Inscribed Rectangle Coincidences

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November 16, 2019

Abstract

We prove an integral formula for continuous paths of rectangles inscribed in a piecewise smooth loop. We use this integral formula to prove the inequality

$$M(\gamma) \geq \Delta(\gamma)/2 - 1.$$

Here $M(\gamma)$ denotes the total multiplicity of rectangle coincidences – i.e., pairs, triples, etc., of isometric rectangles inscribed in γ. The number $\Delta(\gamma)$ denotes the number of stable diameters of γ – i.e. critical points of the distance function on γ.

1 Introduction

In 1911, Toeplitz asked if every Jordan loop has an inscribed square. This problem, often called the Square Peg Problem, has a long history. For instance, In 1944 L. G. Shnirlmann [**Shn**] proved that every smooth Jordan loop has an inscribed square. This result has seen a number of improvements, but none proving that an arbitrary Jordan loop has an inscribed square. See [**Ma1**] and [**P**] for an extensive discussion of the Square Peg Problem, as well as many references.

Some work has also been done concerning rectangles inscribed in Jordan loops. In 1977, H. Vaughan [**Va**] gave a proof that every Jordan loop has an inscribed rectangle. A recent paper of C. Hugelmeyer [**H**] combines Vaughan's basic idea with some very modern knot theory results to show that

* Supported by N.S.F. Research Grant DMS-1204471

a smooth Jordan loop always has an inscribed rectangle of aspect ratio $\sqrt{3}$. The recent paper [**AA**] proves that any quadrilateral inscribed in a circle can (up to similarity) be inscribed in any convex smooth curve. See also [**Ma2**]. In the recent paper [**ACFSST**], the authors show that every Jordan Loop contains a dense set of points which are vertices of inscribed rectangles. In my paper [**S1**] I show, among other things, that all but at most 4 points of any Jordan loop are vertices of inscribed rectangles. For additional work on inscribed rectangles, see [**Mak1**], [**Mak2**], and [**NW**].

The purpose of this paper is to prove a result about *rectangle coincidences*. Given a polygon γ, we define $M(\gamma)$ to be the total number of rectangle coincidences. Informally, γ scores one point for each pair of isometric inscribed rectangles, and 2 points for each triple of isometric inscribed rectangles, and so on. Informally, a *diameter* of γ is a critical point for the distance function $d : \gamma \times \gamma \to \boldsymbol{R}$, and the diameter is *stable* if it persists under small perturbation. Let $\Delta(\gamma)$ be the number of stable diameters of γ.

Theorem 1.1 *For an arbitrary polygon γ, we have $M(\gamma) \geq \Delta(\gamma)/2 - 1$.*

Now we formally define $M(\gamma)$ and $\Delta(\gamma)$.

Definition of Rectangle Coincidences: For each isometry class c of rectangle, we let $N(\gamma, c)$ denote the number of distinct rectangles isometric to c which are inscribed in γ. We define

$$M(\gamma) = \sum_c \max(0, N(\gamma, c) - 1). \tag{1}$$

Here we sum over all isometry types. Generically this sum is finite.

Definition of a Stable Diameter: The complement of any line B in the plane is a union of 2 open halfplanes. Say that a set S lies *on one side* of B if S does not intersect both of these open halfplanes. If, additionally, $S \cap B = \emptyset$, we say that S lies *strictly on one side* of B. Let A be a chord of γ having at least one endpoint which is a vertex of γ. Let A_1 and A_2 be the two endpoints of A. Let B_j be the line perpendicular to A at A_j. Let γ_j be the union of edges of γ, either one or two, which contain A_j. We call A a *diameter* if γ_j lies on one side of B_j for $j = 1, 2$. We call A *stable* if $\gamma_j - A_j$ lies strictly on one side of B_j whenever A_j is a vertex of γ.

It is worth recasting Theorem 1.1 in terms of the distance function. For this, we will not try for the most general result. Say that an *extreme diameter* of γ is a positive local extremum for the distance function. Generically, half the diameters of γ are extreme diameters and the other half are saddles, and all the diameters are stable. The diameter count essentially follows from the fact that the Euler characteristic of $\gamma \times \gamma$ is 0. Let $K(\gamma)$ denote the number of extreme diameters of γ.

Corollary 1.2 *When γ is a generic polygon, we have $M(\gamma) \geq K(\gamma) - 1$. In particular, if γ has at least 2 extreme diameters, then γ has an isometric pair of distinct inscribed rectangles.*

This result is sharp. For instance, when γ is an obtuse isosceles triangle, we have $K(\gamma) = 1$ and $M(\gamma) = 0$.

