Notes on Antoine's Necklace

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Antoine's necklace refers to a family of embeddings of a Cantor set into $\boldsymbol{R}^3$. I will construct a self-similar example in this family, which I'll call A. The construction depends on an even integer K and works when K is sufficiently large. Probably all you need is $K \geq 24$. After giving the construction I'll prove two main things. First, A is homeomorphic to the middle third Cantor set. Second, $\boldsymbol{R}^3 - A$ is not simply connected. This is a rather amazing thing: You can stick a Cantor set into $\boldsymbol{R}^3$ in such a way that some loops get inextricably tangled up in it.

1 Construction

Let Π be the xy plane in $\boldsymbol{R}^3$. Let $\widehat{C}_0 \subset \Pi$ be a circle of radius $4K$. Let A_0 denote the torus consisting of points having distance at most K from $\widehat{C}_0$. The shape of A_0 (i.e. similarity equivalence class) is independent of K. Let $P \subset \widehat{C}_0$ be an inscribed regular K-gon. Let $P_0, ..., P_{K-1}$ be the vertices of P. The distance $8K\sin(\pi/K)$ between successive vertices of P converges to 8π as $K \to \infty$.

For $k \in \{0, ..., K-1\}$ even let C_k be the circle of radius 6π centered at P_k and contained in Π. For k odd let C_k be the circle of radius 6π centered at P_k and contained in the plane perpendicular to the line through the origin containing P_k. Adjacent circles are linked and non-adjacent circles are unlinked. Moreover, the minimum distance between points on distinct circles converges to 4π as $K \to \infty$ and in particular exceeds 3π when K is large. Let τ_k denote the torus consisting of all points within $3\pi/2$ from C_k. These tori have the same shape as A_0. Let A_1 be the union of these tori. Figure 0 shows an fairly accurate projection (into Π) of 5 consecutive tori when K is very large.

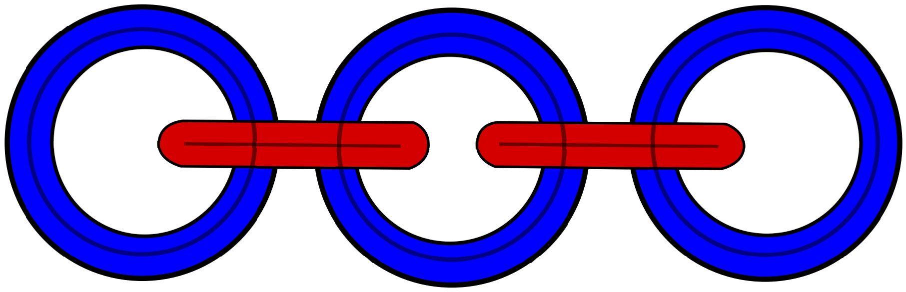

Figure 0: Projections of 5 consecutive tori.

By construction the tori are disjoint, adjacent tori are linked, and non-adjacent tori are unlinked. All points of A_1 are within $(15/2)\pi$ of $\widehat{C}_0$. Hence $A_1 \subset A_0$ once $K > (15/2)\pi$. At this point we fix K large enough to have all the above properties. Again, I think that any $K \geq 24$ will work.

There are similarities $S_0, ..., S_{K-1}$ such that $S_k(A_0) = \tau_k$ for each index $k = 0, ..., K-1$. Let A_m consist of all tori of the form $S_W(A_0)$ where W is a length m composition of these simiarities. Thus A_m consists of K^m disjoint tori, and these tori are partitioned into "necklaces" of linked tori having length K, each contained in a torus of A_{m-1}. In other words, we get A_m by replacing each torus of A_{m-1} by a K-necklace that sits inside this torus in the same way that A_1 sits inside A_0. Antoine's necklace is the nested intersection:

$$A = \bigcap_{m=0}^{\infty} A_m.$$

Remark: What is the best value of K that will work for a self-similar construction like this? With a modified construction one can get $K = 20$. I don't think it is possible to get $K < 20$.

