Combinatorics of the tropical Torelli map

`arxiv:1012.4539`

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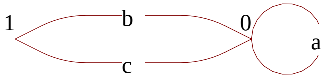
What is a tropical curve?

A **tropical curve** C is a triple (G, l, w), where (G, l) is a metric graph, and w is a weight function

$$w : V(G) \to \mathbb{Z}_{\geq 0}$$

on the vertices of G, with the property that every weight zero vertex has degree at least 3.



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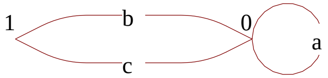

Its **genus** is $g(G) + \sum_{v \in V} w(v)$.

Its **combinatorial type** is the pair (G, w).

The Jacobian of a tropical curve

Given a genus g tropical curve $C = (G, l, w)$, with edges of G oriented for reference, let $H_1(G, \mathbb{R})$ = formal sums of edges of G with zero boundary.

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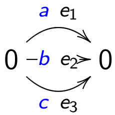
Now define a positive semidefinite form Q on $H_1(G, \mathbb{R}) \oplus \mathbb{R}^{\sum w(v)}$ which is 0 on $\mathbb{R}^{\sum w(v)}$ and on $H_1(G, \mathbb{R})$ is

$$Q\Big(\sum_{e \in E(G)} \alpha_e \cdot e, \sum_{e \in E(G)} \beta_e \cdot e\Big) = \sum_{e \in E(G)} \alpha_e \cdot \beta_e \cdot l(e).$$

Choosing a $\mathbb{Z}$-basis for $H_1(G, \mathbb{Z})$ defines Q as a $g \times g$ positive semidefinite matrix with rational nullspace.



$e_1 - e_2, e_2 - e_3$

$$\begin{pmatrix} a+b & -b \\ -b & b+c \end{pmatrix}$$

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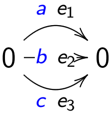

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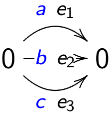
Choosing a different $\mathbb{Z}$-basis for $H_1(G, \mathbb{Z})$ changes Q by a $GL_g(\mathbb{Z})$-action:



$$e_1 - e_2, e_1 - 2e_2 + e_3$$

$$\begin{pmatrix} a+b & a+2b \\ -a+2b & a+4b+c \end{pmatrix}$$

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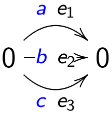

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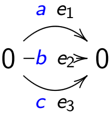

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$$\begin{pmatrix} a+b & a+2b \\ -a+2b & a+4b+c \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^T \begin{pmatrix} a+b & -b \\ -b & b+c \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

The Jacobian of a tropical curve

So we obtain a well-defined element of

$$\frac{\widetilde{S}^g_{\geq 0}}{GL_g(\mathbb{Z})} := \frac{\text{psd matrices with rational nullspace}}{Q \sim X^T QX \text{ for all } X \in GL_g(\mathbb{Z})},$$

and this point in $\widetilde{S}^g_{\geq 0}/GL_g(\mathbb{Z})$ is called the **Jacobian** of the curve.

The tropical Torelli map

Classically, the Torelli map, from the moduli space of curves to the moduli space of principally polarized abelian varieties, sends a curve to its Jacobian.

We will construct a tropical analogue: a tropical Torelli map

$$t_g^{\text{tr}} : M_g^{\text{tr}} \to A_g^{\text{tr}}$$

from the moduli space of tropical curves to the moduli space of principally polarized tropical abelian varieties that takes a tropical curve to its Jacobian.

Brannetti-Melo-Viviani `arXiv:0907.3324`

Towards a moduli space of tropical curves

Warm up: what are the possible combinatorial types of genus 2 tropical curves?

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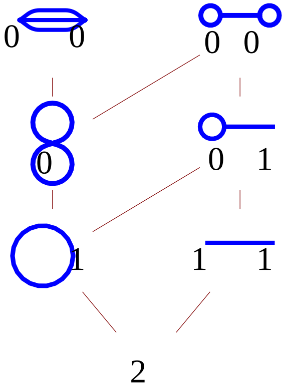
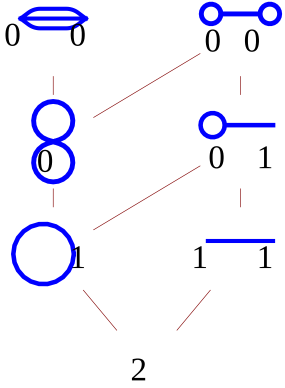


This is the poset of combinatorial types of genus 2 tropical curves, ordered by contraction. Note: contracting a loop at a vertex increases its weight by 1.