Now we discuss the idea behind the proof of Theorem 1.1. Let $I(\gamma)$ be the space of labeled rectangles inscribed in γ. The labeling always goes clockwise around the rectangle. Generically $I(\gamma)$ is a piecewise smooth manifold when γ is a generic polygon. (There is a proof in [**S1**] which we sketch in §3 of this paper.) Some of the components of $I(\gamma)$ are loops and some are arcs. The arc components are proper in the sense that as one goes out the end of an arc, the corresponding rectangles accumulate on a diameter of γ. Indeed, generically there is a bijection between the diameters of γ and the ends of arcs in $I(\gamma)$.

For each arc component α of $I(\gamma)$, we define a map $Z : \alpha \to \boldsymbol{R}^2$ as follows. We let $Z(p) = (X, Y)$ where X and Y respectively are the length and width of the rectangle represented by $p \in \alpha$. We call the image $Z(\alpha)$ the *shape curve*. There is a compact region $\Omega(\alpha)$ bounded by $Z(\alpha)$ and portions of the coordinate axis. Using an integral formula, we prove the following result.

Theorem 1.3 *The following is true for a generic polygon γ. For each arc component α of $I(\gamma)$, the signed area of the region $\Omega(\alpha)$ is either 0 or equal (up to sign) to the area of the region bounded by γ.*

In case the rectangles have constant aspect ratio, our integral formula is quite similar to the one which appears in [**Ta**]. Compare also [**AA**]. I presented this formula in [**S2**], in a slightly different form. Here I will prove exactly what is needed for Theorem 1.3.

Consider the implications of Theorem 1.3. If α is such that $\Omega(\alpha)$ has signed area 0, then $Z(\alpha)$ must have a self-intersection. This intersection point corresponds to a pair of inscribed rectangles which are isometric to each other. If α_1 and α_2 are two arc components such that $\Omega(\alpha_1)$ and $\Omega(\alpha_2)$ both have nonzero signed area, then these two regions have the same signed area, up to sign. But then $Z(\alpha_1)$ and $Z(\alpha_2)$ intersect each other or else at least one of $Z(\alpha_j)$ has a self-intersection. When we do the count carefully we get Theorem 1.1.

In connection with the argument above, I want to point out one other phenonomenon I observed. Call a rectangle R *gracefully inscribed* in γ the counterclockwise cyclic ordering on γ induces the counterclockwise cyclic ordering on the vertices of R. Let $G(\gamma)$ denote the subspace of $I(\gamma)$ consisting of gracefully inscribed rectangles. After thousands of trials, I observed that an arc component of $G(\gamma)$ always has one end which is an extreme diameter and one end which is a saddle. I didn't test the components of $I(\gamma) - G(\gamma)$ but I presume that the same thing is true. I think that this observation points to a deeper structure underlying the space of inscribed rectangles, but I can't put my finger on it.

This paper is organized as follows. In §2, I will establish the integral formula and use it to prove Theorem 1.3 modulo a structral statement about the arc components of the space $I(\gamma)$. Theorem 1.3 also relies on a second result from [**S1**], but at the end of §4 I will explain how one can ignore this other result and still get a theorem almost as sharp as Theorem 1.1.

In §3, I will sketch the proof of the structral result about $I(\gamma)$ used in §2.

In §4, I will use Theorem 1.3 to prove Theorem 1.1.

I would like to thank Arseniy Akopyan and Peter Doyle for conversations related to this paper. This work was supported by a grant from the National Science Foundation. I would like to thank the N.S.F. for their support.

2 The Integral Formula

2.1 The Differential Version

Let γ be a counterclockwise oriented piecewise smooth Jordan loop and let R be a labeled rectangle inscribed in γ. We label the vertices of R so that they go counterclockwise around R. We denote these vertices as R_1, R_2, R_3, R_4. The simplest case to picture is when the counterclockwise ordering on γ induces the given labeling of the vertices of R. In [**S1**] we called such rectangles gracefully labeled. For the sake of drawing nice pictures, we will consider the graceful case until the last section.

