The Gauss-Salamin Algorithm

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1 Introduction

The Gauss-Salamin algorithm is a simple algorithm for computing the digits of π very rapidly. Every step of the algorithm approximately doubles the number of digits of accuracy! These notes give a proof. Very little in my notes is original. I took this proof directly from the paper:
Nick Lord, *Recent Calculations of π: The Gauss-Salamin Algorithm*, The Mathematical Gazette, Vol 76 No. 476 (1992)
All I did was put Lord's proof in a more direct order, omitting all the cool (but extraneous) stuff related to the simple pendulum, lemniscates, etc. Also, I fill in details about the Γ and β functions, and briefly discuss convergence.

2 The Formula and the Algorithm

Given $a_0 \geq b_0 \geq 0$ define recursively:

$$a_n = \frac{a_{n-1} + b_{n-1}}{2}, \qquad b_n = \sqrt{a_{n-1} b_{n-1}}, \tag{1}$$

The *arithmetic-geometric mean* is:

$$\mathrm{AGM}(a_0, b_0) = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n. \tag{2}$$

Define the series:

$$S(a_0, b_0) = \frac{1}{2} \sum_{k=0}^{\infty} 2^k (a_k^2 - b_k^2). \tag{3}$$

The equality behind the algorithm is:

$$\pi = \frac{\big(\mathrm{AGM}(\sqrt{2},1)\big)^2}{1 - S(\sqrt{2},1)}. \tag{4}$$

Convergence: Define

$$\pi_n = \frac{a_n^2}{1 - S_n(\sqrt{2},1)}, \qquad S_n(a_0,b_0) = \frac{1}{2}\sum_{k=0}^{n} 2^k (a_k^2 - b_k^2). \tag{5}$$

We have, for instance,

$$|\pi - \pi_n| < 10^{-(4/3)2^n}$$

provided that $n \geq 6$. The convergence is a bit faster, but this is a nice simple expression. The rapid convergence comes from the fact that

$$\kappa_n = \frac{\kappa_{n-1}^2}{4(a_{n-1}+b_{n-1})^2} < \frac{\kappa_{n-1}^2}{16\mathrm{AGM}(\sqrt{2},1)^2} = \frac{\kappa_{n-1}^2}{22.96864...}. \tag{6}$$

Here $\kappa_n = a_n^2 - b_n^2$.

Some Mathematica Code:

```
a[0]:=Sqrt[2]; b[0]:=1; d[0]:=1/2; s[0]:=1/2;
a[n_]:=(a[n-1]+b[n-1])/2; b[n_]:=Sqrt[a[n-1] b[n-1]];
d[n_]:=2 d[n-1]; s[n_]:=s[n-1]+d[n] (a[n] a[n]-b[n] b[n]);
pi[n_]:=a[n] a[n]/(1-s[n]);
```

As a test, the first 10 iterations compute π to 1395 digits. The command `SetPrecision[pi[10]-Pi,1410]` returns $1.90043721 \times 10^{-1396}$.

Remarks:
(1) As Nick Lord explains in his paper, the algorithm above is a variant of the Gauss-Salamin algorithm. A simple change of variables gives it exactly.
(2) To make this an honest algorithm, from scratch, you would want a good method for extracting square roots. The iteration

$$x_0 = 1, \qquad x_n = \frac{1}{2}\left(x_{n-1} + \frac{ab}{x_{n-1}}\right)$$

converges to $\sqrt{ab}$ and has the same precision-doubling feature as the algorithm above. This is basically just Newton's method.

3 Deriving the Formula

In this section we derive Equation 4 modulo 4 statements. Define the elliptic integrals:

$$I(a_0, b_0) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a_0^2 \cos^2(\theta) + b_0^2 \sin^2(\theta)}}. \tag{7}$$

$$L(a_0, b_0) = \int_0^{\pi/2} \frac{\cos^2(\theta) d\theta}{\sqrt{a_0^2 \cos^2(\theta) + b_0^2 \sin^2(\theta)}}. \tag{8}$$

Lemma 3.1 *The following is true.*

1. $L(a_0, b_0) + L(b_0, a_0) = I(a_0, b_0)$.
2. $I(a_0, b_0) = \frac{\pi}{2\mathrm{AGM}(a_0, b_0)}$.
3. $(a_0^2 - b_0^2) L(b_0, a_0) = S(a_0, b_0) I(a_0, b_0)$.
4. $\frac{\pi}{4} = L(\sqrt{2}, 1) I(\sqrt{2}, 1)$.

Set $I = I(\sqrt{2}, 1)$ and $L = L(\sqrt{2}, 1)$, etc. When $a_0 = \sqrt{2}$ and $b_0 = 1$ we have $a_0^2 - b_0^2 = 1$. So, Statements 1 and 3 give $L = (1 - S)I$. Statements 2 and 4 now give

$$\frac{\pi}{4} = (1 - S) I^2 = (1 - S) \frac{\pi^2}{4\mathrm{AGM}^2}.$$

Dividing both sides by $\pi/4$ and rearranging gives Equation 4. Now we prove the 4 statements above.