Motivation: stratification of $\overline{\mathcal{M}_g}$ by dual graphs



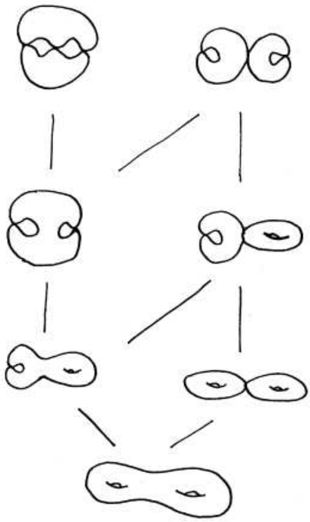


Figure: Posets of cells of M_2^{tr} (left) and of $\overline{\mathcal{M}_2}$ (right). Vertices record irreducible components, weights record genus, edges record nodes.

Construction of M_g^{tr}

Our goal is to construct a **moduli space** M_g^{tr} for genus g tropical curves, that is, a space whose points correspond to tropical curves of genus g and whose geometry reflects the geometry of the tropical curves in a sensible way.

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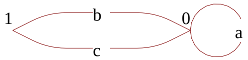
Construction due to B-M-V.

Fix a combinatorial type (G, w) of genus g. What is a parameter space for all tropical curves of this type?



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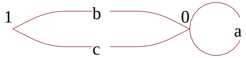


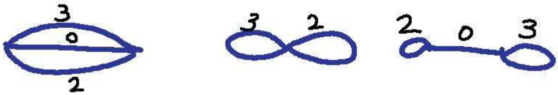
$$\frac{\mathbb{R}^3_{\geq 0}}{(a,b,c)\sim(a,c,b)} = \frac{\mathbb{R}^3_{\geq 0}}{S_2}$$

Construction of M_g^{tr} continued

Strategy: each combinatorial type of genus g gets a cell

$$\frac{\mathbb{R}_{\geq 0}^{|E(G)|}}{\mathrm{Aut}(G, w)}.$$

Now identify two graphs in the disjoint union of all such cells if they are the same after contracting all edges of length zero.



The resulting space, denoted M_g^{tr}, has points in bijection with genus g tropical curves. It is a Hausdorff topological space (Caporaso 2010).

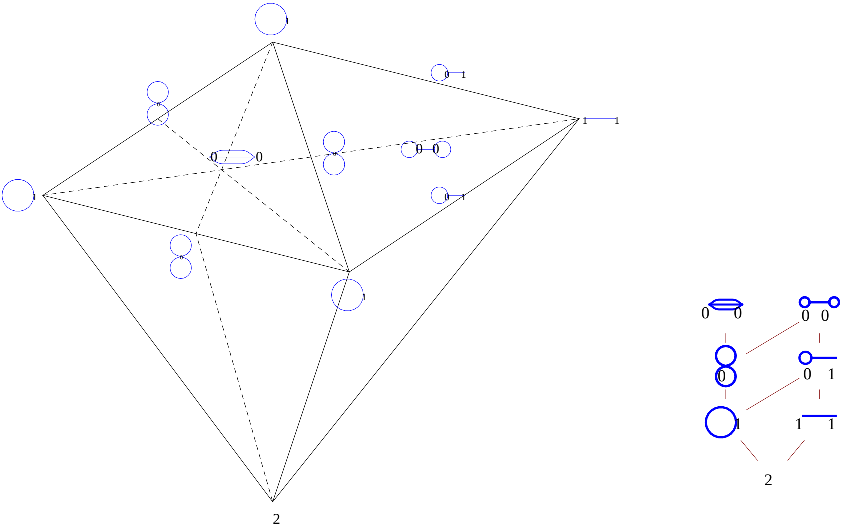
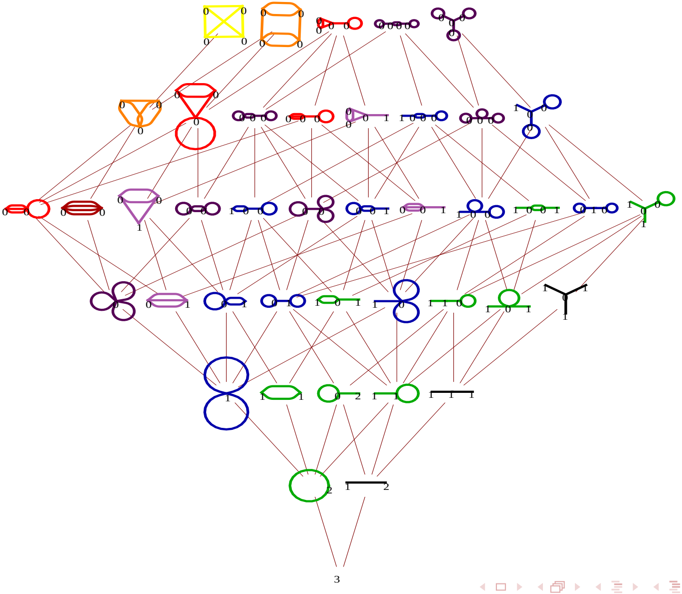