2 Cantor Set Property

The middle third Cantor set $\mathcal{C}$ is the subset of $[0,1]$ consisting of all points whose base 3 expansion has no 1s in it. Here I will show in an elementary way that A is homeomorphic to $\mathcal{C}$.

Let T_K denote the subset of all infinite words in the symbols $\{0,...,K-1\}$. The distance between two words in T_K is $2^{-\ell}$ where ℓ is the number of initial spots where the two words agree. For example, the distance between 123123... and 121212... is 2^{-2} because these words agree in the first two positions and then disagree. A basis for the topology on T_K is given by subsets of words all having the same m-prefix for some m. In other words, you fix the first m digits and then let the rest vary. These are the open metric balls of T_K.

Lemma 2.1 *T_K is homeomorphic to A.*

Proof: Each point in A is the nested intersection $\cap\tau_m$, where τ_m is one of the tori in the union A_m. Thus each point of A defines a K-ary sequence, which is to say a point of T_K. We let $\phi : A \to T_K$ be this map. The map ϕ is surjective because, from the construction of A, we can realize any K-ary sequence. The map ϕ is injective because the diameters the tori in A_m tends to 0 as $m \to \infty$. Thus, distinct points of A define distinct K-ary sequences. The subsets of A having the form $A \cap \tau$, for τ a torus in A_m, form a basis for the topology of A. The reason: these sets are open and the intersection of any two of them, if nonempty, is another one. Hence ϕ maps the basis for the topology of A to the basis for the topology of T_K. Hence ϕ is a homeomorphism. ♠

Lemma 2.2 *T_2 is homeomorphic to $\mathcal{C}$.*

Proof: We define special subsets of $\mathcal{C}$ just as we did for T_K. These are subsets having the same m-prefix. These subsets are open (and closed). Moreover, the intersection of any two of them, if non-empty, is a third. Hence these special subsets form a basis for the topology on $\mathcal{C}$. The map $\phi : T_2 \to \mathcal{C}$ is given by $\phi(a_0, a_1, ...) = .b_0, b_1, ...$ where $b_j = 2a_j$. That is, we just change the 1 digits to 2s. By construction, ϕ maps basis elements of T_2 to basis elements of $\mathcal{C}$ bijectively. Hence ϕ is a homeo. ♠

Remark: Before reading the next proof, consider the fact that we could take K to be a power of 2 in our construction of A. The choice $K = 32$ works. In this case, it is easy to show that T_2 and T_{2^k} are homeomorphic. I am including the next proof mainly to show how to prove that T_2 and T_K are homeomorphic in general. The proof suggests how one might prove in general that any compact, perfect, totally disconnected metric space is homeomorphic to T_2.

Lemma 2.3 *T_2 is homeomorphic to T_K for all $K \geq 2$.*

Proof: Each ball of T_2 is a union of 2 balls $B(0)$ and $B(1)$ having half the diameter. There is a canonical homeomorphism from T_2 to each of these: For instance, $\phi_0 : T_2 \to B(0)$ is given by $\phi_0(a_0, a_1, ...) = (0, a_0, a_1, ...)$. This is the "padding map". More generally, T_2 is homeomorphic to any of its metric balls *via* a (iterated) padding map.

By recursively subdividing, we can partition T_2 into K metric balls. We do this, and we let S_k denote the padding map which carries T_2 to the kth ball in the partition. In general, T_2 is partitioned into K^n metric balls. These metric balls have the form $S_{a_{n-1}} \circ ...S_{a_0}(T_2)$, where $a_0, ..., a_{n-1}$ is any binary sequence of length n. Call this partition B_n. The diameter of each ball in B_n is at most 2^{-n}.

Since we have a sequence of partitions (refining each other), each point in T_2 can be uniquely described as the limit point of a nested intersection of the form $\cap\beta_n$ where β_n is a metric ball of B_n. Thus each point in T_2 defines a K-ary sequence, namely a point of T_K. We let $\phi : T_2 \to T_K$ be the map which has this description.