For each $j = 1, 2, 3, 4$ we let A_j denote the signed area of the region R_j^* bounded by the segment $\overline{R_j R_{j+1}}$ and the arc of γ that connects R_j to R_{j+1} and is between these two points in the counterclockwise order. Figure 2.1 shows a simple example. We compute the signs as follows. If we reverse the orientation on R, so that it goes clockwise around R, then the boundary of R_j^* has a consistent orientation. We then assign to each point of R_j^* the number of times the boundary winds counterclockwise around this point. The signed area of R_j^* is then the integral of the winding number function over R_j^*. When γ is convex, all the signed areas are positive.

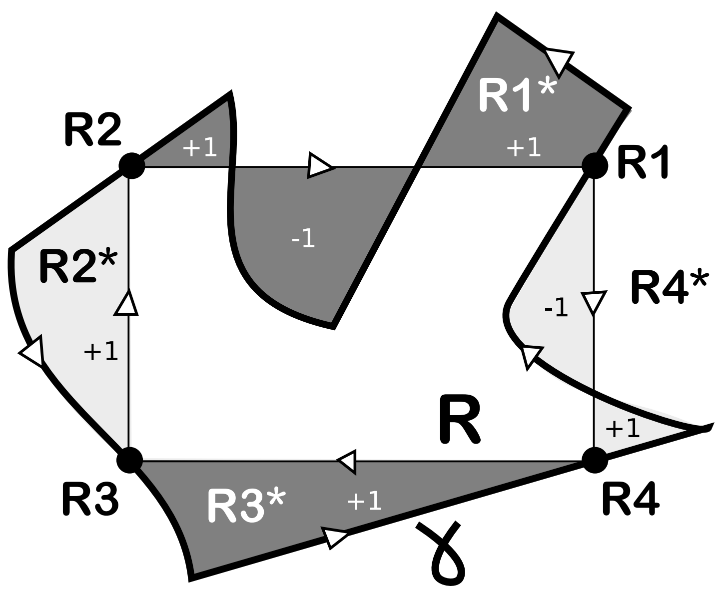

Figure 2.1: The various regions.

Assuming that γ is fixed, we introduce the quantity

$$A(R) = (A_1 + A_3) - (A_2 + A_4). \tag{2}$$

We also have the point $(X, Y) \in \boldsymbol{R}^2$, where

$$X = \text{length}(\overline{R_1 R_2}), \qquad Y = \text{length}(\overline{R_2 R_3}), \tag{3}$$

Assuming that we have a piecewise smooth path $t \to R(t)$ of rectangles gracing γ, we have the two quantities

$$A(t) = A(R(t)), \qquad (X(t), Y(t)) = (X(R(t)), Y(R(t))). \tag{4}$$

If t is a point of differentiability, we may take derivatives of all these quantities. Here is the main formula.

Lemma 2.1

$$\frac{dA}{dt} = Y\frac{dX}{dt} - X\frac{dY}{dt}. \tag{5}$$

Proof: It suffices to prove this result for $t = 0$. This formula is rotation invariant, so we rotate the picture so that the first side of $R(0)$ is contained in a horizontal line, as shown in Figure 2.2. When we differentiate, we evaluate all derivatives at $t = 0$. We write $dR_j/dt = (V_j, W_j)$.

Up to second order, the region $R_1^*(t)$ is obtained by adding a small quadrilateral with base $X(0)$ and adjacent sides parallel to $t(V_1, W_1)$ and $t(V_2, W_2)$, and the area of this quadrilateral is $tX(W_1 + W_2)/2$.

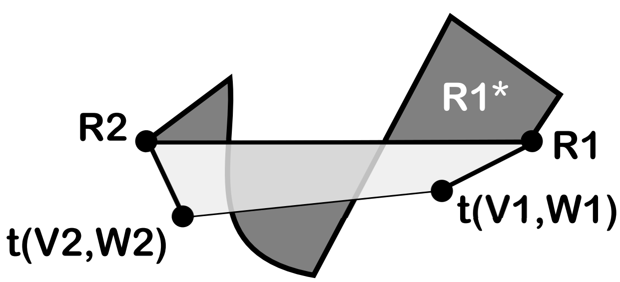

Figure 2.2: The change in area.