Figure: Cells of M_2^{tr}.

Theorem (C, also Maggiolo-Pagani 2010)

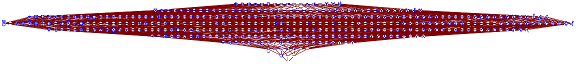
The moduli space M_3^{tr} *has 42 cells and* f*-vector* $(1, 2, 5, 9, 12, 8, 5)$.



Theorem (C, also Maggiolo-Pagani 2010)

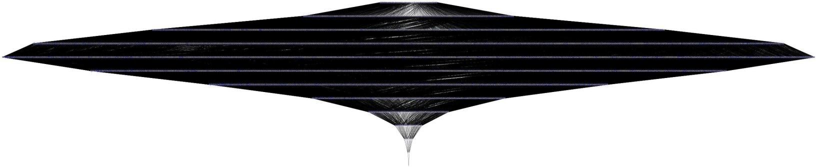
- *The moduli space M_4^{tr} has 379 cells and f-vector*

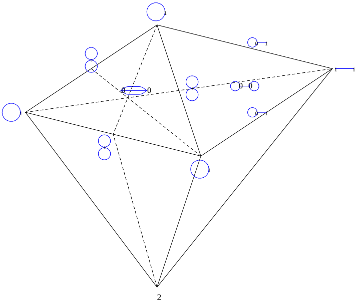
$$(1, 3, 7, 21, 43, 75, 89, 81, 42, 17).$$



- *The moduli space M_5^{tr} has 4555 cells and f-vector*

$$(1, 3, 11, 34, 100, 239, 492, 784, 1002, 926, 632, 260, 71).$$



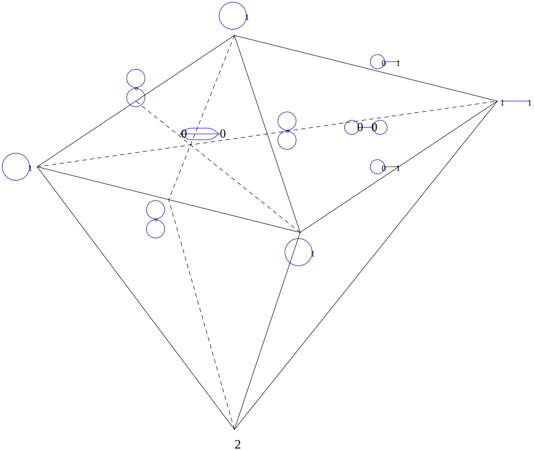


Note: does M_g^{tr}, the moduli space of tropical curves, really deserve to be called that?

That is, we saw a poset correspondence between $\overline{\mathcal{M}_g}$ and M_g^{tr}, but what about a tropicalization map $\overline{\mathcal{M}_g} \to M_g^{\text{tr}}$?

This point is not addressed in my work, but see work on Berkovich spaces by Baker, Payne, and Rabinoff.

What kind of space is M_g^{tr}?



It consists of rational open polyhedral cones modulo symmetries, glued along boundaries via integral linear maps. We will make this precise by defining a category of **stacky fans.**

What is a Stacky Fan?