The map ϕ is surjective because we can realize any K-ary sequence. The map ϕ is injective because the diameters the metric balls in B_n tends to 0. The inverse image of any basis element of T_K is a finite union of metric balls in T_2. Hence this inverse image is open. Note finally that ϕ maps each metric ball in T_2 is a union of metric balls of T_n once n is sufficiently large. Thus ϕ maps metric balls to finite unions of metric balls. This shows that ϕ^{-1} is continuous. All in all ϕ is a homeo. ♠

Stringing these lemmas together we see that A is homeomorphic to $\mathcal{C}$.

3 Structure of the Complement

Let A be Antoine's necklace. The rest of these notes are devoted to proving that $\boldsymbol{R}^3 - A$ is not simply connected. The argument will show more strongly that a particular element $[\beta_0]$ of $\pi_1(\boldsymbol{R}^3 - A)$ has infinite order. With some modification the proof below would work for more general versions of Antoine's necklace.

In this section we reduce the main result to something we call the Linking Lemma. Let A_n denote the union of the K^n linked tori as above. Let L_n denote the link of circles obtained by replacing each torus in A_n by its core circle. Let τ be some torus used in our construction. The fundamental group $\pi_1(\partial\tau) = \boldsymbol{Z}^2$ has a canonical basis:

- $\alpha(\tau)$ is represented by a curve on $\partial\tau$ parallel to the core of τ.
- $\beta(\tau)$ is represented by a curve on $\partial\tau$ perpendicular to the core of τ.

The curve $\beta(\tau)$ links the core of τ.

Let $\beta_0 = \beta(A_0)$, the boundary of the big outer torus. We take the base-point p of $\boldsymbol{R}^3 - A$ on β_0 and think of $[\beta_0]$ as an element of $\pi_1(\boldsymbol{R}^3 - A, p)$. We suppress p from our notation. Here is the main technical step:

Lemma 3.1 (Linking) *$[\beta_0]$ has infinite order $\pi_1(\boldsymbol{R}^3 - L_n)$ for each n.*

Let us deduce the main result from the Linking Lemma. Let m be an arbitrary nonzero integer. Define the unit square $Q = [0,1]^2$. Suppose that $F : Q \to \boldsymbol{R}^3 - A$ is a homotopy from β_0^m to the trivial loop. We just have to produce some $q \in Q$ such that $F(q) \in A$. By the Linking Lemma, $[\beta_0^m]$ is nonzero in $\pi_1(\boldsymbol{R}^3 - L_n)$. Hence there is some point $q_n \in Q$ such that $F(q_n) \in L_n$. In particular, $F(q_n) \in A_j$ for all $j = 0, ..., n$. Since Q is compact the sequence $\{q_n\}$ has an accumulation point q. By construction $F(q)$ is an accumulation point of A_j for all j. Hence $F(q) \in A$. Hence $[\beta_0^m]$ is nontrivial in $\pi_1(\boldsymbol{R}^3 - A)$. The rest of the notes are devoted to proving the Linking Lemma.

Remark: If you are just interested in showing that the element $[\beta_0]$ is nonzero in $\pi_1(\boldsymbol{R}^3 - A)$, you could get use a weaker form of the Linking Lemma which just says that $[\beta_0]$ is nonzero in $\pi_1(\boldsymbol{R}^3 - A_n)$. However, the weaker version of the Linking Lemma does not have the nice inductive proof that the stronger version does. I will discuss this more at the end of the proof.

4 The Base of the Induction

Here we prove that $[\beta_0]$ is an infinite order element in $\pi_1(\boldsymbol{R}^3 - L_1)$. Figure 1 shows a projection of L_1, in the case $K = 8$. The general case is very similar. The labels a_0, b_0 and a_1, b_1 are meant to suggest that the pattern continues around the loop, with additional elements a_2, b_2 and a_3, b_3 and so on.