From this equation, we conclude that

$$\frac{dA_1}{dt} = -\frac{X(W_1 + W_2)}{2}. \tag{6}$$

We get the negative sign because the area of the region increases when W_1 and W_2 are negative. A similar derivation gives

$$\frac{dA_3}{dt} = +\frac{X(W_3+W_4)}{2}. \tag{7}$$

Adding these together gives

$$\frac{dA_1}{dt} + \frac{dA_3}{dt} = X \times \left[\frac{W_3 - W_1}{2}\right] + X \times \left[\frac{W_4 - W_2}{2}\right] =$$

$$-X \times \left[\frac{1}{2}\frac{dY}{dt}\right] + -X \times \left[\frac{1}{2}\frac{dY}{dt}\right] = -X\frac{dY}{dt}. \tag{8}$$

A similar derivation gives

$$\frac{dA_2}{dt} + \frac{dA_4}{dt} = -\frac{X(V_2+V_3)}{2} + \frac{X(V_4+V_1)}{2} = -Y\frac{dX}{dt}. \tag{9}$$

Subtracting Equation 9 from Equation 8 gives the desired result. ♠

2.2 The Integral Version

Continuing with the notation above, we define the *shape curve*

$$Z(t) = (X(t), Y(t)). \tag{10}$$

Let $\omega = -XdY + YdX$. Here we think of ω as a 1-form. Integrating Equation 5 over the piecewise smooth path, we see that

$$A(1) - A(0) = \int_Z \omega. \tag{11}$$

We can interpret this integral geometrically. Letting $O = (0,0)$, consider the closed loop

$$Z' = \overline{O, Z_0} \cup Z \cup \overline{Z_1, O}. \tag{12}$$

Since ω vanishes on vectors of the form (h,h), we see that

$$A(1) - A(0) = \int_Z \omega = \int_{Z'} \omega = -\int\int_\Omega 2dxdy = -2\ \text{area}(\Omega). \tag{13}$$

Here Ω is the region bounded by Z'. The last line of the equation refers to the signed area of Ω.

2.3 Other Orderings

So far we have considered the case of gracefully inscribed labelings, but this is only for the convenience of drawing nice pictures. Up to cyclic relabeling, there are two other ways that one can inscribe a rectangle in R. One way is that the counterclockwise ordering on γ induces the clockwise ordering on the vertices of R. We call this *anti-graceful*. This third way is the way which is neither graceful nor antigraceful. For lack of a better word, we call this third way *ungraceful*. Figure 2.3 shows the antigraceful and ungraceful cases.

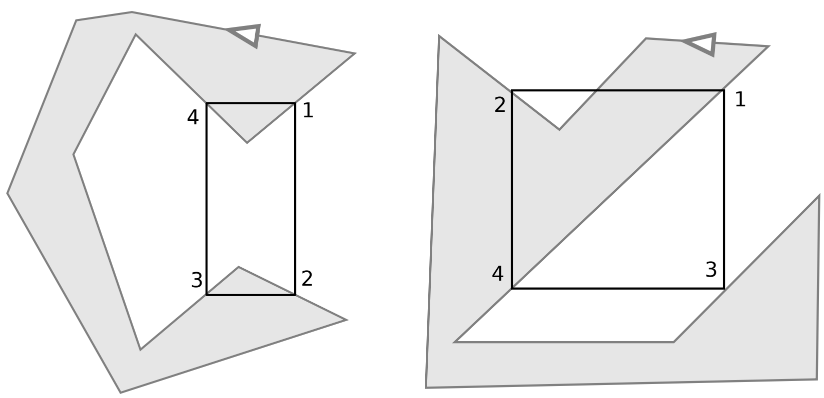

Figure 2.3: Antigracefully and ungracefully inscribed rectangles.

If we define the regions exactly as above, the same differential and integral formulas hold. Again, the recipe is to equip the rectangle R with its clockwise orientation, and then observe that $R \cup \gamma$ defines four consistently oriented loops which meet in pairs at the vertices of R. Figures 2.4 shows how this works for an ungraceful example.