Definition (C) Let

$$X_1 \subseteq \mathbb{R}^{m_1}, \ldots, X_k \subseteq \mathbb{R}^{m_k}$$

be full-dimensional rational open polyhedral cones and

$$G_1 \subseteq GL_{m_1}(\mathbb{Z}), \ldots, G_k \subseteq GL_{m_k}(\mathbb{Z})$$

be subgroups such that the action of each G_i on $\mathbb{R}^{m_i}$ fixes X_i. Let

$$X_i/G_i \quad \text{and} \quad \overline{X_i}/G_i$$

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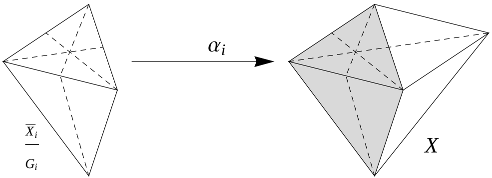
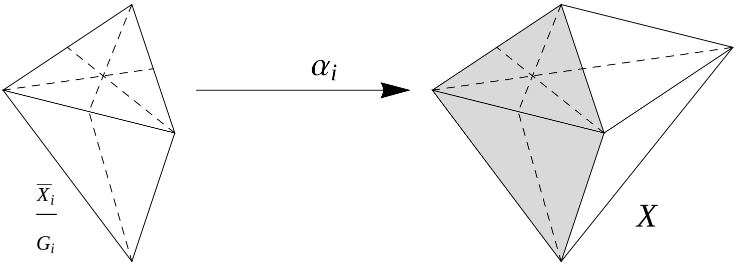
Suppose that we have a topological space X and, for each $i = 1, \ldots, k$, a continuous map $\alpha_i : \overline{X_i}/G_i \to X$.



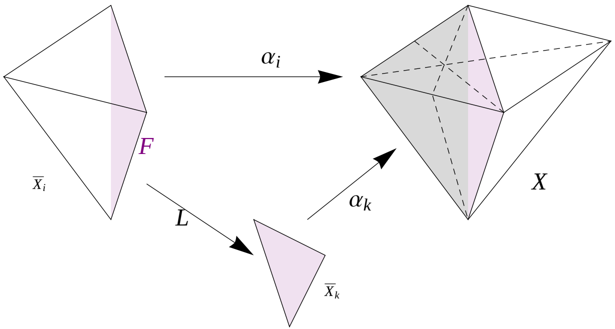
Then X is a **stacky fan**, with cells X_i/G_i, if the following four properties hold:

1. The restriction of α_i to $\frac{X_i}{G_i}$ is a homeomorphism onto its image,



2. We have an equality of sets $X = \coprod \alpha_i(X_i/G_i)$,

3. For each face F of any cone $\overline{X_i}$, there exists k such that $\alpha_i(F) = \alpha_k(\overline{X_k}/G_k)$, and an invertible, lattice point-preserving linear map L taking F to $\overline{X_k}$, such that the following diagram commutes:



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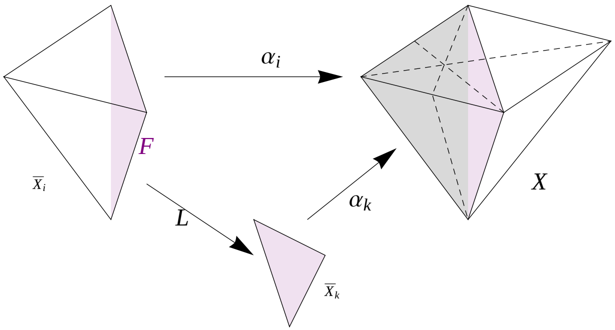

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4. For each pair i, j,

$$\alpha_i(\overline{X_i}/G_i) \cap \alpha_j(\overline{X_j}/G_j) = \alpha_{k_1}(X_{k_1}/G_{k_1}) \cup \cdots \cup \alpha_{k_t}(X_{k_t}/G_{k_t})$$

where the union ranges over the common stacky faces.

Theorem (B-M-V,C)

The moduli space M_g^{tr} is a stacky fan with cells corresponding to combinatorial types of genus g.

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The moduli space M_g^{tr} is a stacky fan with cells corresponding to combinatorial types of genus g.

We have constructed the moduli space M_g^{tr} and shown that it is a stacky fan. Next, we will construct the moduli space of principally polarized tropical abelian varieties, denoted A_g^{tr}, and then show that the tropical Torelli map is a stacky morphism.