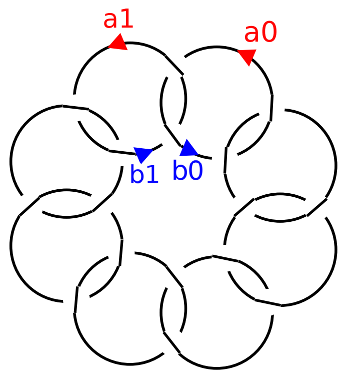

Figure 1: L_1.

Let $\Gamma = \pi_1(\boldsymbol{R}^3 - L_1)$. Using the Wirtinger representation, we see that Γ is generated by elements $a_1, ..., a_K$ and $b_1, ..., b_K$ subject to the relations

$$a_0 = a_1 b_0 a_1^{-1}, \qquad b_1 = b_0 a_1 b_0^{-1}, \tag{1}$$

and all cyclic permutations of these: $a_1 = a_2 b_1 a_2^{-1}$ and $b_2 = b_1 a_2 b_1^{-1}$, etc. Note that

$$a_1 b_1^{-1} = a_1 (b_0 a_1 b_0^{-1}) = a_1 b_0 a_1^{-1} b_0^{-1} = a_0 b_0^{-1},$$

and similarly for cyclic permutations. Thus we can see directly from the presentation of Γ the element $a_k b_k^{-1}$ is independent of k. We have

$$[\beta_0] = a_k b_k^{-1}, \quad \forall k. \tag{2}$$

Geometrically, this is the element that starts from your nose, runs through the middle of the necklace, links it, and returns to your nose.

To prove that $a_0 b_0^{-1}$ has infinite order in Γ it suffices to produce a group H and a homomorphism $\phi : \Gamma \to H$ such that $\phi(a_0 b_0^{-1})$ has infinite order. The group H will be the *Heisenberg group*. As a set H is $\boldsymbol{C} \times \boldsymbol{R}$ but the group law is given by

$$(z_1, t_1) * (z_2, t_2) = (z_1 + z_2, t_1 + t_2 + \mathrm{Im}(z_1 \overline{z}_2)).$$

The identity is $(0,0)$. Inverses are given by $(z,t)^{-1}=(-z,-t)$. We compute

$$(z_1,t_1)*(z_2,t_2)*(z_1,t_1)^{-1}=(z_2,t_2+2\mathrm{Im}(z_1\overline{z}_2)). \qquad (3)$$

Define

$$\phi(a_k)=(\omega^k,\mu), \quad \phi(b_k)=(\omega^k,-\mu), \quad \omega=e^{i\mu}, \quad \mu=2\pi/K. \qquad (4)$$

To check that ϕ is really a homomorphism we just need to check that ϕ respects the relations in Equation 1. It follows from symmetry (or similar calculations) that ϕ respects the cyclic permutations of these relations as well. We compute

$$\phi(a_1b_0a_1^{-1})=(1,-\mu+2\mathrm{Im}(\omega))=(1,-\mu+2\mu)=(1,\mu)=\phi(a_0),$$

$$\phi(b_0a_1b_0^{-1})=(\omega,\mu-2\mu)=(\omega,-\mu)=\phi(b_1).$$

It works! Finally, we compute that

$$\phi(a_0b_0^{-1})=(0,2\mu).$$

The element $(0,2\mu)$ has infinite order in H because $(0,2\mu)^m=(0,2m\mu)$. This completes the proof of the base case.

Remark: Where the hell did that come from? Well, $a_0b_0^{-1}$ is a commutator, so you need to take H to be non-abelian and of infinite order. The Heisenberg group is one of the simplest infinite order non-abelian groups, so it ought to be on any reasonable menu of choices.

The link L_1 has K-fold rotational symmetry, and this suggests that you might want to map the generators of Γ to elements which have some kind of K-fold rotational symmetry. A natural choice would be to map these elements to Kth roots of unity in $\boldsymbol{C}$. This has nice symmetry properties but unfortunately $\boldsymbol{C}$ is abelian. This won't work, but H is a non-abelian "extension" of $\boldsymbol{C}$. There is an exact sequence $0\to\boldsymbol{R}\to H\to\boldsymbol{C}\to 0$. Put in a more elementary way, there is a surjective homomorphism from H to $\boldsymbol{C}$ whose kernel is $\boldsymbol{R}$.