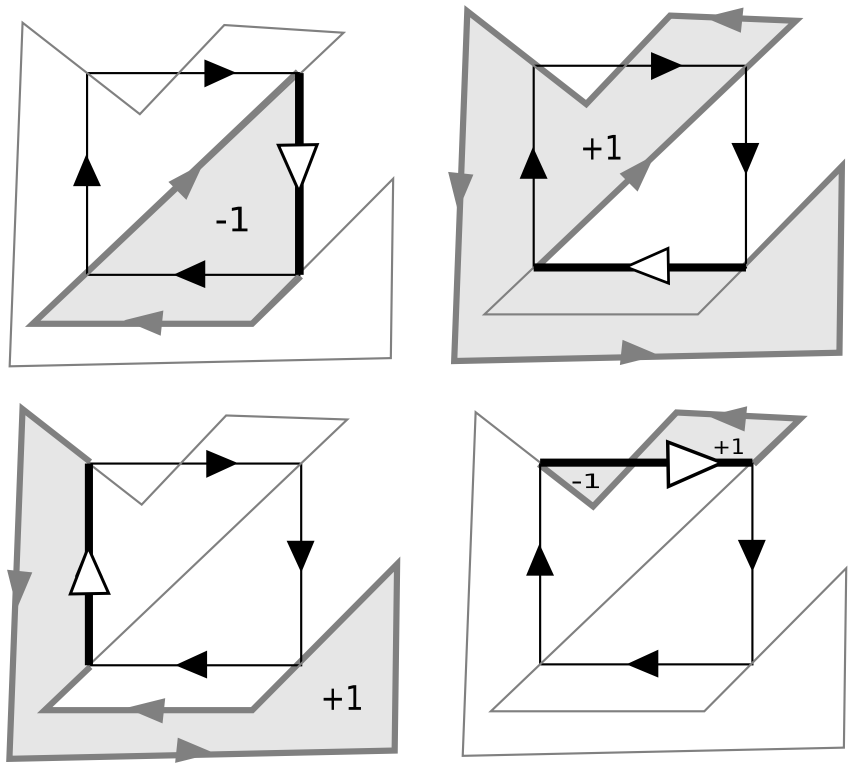

Figure 2.4: The regions for an ungracefully inscribed rectangle.

Now we draw some general conclusions that are independent of the labeling. Let A_γ be the area of the region bounded by γ. When two sides of R are very close together, the corresponding sum $A_i + A_{i+2}$ is very near $\pm A_\gamma$. The equation is exact in the degenerate limit. When one side of R is very short, the corresponding term A_i is either close to 0 or close to $\pm A_\gamma$. Again, the equation is exact in the degenerate limit. Therefore, if the aspect ratio of R is very near 0 or very near ∞, the sum $(A_1 + A_3) - (A_2 + A_4)$ is very close to kA_γ for some integer k with $|k| \leq 3$. The result is exact in the degenerate limit. Moreover, if we have a path of rectangles $t \to R(t)$, for $t \in [0, 1]$, which starts and ends with rectangles having aspect ratio 0, then the two integers $k(0)$ and $k(1)$ corresponding to each end of the path coincide. Thus, the region Ω corresponding to this path has signed area 0. The same goes if the aspect ratios at either end tend to ∞.

2.4 Consequences of the Integral Formula

Let γ be a polygon and let $I(\gamma)$ be the space of rectangles inscribed in γ. We think of $I(\gamma)$ as a subset of $\boldsymbol{R}^8 = (\boldsymbol{R}^2)^4$. We say that a *proper arc* of $I(\gamma)$ is a connected component homeomorphic to an arc, with the following property. As one moves towards an endpoint of an arc component in $I(\gamma)$, the aspect ratio tends either to 0 or to ∞. These chords turn out to be diameters, and they also turn out in the generic case to be distinct from each other.

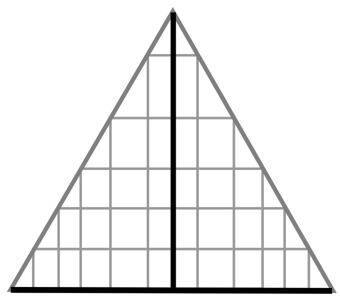

Figure 2.5: An arc component of $I(\gamma)$.

Theorem 2.2 *There is an open dense subset $\mathcal{P}$ of polygons with the following property. For each $\gamma \in \mathcal{P}$ the space $I(\gamma)$ is a piecewise smooth 1-manifold whose arc components are proper.*

We will sketch the proof in the next chapter. Now we prove Theorem 1.3 for all polygons in $\mathcal{P}$. Let γ be such a polygon. We call an arc component of $I(\gamma)$ *hyperbolic* if the rectangles going out one end have aspect ratio tending to 0 and the rectangles going out the other have aspect ratio tending to 0. Otherwise we call the arc *null*. In [**S1**] we proved the following theorem.

Theorem 2.3 *The hyperbolic ends of $I(\gamma)$ are all graceful.*

Let A be some arc component of $I(\gamma)$. Let $t \to R(t)$ be a parametrization of A. Here $t = 0$ corresponds to one end of A and $t = 1$ corresponds to the other. Recall that $X(t)$ and $Y(t)$ represent the lengths of the first two sides of $R(t)$ respectively. The aspect ratio of $R(t)$ is $Y(t)/X(t)$. Recall that $Z(t) = (X(t), Y(t))$ is a path in the plane, and the region Ω is bounded by the loop we get by suitably closing off Z. Referring to §2.2 for notation, we see that there are two possibilities.