Construction of the moduli space A_g^{tr}

A **principally polarized tropical abelian variety** is a point in

$$\frac{\widetilde{S}^g_{\geq 0}}{GL_g(\mathbb{Z})} := \frac{\text{psd matrices with rational nullspace}}{Q \sim X^T Q X \text{ for all } X \in GL_g(\mathbb{Z})}.$$

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What is a good moduli space of principally polarized tropical abelian varieties?

$\widetilde{S}^g_{\geq 0}/GL_g(\mathbb{Z})$ itself?

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$\widetilde{S}^g_{\geq 0}/GL_g(\mathbb{Z})$ itself?

Not good enough: it's not even Hausdorff, and does not admit stacky fan structure.

Instead, we will use the beautiful combinatorics of Voronoi reduction theory (Voronoi, 1908) to break $\widetilde{S}^g_{\geq 0}/GL_g(\mathbb{Z})$ into a finite number of polyhedral pieces, then glue them back together.

Voronoi reduction theory

Given $Q \in \widetilde{S}^g_{\geq 0}$, the **Delone subdivision** $\mathrm{Del}(Q)$ is the infinite-periodic regular subdivision of $\mathbb{R}^g$ obtained by lifting each lattice point $x \in \mathbb{Z}^g$ to the height $x^T Q x$, then taking lower faces of the convex hull of the lifted points.

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Now, given a Delone subdivision D, let

$$\sigma_D = \{Q \in \widetilde{S}^g_{\geq 0} : \mathrm{Del}(Q) = D\}.$$

Then σ_D is an open rational polyhedral cone, called the **secondary cone** of D.

Voronoi reduction theory

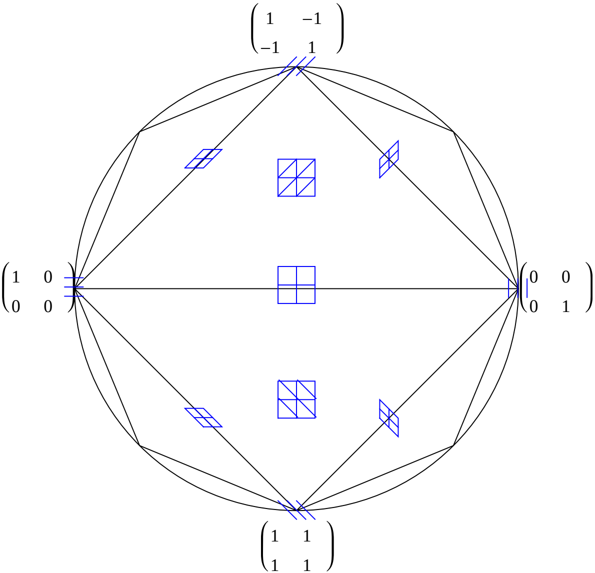

Figure: Infinite decomposition of $\widetilde{S}^2_{\geq 0}$ into secondary cones.

Theorem (Main theorem of Voronoi reduction theory)

The set of closed secondary cones

$$\{\overline{\sigma_D} : D \text{ is a Delone subdivision of } \mathbb{R}^g\}$$

yields an infinite polyhedral fan whose support is $\widetilde{S}^g_{\geq 0}$. *There are only finitely many* $GL_g(\mathbb{Z})$*-orbits of this set.*

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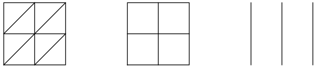
yields an infinite polyhedral fan whose support is $\widetilde{S}^g_{\geq 0}$. There are only finitely many $GL_g(\mathbb{Z})$-orbits of this set.

For example, when $g = 2$, there are four $GL_g(\mathbb{Z})$-classes of Delone subdivisions, with representatives shown below. They give rise to secondary cones of dimensions 3, 2, 1, and 0, respectively.



D_1 D_2 D_3 D_4

The moduli space A_g^{tr}

Pick Delone subdivisions $D_1, \ldots, D_k$ that are representatives for the $GL_g(\mathbb{Z})$-equivalence classes. Let $\mathrm{Stab}(\sigma_D)$ denote the subgroup of elements of $GL_g(\mathbb{Z})$ that fix σ_D as a set.

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Pick Delone subdivisions $D_1, \ldots, D_k$ that are representatives for the $GL_g(\mathbb{Z})$-equivalence classes. Let $\mathrm{Stab}(\sigma_D)$ denote the subgroup of elements of $GL_g(\mathbb{Z})$ that fix σ_D as a set.