4 Proof of Statement 1

Set $a = a_0$ and $b = b_0$. Making the substitution $\vartheta = \pi/2 - \theta$, we see that

$$L(b, a) = \int_0^{\pi/2} \frac{\sin^2(\vartheta) d\vartheta}{\sqrt{a^2 \cos^2(\vartheta) + b^2 \sin^2(\vartheta)}}.$$

Therefore

$$L(a, b) + L(b, a) = \int_0^{\pi/2} \frac{\cos^2(\vartheta) + \sin^2(\vartheta)}{\sqrt{a \cos^2(\vartheta) + b^2 \sin^2(\vartheta)}} d\vartheta = I(a, b).$$

5 Proof of Statement 2

It is convenient to write

$$I(a,b) = \int_0^{\pi/2} \frac{d\theta/\cos^2(\theta)}{\sqrt{a^2 + (b\tan(\theta))^2} \times (1/\cos(\theta))}.$$

The substitutions

$$u = b\tan(\theta), \qquad du = b\ d\theta/\cos^2(\theta), \qquad 1/\cos(\theta) = \frac{\sqrt{u^2+b^2}}{b}$$

lead to

$$I(a,b) = \int_0^{\infty} \frac{du}{\sqrt{(a^2+u^2)(b^2+u^2)}} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{du}{\sqrt{(a^2+u^2)(b^2+u^2)}}.$$

The second integral is the same as the first, by symmetry.

If we start with the integral

$$I(a_1,b_1) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{du}{\sqrt{(a_1^2+u^2)(b_1^2+u^2)}},$$

and make the substitutions

$$a_1 = \frac{a_0+b_0}{2}, \quad b_1 = \sqrt{a_0 b_0}, \quad u = \frac{1}{2}\left(v - \frac{a_0 b_0}{v}\right), \quad du = \frac{1}{2}\left(1 + \frac{a_0 b_0}{v^2}\right), \quad (9)$$

then simplify the mess, we get

$$I(a_1,b_1) = \int_0^{\infty} \frac{dv}{\sqrt{(a_0^2+v^2)(b_0^2+v^2)}} = I(a_0,b_0).$$

Iterating, we see that

$$I(a_0,b_0) = I(a_1,b_1) = I(a_2,b_2) = ... = I(A,A) = \int_0^{\pi/2} d\theta/A = \frac{\pi}{2A}.$$

Here $A = \mathrm{AGM}(a_0,b_0)$.

6 Proof of Statement 3

Write $c_k = \sqrt{a_k^2 - b_k^2}$. The key step is showing that

$$2c_0^2 L(b_0, a_0) - 4c_1^2 L(b_1, a_1) = c_0^2 I(a_0, b_0). \tag{10}$$

Setting $I = I(a_0, b_0) = I(a_1, b_1)...$, we iterate Equation 10, multiplying the relation by 2 each time:

$$\begin{aligned}
2c_0^2 L(b_0, a_0) - 4c_1^2 L(b_1, a_1) &= 2c_0^2 I(a_0, b_0) = c_0^2 I. \\
4c_1^2 L(b_1, a_1) - 8c_2^2 L(b_2.a_2) &= 4c_1^2 I(a_1, b_1) = 2c_0^2 I. \\
8c_2^2 L(b_2, a_2) - 16c_3^2 L(b_3.a_3) &= 8c_2^2 I(a_2, b_2) = 4c_0^2 I. \\
&\cdots
\end{aligned} \tag{11}$$

The bound in Equation 6 implies that the infinite series made from the terms on the right side of Equation 11 converges. Summing these terms, we get $2c_0^2 L(a_0, b_0) = 2S(a_0, b_0)I$. Dividing by 2 we get Statement 3.

Now for Equation 10. The substitution $u = b\tan\theta$ used above gives

$$L(a, b) = \int_0^\infty \frac{b^2 du}{(u^2 + b^2)\sqrt{(u^2 + a^2)(u^2 + b^2)}}.$$

Just as in the proof of Statement 1, we write out

$$L(b_1, a_1) = \int_0^\infty \frac{a_1^2 du}{(u^2 + b^2)\sqrt{(u^2 + a_1^2)(u^2 + b_1^2)}},$$

and make the substitutions from Equation 9. This gives

$$L(b_1, a_1) = \frac{a_0 + b_0}{a_0 - b_0}(L(b_0, a_0) - L(a_0, b_0)). \tag{12}$$

Combining this the fact that $L(a, b) + L(b, a) = I(a, b)$, we have

$$L(b_1, a_1) = \frac{a_0 + b_0}{a_0 - b_0}(2L(b_0, a_0) - I(a_0, b_0))$$

Multiplying through by $(a_0 - b_0)^2$ and rearranging, we have

$$2(a_0^2 - b_0^2)L(b_0, a_0) - (a_0 - b_0)^2 L(b_1, a_1) = (a_0^2 - b_0^2)I.$$

Now we observe that $c_0^2 = a_0^2 - b_0^2$ and $4c_1^2 = (a_0 - b_0)^2$. Once we make these substitutions, we get Equation 10.