1. Suppose that the aspect ratio of $R(t)$ either tends to 0 at both ends of A or tends to ∞ at both ends of A. Then, as we discussed at the end of the last section, the region Ω has signed area 0.

2. Suppose that the aspect ratio of $R(t)$ tends to 0 at one end of A and to ∞ at the other. Then, in this case, we can see by direct inspection that $A(0) = -A(1) = \pm A_\gamma$, where A_γ is the area of the region bounded by γ. So, in this case, we have $A_0 = -A_1 = \pm A_\gamma$. This time the signed area of Ω is exactly A_γ.

This enumeration proves Theorem 1.3.

Figure 2.6 shows the two possibilities.

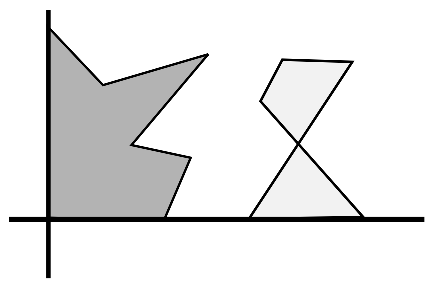

Figure 2.6: Two kinds of shape curves

Note that it can also happen that the shape curve starts and ends on the Y-axis instead. Also, these curves can be much more complicated, with many messy self-intersections.

Remark: If we to not assume Theorem 2.3 then we get a theorem which is almost as strong as Theorem 1.3. The slightly weaker theorem would say that the region Ω has area kA_γ for some integer k with $|k| \leq 6$. In §4.3 we will explain how to use this result to prove a version of Theorem 1.1 that is almost as strong.

3 Spaces of Inscribed Rectangles

3.1 Inscribing Rectangles in Four Lines

The way we understand the space $I(\gamma)$ of rectangles inscribed in a polygon γ is to first study how to inscribe rectangles in quadruples of lines. This is relevant because every rectangle inscribed in γ is inscribed in some collection of at most 4 lines. We studied this question in detail in [**S1**] and [**S2**]. Here we give an abbreviated account of the material in [**S1**].

Consider quadruple $L = (L_0, L_1, L_2, L_3)$ of general position lines in the complex plane $\boldsymbol{C}$. We say that a rectangle R *is inscribed in* L if the vertices (R_0, R_1, R_2, R_3) go cyclically around R (either clockwise or counterclockwise) and satisfy $R_i \in L_i$ for $i = 0, 1, 2, 3$. We let $G(L)$ denote the set of rectangles inscribed in L. We think of $G(L)$ as a subset of $\boldsymbol{R}^8$.

We define the *aspect ratio* $\rho : G(L) \to \boldsymbol{R}$ by the formula

$$\rho(R) = \pm\frac{|R_2 - R_1|}{|R_1 - R_0|}. \qquad (14)$$

The sign is -1 if R is clockwise ordered and $+1$ if R is counterclockwise ordered. We allow both signs for the aspect ratio but eventually we will be interested in the positive case.

When R has aspect ratio ρ, the vertices satisfy

$$R_2 - R_1 = i\rho(R_1 - R_0), \qquad R_3 - R_2 = R_0 - R_1.$$

This leads to the matrix equation $(R_2, R_3) = M(R_0, R_1)$, where

$$M = \begin{bmatrix} -i\rho & 1 + i\rho \\ 1 - i\rho & i\rho \end{bmatrix}. \qquad (15)$$

To find a rectangle of aspect ratio ρ inscribed in L, we find the intersection points of $M(\Pi_{01}) \cap \Pi_{23}$. Here $\Pi_{ij} = L_i \times L_j \subset \boldsymbol{C}^2$.

A dimension count shows that, for a generic choice of L, there is no value of ρ for which there are infinitely many solutions. Hence, for a generic choice of L, Equation 15 either has 0 or 1 solutions for each choice of ρ. This allows us to speak of *the* rectangle R_ρ in $G(L)$ when one exists. When R_ρ exists, the two planes $M(\Pi_{12})$ and Π_{34} intersect transversely. This means that $R_{\rho'}$ also exists for all nearby ρ'. The vertices of R_ρ vary analytically with L and

ρ. These observations imply that $\rho : G(L) \to \boldsymbol{R}$ is a homeomorphism from $G(L)$ onto an open subset of $\boldsymbol{R}$.

