\mathcal{A})$ is then the 1-skeleton of the associahedron. Two triangulations are adjacent in $G(A)$ if they differ by a single flip of a diagonal.

Now let $\mathcal{A}$ be the cluster algebra of type A_{n-3} defined as follows. We let

$$\mathbb{P} = \mathrm{Trop}(p_{ab} \ : \ ab \text{ is a side of the polygon } P_n).$$

We associate, to each diagonal ij of P_n used in a triangulation, the cluster variable x_{ij}, so that each triangulation is labeled with a cluster of size n. For convenience, we set $x_{ij} = 1$ if ij is a side, not a diagonal, of P_n. Then the exchange relations (2) for $\mathcal{A}$ are given by

$$x_{ac}x_{bd} = p^{+}_{ac,bd}x_{ab}x_{cd} + p^{-}_{ac,bd}x_{ad}x_{bc}$$

where a, b, c, d are distinct vertices of P_n occurring in counterclockwise order. Here, we define

$$p^{+}_{ac,bd} = p_{ab}p_{cd} \text{ and } p^{-}_{ac,bd} = p_{ad}p_{bc},$$

where again, for convenience of notation, we let $p_{ij} = 1$ if ij is a diagonal, and not a side, of P_n.

Finally, given a triangulation T of P_n, we let $B(T)$ be the square matrix indexed by the n diagonals of P_n occurring in T, with entries given as follows. We set $B_{xy} = 1$ if x and y are diagonals of P_n that occur in clockwise order as sides of a triangle in T, and we set $B_{xy} = -1$ if x and y occur in counterclockwise order as sides of a triangle in T. Otherwise, we set $B_{xy} = 0$.

We may check that these coefficients and exchange matrices satisfy the exchange relations (3) and (4), so $\mathcal{A}$ is well-defined. Finally, we can make use of the following criterion from [2, Proposition 11.1], as formulated in [8, Proposition 1]:

Proposition 5.1. *Let $\mathcal{A}$ be a cluster algebra of rank n, let X be the set of cluster variables of $\mathcal{A}$, and let C be the set of coefficients (satisfying the mild assumption of* [2, (11.2)]. *Let Y be a rational quasi-affine irreducible algebraic variety over $\mathbb{C}$. Suppose we are given elements of $\mathbb{C}[Y]$: one for each $x \in X$, say ϕ_x, and one for each $c \in C$, say ϕ_c, such that the elements*

$$\{\phi_x \ : \ x \in X\} \cup \{\phi_c \ : \ c \in C\}$$

generate $\mathbb{C}[Y]$. Suppose further that $dim(Y) = n + |C|$ and that each exchange relation in $\mathcal{A}$ remains valid in $\mathbb{C}[Y]$ under the maps $x \mapsto \phi_x$ and $c \mapsto \phi_c$.

Then the correspondences $x \mapsto \phi_x$ and $c \mapsto \phi_c$ extend uniquely to an algebra isomorphism from $\mathcal{A}$ to $\mathbb{C}[Y]$.

We can check that the cluster algebra we have constructed satisfies the conditions of the above proposition with respect to the coordinate ring $\mathbb{C}[\mathrm{Gr}(2,n)]$. In particular, the exchange relations remain valid in $\mathbb{C}[\mathrm{Gr}(2,n)]$ precisely because they become 3-term Plücker relations. Thus $\mathbb{C}[\mathrm{Gr}(2,n)]$ has the structure of a cluster algebra of type A_{n-3}.

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