Then define the **moduli space of principally polarized tropical abelian varieties**, denoted A_g^{tr}, to be the topological space

$$A_g^{\mathrm{tr}} \quad = \quad \left(\coprod_{i=1}^{k} \overline{\sigma_{D_i}} / \, \mathrm{Stab}(\sigma_{D_i}) \right) / \sim,$$

where $\sim$ denotes gluing by $GL_g(\mathbb{Z})$-equivalence.

The moduli space A_g^{tr}

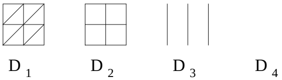
Theorem (B-M-V, C)

The moduli space A_g^{tr} is a stacky fan. Its cells correspond to $GL_g(\mathbb{Z})$-equivalence classes of Delone subdivisions.

Theorem (C)

A_g^{tr} is a Hausdorff topological space. It is independent of the choice of representative Delone subdivisions in its construction. That is, choosing different representatives produces an isomorphic stacky fan.

Example: A_2^{tr}



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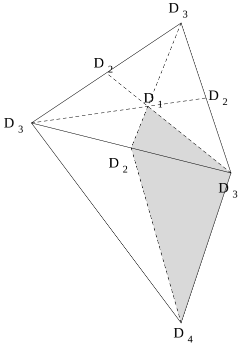

A_2^{tr} is homeomorphic to a closed, 3-dimensional simplicial cone.

The tropical Torelli map

Definition

We define the **tropical Torelli map**

$$t_g^{\text{tr}} : M_g^{\text{tr}} \longrightarrow A_g^{\text{tr}}$$

to send a tropical curve $C \in M_g^{\text{tr}}$ to its Jacobian $Jac(C) \in A_g^{\text{tr}}$.

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Theorem (B-M-V)

The map t_g^{tr} is a morphism of stacky fans. That is, it takes each cell of M_g^{tr} to a cell of A_g^{tr}, and this map is induced by an integral-linear map on the relevant cones.

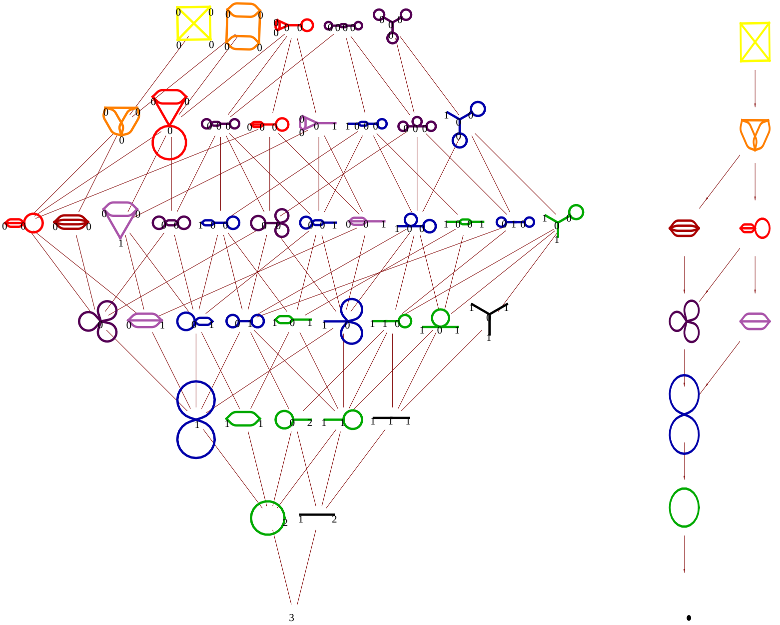

Figure: Cells of M_3^{tr} and of A_3^{tr}, color-coded according to t_g^{tr}.

The tropical Schottky locus

The tropical Torelli map t_g^{tr} is surjective when $g = 2, 3$, but not when $g \geq 4$.

Thus, it becomes interesting to study the **tropical Schottky locus**, i.e. the image of t_g^{tr} inside A_g^{tr}.

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Theorem (C)

We obtained the following computational results:

1. *The tropical Schottky locus A_3^{cogr} has nine cells and f-vector* $(1, 1, 1, 2, 2, 1, 1)$.
2. *The tropical Schottky locus A_4^{cogr} has 25 cells and f-vector* $(1, 1, 1, 2, 3, 4, 5, 4, 2, 2)$.
3. *The tropical Schottky locus A_5^{cogr} has 92 cells and f-vector* $(1, 1, 1, 2, 3, 5, 9, 12, 15, 17, 15, 7, 4)$.