The nice feature of H is that all commutators in H lie in the $\boldsymbol{R}$ direction. We need the images of $a_kb_k^{-1}$ to be the same, independent of k, which means that we want a whole bunch of commutators to be the same. The group H is perfect for all that. This is really the consideration that led me to H. (I didn't look up a proof; presumably this is the "standard method".)

5 The Induction Step

Now we turn to the inductive step of the Linking Lemma. We take $n \geq 2$ and assume by induction that $[\beta_0]$ has infinite order in $\pi_1(\boldsymbol{R}^3 - L_{n-1})$. Our goal is to show that $[\beta_0]^m$ is nonzero in $\pi_1(\boldsymbol{R}^3 - L_n)$ for all m.

As a preliminary step, we clean up our homotopy. We think of β_0^m as the image of the unit circle S^1 under the continuous mapping $F : S^1 \to \boldsymbol{R}^3$. Our goal is equivalent to showing that no continuous extension of F to the unit disk D^2 maps D^2 disjointly from L_n. We will suppose that there is some choice of F such that $F(D^2) \cap L_n = \emptyset$ and we will derive a contradiction.

The first thing to notice is that since $F(D^2)$ and L_n are both compact, there is some positive $\epsilon > 0$ such that the distance between any point of $F(D^2)$ and any point of L_n is at least ϵ. This means that any continuous map sufficiently close to F also misses L_n. We can replace F by a new map $G : D \to \boldsymbol{R}^3 - L_n$ such that

1. $D \subset D^2$ is a polygon and $G(\partial D)$ is homotopic to β_0^m in $\boldsymbol{R}^3 - A_1$.

2. G is a piecewise linear map, with respect to some triangulation of D.

3. If v is any vertex of the triangulation, $G(v)$ is disjoint from all boundaries of all tori in $A_0, ..., A_n$.

4. If e is any edge or face of the triangulation, $G(e)$ is nowhere tangent to any boundary of any torus of $A_0, ..., A_n$.

To get Condition 1, we restrict F to an n-gon D inscribed in D^2 and then modify F so that it is piecewise linear on ∂D. If we take n large enough then $F(D^2 - D)$ is disjoint from A_1. Thus we can interpret the restriction of F to $D^2 - D$ as a homotopy between β_0 and $F(\partial D)$ in $\boldsymbol{R}^3 - A_1$. We let $G = F$ on ∂D. To get Condition 2 we take a fine triangulation of D. We then let $G = F$ on the vertices of the triangulation and we make G affine (linear composed with translation) on each triangle of the triangulation. Note that G is completely determined by where it sends the vertices of the triangulation. To get Conditions 3 and 4, we perturb the images of the triangulation vertices.

These conditions imply that for each triangle τ in the triangulation, $G(\tau)$ intersects each boundary torus in a finite disjoint union of smooth loops and smooth arcs. The arcs in question have their endpoints in $\partial G(\tau)$. The arcs in adjacent triangles piece together across common endpoints.

Now we get to the main point. By induction, $G(D)$ intersects L_{n-1}. This means that $G(D)$ non-trivially intersects ∂A_{n-1}. Consider the set

$$\Sigma = G^{-1}(G(D) \cap A_{n-1}). \tag{5}$$

From the description of the triangle intersections above, Σ is a finite union of loops. There are no arcs, because such arcs would have their endpoints on ∂D, and ∂D is disjoint from Σ.

Let σ be some loop of Σ. The image $G(\sigma)$ is contained in $\partial\tau$ for some torus τ of A_{n-1}. Interpreting $G(\sigma)$ as an element of $\pi_1(\partial\tau)$, we have

$$[G(\sigma)] = a_\sigma \alpha(\tau) + b_\sigma \beta(\tau). \tag{6}$$

Thus σ determines the two integers a_σ and b_σ. There are three cases.