7 Proof of Statement 4

We have the Γ-function and the β-function:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\ dt, \qquad \beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\ dt. \tag{13}$$

To establish Statement 4 we establish the following facts.

1. $I(\sqrt{2},1) = \frac{1}{4}\beta(1/4,1/2)$ and $L(\sqrt{2},1) = \frac{1}{4}\beta(3/4,1/2)$.
2. $\beta(x,y) = \frac{\Gamma(x)(\Gamma(y)}{\Gamma(x+y)}$.
3. $\Gamma(x+1) = x\Gamma(x)$ when $x > 0$. In particular, $\Gamma(5/4) = \Gamma(1/4)/4$.
4. $\Gamma(1/2) = \sqrt{\pi}$.

These facts give

$$I(\sqrt{2},1)L(\sqrt{2},1) =^1 \frac{\beta(1/4,1/2)\beta(3/4,1/2)}{16} =^2$$

$$\frac{\Gamma(1/4)\Gamma(3/4)\Gamma(1/2)\Gamma(1/2)}{16\Gamma(5/4)\Gamma(3/4)} =^3 \frac{\Gamma(1/4)\Gamma(3/4)\Gamma(1/2)\Gamma(1/2)}{4\Gamma(1/4)\Gamma(3/4)} =^4 \frac{\pi}{4}.$$

7.1 Fact 1

We have

$$I(\sqrt{2},1) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{2\cos^2(\theta)+\sin^2(\theta)}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1+\cos^2(\theta)}} = \int_0^1 \frac{du}{\sqrt{1-u^4}}.$$

The last equality comes from the subst. $u = \cos(\theta)$ and $du = -\sin(\theta)d\theta$. Substituting $v = u^4$ and $dv = 4u^3 du$ we see that

$$I(\sqrt{2},1) = \frac{1}{4}\int_0^1 \frac{dv}{v^{3/4}(1-v)^{1/2}} = \frac{1}{4}\beta(1/2,3/4).$$

The same substitution for L gives us

$$L(\sqrt{2},1) = \int_0^1 \frac{u^2 du}{\sqrt{1-u^4}} = \frac{1}{4}\int_0^1 \frac{dv}{v^{1/4}(1-v)^{1/2}} = \frac{1}{4}\beta(1/2,1/4).$$

7.2 Fact 2

Making the substitution

$$t = \cos^2(\theta), \qquad 1 - t = \sin^2(\theta), \qquad dt = -2\cos(\theta)\sin(\theta)d\theta,$$

we see that

$$\beta(x, y) = 2\int_0^{\pi/2} \cos(\theta)^{2x-1}\sin(\theta)^{2y-1}d\theta.$$

Making the substition $t = u^2$ and $dt = udu$, we have

$$\Gamma(x) = 2\int_0^{\infty} u^{2x-1}e^{-u^2}du.$$

But then

$$\Gamma(x)\Gamma(y) = 4\int_0^{\infty}\int_0^{\infty} e^{-u^2+v^2}u^{2x-1}v^{2y-1}dudv.$$

Integrating this in polar coordinates, with $u = r\cos(\theta)$ and $v = r\sin(\theta)$, we have

$$\Gamma(x)\Gamma(y) = 2\int_0^{\pi/2}\left[2\int_0^{\infty} r^{2(x+y)-1}e^{-r^2}dr\right]\cos(\theta)^{2x-1}\sin(\theta)^{2y-1}d\theta =$$

$$\beta(x, y)\Gamma(x + y).$$

7.3 Fact 3

This is integration by parts. Define

$$u = e^{-t}, \qquad dv = t^{x-1}dt, \qquad du = -e^{-t}, \qquad v = t^x/x.$$

Since $u(0)v(0) = 0$ and $\lim_{n\to\infty} u(n)v(n) = 0$, we have

$$\Gamma(x) = \int_0^{\infty} udv = \int_0^{\infty} vdu = \frac{1}{x}\Gamma(x + 1).$$

7.4 Fact 4

We have $\Gamma(1) = 1$. Hence, by Fact 2,

$$\Gamma(1/2)\Gamma(1/2) = \beta(1/2, 1/2) = \int_0^1 \frac{dt}{\sqrt{t(1-t)}} =^* 2\int_0^{\pi/2} d\theta = \pi.$$

The starred equality comes from $t = \sin^2(\theta)$ and $dt = 2\sin(\theta)\cos(\theta)d\theta$.