We define the 2 *diagonals* of L to be the degenerate rectangles

$$R_0 = (L_{12}, L_{12}, L_{34}, L_{34}), \qquad R_\infty = (L_{14}, L_{34}, L_{23}, L_{14}). \tag{16}$$

These degenerate rectangles are also solutions to Equation 15. If we perturb ρ slightly we still get solutions, by transversality. From this we conclude that $G(L)$ contains rectangles of aspect ratio arbitrarily close to 0 and arbitrarily close to ∞. The rectangles of aspect ratio near 0 are close to R_0 and the rectangles of aspect ratio near ∞ are close to R_∞.

As in [**S1**] There is one additional situation we need to consider, namely quadruples of lines of the form (L_1, L_1, L_2, L_3) up to cyclic permutation. Here $L_1.L_2, L_3$ are in general position. We call these *repeating quadruples*. All the same remarks as above apply to this case. The repeating quadruples arise when we consider rectangles inscribed in a polygon, because two consecutive sides of the rectangle might be on the same side of the polygon. In the repeating case, we define

$$R_0 = (x, x, L_{23}, L_{23}), \qquad R_\infty = (L_{12}, L_{13}, L_{13}, L_{12}). \tag{17}$$

Here x is the interection of the line through L_{23} that is perpendicular to L_1. With these definitions in place, the same remarks as in the non-repeating case apply here. Figure 3.1 below suggests how, in the repeating case, rectangles of aspect ratio 0 and ∞ degenerate to R_0 and R_∞.

3.2 Inscribing Rectangles in Polygons

For the sake of completeness, we now sketch the proof of Theorem 2.2. First of all, generically our polygon has no parallel sides. So, after we appropriately order the lines, every rectangle in $I(\gamma)$ lies in some space $G(L)$ considered above. The intersections $G(L) \cap I(\gamma)$ are essentially coordinate charts for $I(\gamma)$. We write $I(\gamma) = I_0(\gamma) \cup I_1(\gamma)$ where $I_0(\gamma)$ consists of those inscribed rectangles which do not share a vertex with γ and $I_1(\gamma)$ consists of those inscribed rectangles which do. The local structure of $I_0(\gamma)$ is exactly the same as the local structure of $G(L)$ for some quadruple of lines L considered in the previous section. So, $G_0(L)$ is a smooth manifold.

When γ is generic, the rectangles in $I_1(\gamma)$ just share one vertex in common with γ. Let R be such a rectangle and let v be the relevant vertex of R. There are 2 qudruples of lines L and L' such that R belongs to both $G(L)$ and $G(L')$. After relabeling we can arrange that the lines L_0 and L'_0 extend the edges of γ incident to v and the other 3 lines of L and L' are the same. Because γ is generic, we get the following picture. As we vary the aspect ratio the vertex corresponding to R in $G(L)$ moves monotonically along L_0. As we move in one direction, the vertex in question lies on the edge of γ contained in L_0 and in the other direction the vertex lies outside this edge. From this we see that a neighborhood of R in $G(L) \cap I(\gamma)$ is a half-open interval. The same goes for L'. These two half-open neighborhoods fit together to give an arc neighborhood of R in $I(\gamma)$.

When we exit the end of an arc component of $I(\gamma)$, the aspect ratio must tend to 0 or ∞. Otherwise, using the uniform lower bound to the diameter of rectangles inscribed in γ, we could take a limit and get a point that lies in the closure of an arc but not in the arc. This contradicts the manifold nature of $I(\gamma)$. Moreover, once the rectangles in an arc component have very small or large aspect ratio, then all lie in a single $G(L)$ and the end limits on a diagonal of $G(L)$. Hence the arcs are all proper.

There is one more fine point. Suppose that both ends of some arc component α accumulate on the same chord of $I(\gamma)$. Eventually both ends would lie in the same component $G(L)$. Given the uniqueness of rectangles in $G(L)$ having prescribed aspect ratio, the two ends would eventually have rectangles in common, up to relabeling. There is a $\boldsymbol{Z}/4$ action on $I(\gamma)$ coming from cyclically relabelling the rectangles. This action acts freely on $I(\gamma)$. In the situation at hand, some relabeling element ρ would have the property that $\rho(\alpha) \cap \alpha \neq \emptyset$. But then $\rho(\alpha) = \alpha$ and ρ swaps the ends of α. But then, since α is an arc, ρ fixes a point on α. This is a contradiction. Hence the arc components of $I(\gamma)$ always accumulate on distinct chords.