The tropical Schottky locus: computations

g	M_g^{tr}	A_g^{cogr}	A_g^{tr}
2	2	1	1
3	5	1	1
4	17	2	3
5	71	4	222

g	M_g^{tr}	A_g^{cogr}	A_g^{tr}
2	7	4	4
3	42	9	9
4	379	25	61
5	4555	92	179433

Number of maximal cells and total number of cells in the stacky fans M_g^{tr}, the Schottky locus A_g^{cogr}, and A_g^{tr}.

Sources: Balaban 1980s, Engel 2002, Engel-Grishukhin 2002, Vallentin 2003, Maggiolo-Pagani 2010, C 2010

A closer look at the tropical Schottky locus

There is a close relationship between the tropical Schottky locus and cographic matroids.

Let M be a simple regular matroid of rank at most g, and let A be a $g \times n$ totally unimodular matrix that represents M. Let $v_1, \ldots, v_n$ be the columns of A. Then let $\sigma_A \subseteq \mathbb{R}^{\binom{g+1}{2}}$ be the rational open polyhedral cone

$$\mathbb{R}_{>0}\langle v_1 v_1^T, \ldots, v_n v_n^T \rangle.$$

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Example. Let M be the uniform matroid $U_{2,3}$. Then

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

represents M, and σ_A is the open cone generated by matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

A closer look at the tropical Schottky locus

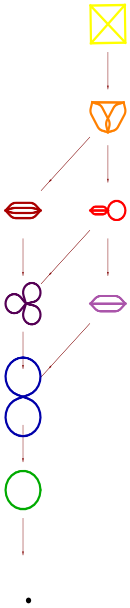
Proposition (B-M-V)

The cone σ_A is a secondary cone in $\widetilde{S}^g_{\geq 0}$. Choosing a different matrix A' to represent M produces a cone $\sigma_{A'}$ that is $GL_g(\mathbb{Z})$-equivalent to σ_A. Thus, we may associate to M a unique cell of A_g^{tr}, denoted $C(M)$.

Proposition (B-M-V)

The tropical Schottky locus is the union of cells

$$\{C(M) : M \text{ a simple cographic matroid of rank} \leq g\}.$$



A closer look at the tropical Schottky locus

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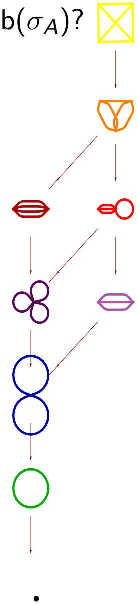

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Example. Each cell of A_3^{tr} is cographic, and A_3^{tr} is a 6-dimensional closed simplicial cone modulo the automorphisms of the matroid $M(K_4)$, plus some additional identifications along the boundary.

A tropical cover for A_3^{tr}

One problem with the spaces M_g^{tr} and A_g^{tr} is that although they are tropical moduli spaces, they do not "look" very tropical: they do not satisfy a tropical balancing condition. In other words: stacky fans, so far, are not tropical varieties.

But what if we allow ourselves to consider finite-index covers of our spaces – can we then produce a more tropical object?

We can do this for A_3^{tr}, using the Fano matroid F_7.

A tropical cover for A_3^{tr}

Theorem (C)

Let $\mathbb{FP}^6$ denote the complete polyhedral fan in $\mathbb{R}^6$ usually associated to the toric variety $\mathbb{P}^6$, e.g. with rays

$$e_1, \ldots, e_6, \quad e_7 := -e_1 - \cdots - e_6.$$

Then there is a surjective morphism of stacky fans

$$\mathbb{FP}^6 \to A_3^{tr}$$

mapping each of the seven maximal cells of $\mathbb{FP}^6$ surjectively onto the maximal cell of A_3^{tr}.

A tropical cover for A_3^{tr}

Proof Sketch.

We would like to send each maximal cone of $\mathbb{FP}^6$ to the unique maximal cell of A_3^{tr}, with maps that agree on the lower-dimensional cones of $\mathbb{FP}^6$. The only possible obstacle is that not all 3-dimensional and 4-dimensional cells of A_3^{tr} look alike.

However, the Fano matroid precisely gives a way to coherently identify each 6-element set of $\{1, \ldots, 7\}$ with the matroid $M(K_4)$. □