Case 1: Suppose that $a_\sigma = b_\sigma = 0$ for all loops σ of Σ. Let D_σ be the disk bounded by σ. In this case, $G(\sigma)$ is trivial in $\pi_1(\partial\tau)$. Hence there is a continuous map $H_\sigma : D_\sigma \to \partial\tau$ which extends $G|_\sigma$. In other words, we can shrink $G(\sigma)$ to a point inside $\partial\tau$. We pick some ordering $\sigma_1, ..., \sigma_n$ on the components of Σ and then, when applicable, we modify the map G so that it implements H_{σ_k} on Δ_{σ_k}. (The reason why we say "when applicable" is that the modification made with respect to σ_1 might eliminate some of the other σ_j, and so on.) When we are done, the new map G' has the property that $G'(D)$ is disjoint from the interior of A_{n-1}. The reason: $G'(D)$ contains points in the complement of A_{n-1} and also this image never crosses ∂A_{n-1}. But now we see that $G'(D)$ is disjoint from L_{n-1}. This is a contradiction.

Case 2: Suppose that there is some σ such that $b_\sigma \neq 0$. Let τ be the torus of A_{n-1} whose boundary contains σ, as above. With respect to a suitably chosen basepoint, we interpret $G(\sigma)$ as an element of

$$\pi_1(\Omega), \qquad \Omega = \boldsymbol{R}^3 - (L_n \cap \tau).$$

The element $\alpha(\tau)$ is trivial in $\pi_1(\boldsymbol{R}^3 - \tau)$ and so it is *a fortiori* trivial in $\pi_1(\Omega)$. Thus, $G(\sigma)$ represents a multiple of $\beta(\tau)$ in $\pi_1(\Omega)$. By the base case of the Linking Lemma, $\beta(\tau)$ has infinite order in $\pi_1(\Omega)$. Hence $G(\sigma)$ is nonzero in $\pi_1(\Omega)$. Since $\boldsymbol{R}^3 - L_n \subset \Omega$ we see that $G(\sigma)$ is nonzero in $\pi_1(\boldsymbol{R}^3 - L_n)$ as well. But we can interpret $G|_{\Delta_\sigma}$ as a homotopy from $G(\sigma)$ to the constant loop in $\boldsymbol{R}^3 - L_n$. This is a contradiction.

Case 3: Suppose that there is some σ such that $a_\sigma \neq 0$. Let τ be the torus of A_{n-1} whose boundary contains σ, as above. Let τ' be one of the two tori of A_{n-1} which links τ. Now we interpret $G(\sigma)$ as an element of

$$\pi_1(\Omega'), \qquad \Omega' = \boldsymbol{R}^3 - (L_n \cap \tau').$$

In the set $\boldsymbol{R}^3 - \tau - \tau'$ we can move $\alpha(\tau)$ to $\beta(\tau')$ by a homotopy. Likewise, in $\boldsymbol{R}^3 - \tau - \tau'$ we can move $\beta(\tau)$ to $\alpha(\tau')$ by a homotopy. This $G(\sigma)$ represents the same element as $b_\sigma \alpha(\tau') + a_\sigma \beta(\tau')$ in $\pi_1(\Omega')$. Now the same argument as in Case 2, using Ω' in place of Ω, finishes the proof.

In all cases, we get a contradiction. The only way out of the contradiction is that the original map F is such that $F(D^2) \cap A_n \neq \emptyset$. This completes the proof of the induction step of the Linking Lemma. Hence, the Linking Lemma is true.

Remark: Go back to Step 2 and look carefully at the underlined word multiple. At this point in the proof we lose control over which multiple we are taking about. So, if we only had the weaker version of the Linking Lemma, we could not use the fact that $\beta(\tau)$ is nontrivial in $\pi_1(\Omega)$ to conclude that so is $G(\sigma)$. This is why we need to run an induction argument on the form of the Linking Lemma we have given.