3.3 The Structure of the Ends

Let $\overline{I}(\gamma)$ be the quotient of $I(\gamma)$ by the $\boldsymbol{Z}/4$ cyclic relabeling action.

Lemma 3.1 *For a polygon γ in $\mathcal{P}$ the following is true. Each end of each arc component of $\overline{I}(\gamma)$ accumulates on a unique diameter, and each diameter is the accumulation set of a unique end of a unique arc component of $\overline{I}(\gamma)$.*

Proof: Each end of a proper arc is a chord of γ. The perpendiculars of this chord are limits of lines extending the edges of rectangles converging to this chord. These edges intersect γ in a pair of points which converge to the endpoint of the chord, as shown in Figure 3.2. From this picture we conclude that these limiting chords are diameters of γ, which means that the perpendiculars at the endpoints do not locally separate γ. In short, each end of a proper arc of $I(\gamma)$ is a diameter. Generically these diameters are all stable.

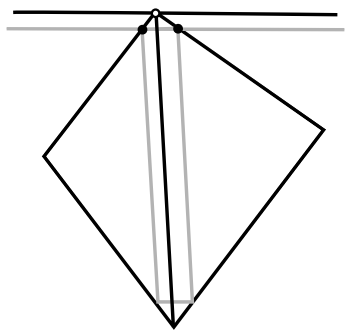

Figure 3.2: The limiting chord and its perpendicular

Conversely, if we have a diameter A of γ, then we can consider the configuration L of lines extending the edges of γ incident to A. We will either have a quadruple of distinct lines or else a repeating quadruple. In either case, we can order the lines so that A is a diagonal of L. But then $G(L)$ contains rectangles that accumulate on A. But this is only possible if A is the end of some arc component of $I(\gamma)$. In short, every diameter arises as the end of a proper arc of $I(\gamma)$.

Suppose now that one diameter of γ is the end of two arc components α_1 and α_2 of $G(\gamma)$, then $\alpha_1 = \alpha_2$ up to cylic relabeling. If this is false, then the rectangles on both ends belong to the same space $G(L)$. But then, given the uniqueness of rectangles with prescribed aspect ratio in $G(L)$, the end α_1 would intersect some end α_2' which is simply a cyclic relabeling of α_2. This would force $\alpha_1 = \alpha_2'$. So each diameter of $I(\gamma)$ corresponds to 4 ends of arc components of $I(\gamma)$.

To finish the proof, we just have to see that the $\boldsymbol{Z}/4$ relabeling action freely permutes the arc components of $I(\gamma)$, so that there are 4 such components in each orbit. If this is false, then some cyclic relabeling would stabilize an arc component and necessarily swap the ends. But then, as discussed at the end of the last section, the cyclic relabeling action would have a fixed point. This is a contradiction. ♠

3.4 Kinds of Diameters

There are 3 kinds of inscribed rectangles corresponding to points in $I(\gamma)$: graceful, antigraceful, and ungraceful. By continuity, the set of all rectangles of a given type is a union of components of $I(\gamma)$. We call a diameter of γ graceful, antigraceful, or ungraceful according to the kind of component it corresponds to. Thus, the graceful components of $I(\gamma)$ pair up the graceful diameters, the antigraceful components pair up the antigraceful diameters, and the ungraceful components pair up the ungraceful diameters.

The way we have defined the type of a diameter makes it look like a fairly global notion, but here we give a local criterion. (This material is not needed for the proof of Theorem 1.1.) Given a diameter A, let Q_ϵ be the quadrilateral whose vertices are precisely ϵ away from the endpoints of A. In the stable case, which holds generically, there is a positive angle, independent of ϵ, between the short sides of Q_ϵ and A. Thus Q_ϵ is convex for small ϵ. The following result is not needed for the proof of

Lemma 3.2 *The diameter A is graceful (respectively antigraceful or ungraceful) if and only if Q_ϵ is gracefully (respectively antigracefully or ungracefully) inscribed.*

Proof: For ease of exposition, we treat the case when both ends of A are vertices of γ. The other case is similar. We rotate so that A is vertical. Let R_ϵ be some rectangle corresponding to an end of $I(\gamma)$ that is close to A.

Each vertex of Q_ϵ has a partner vertex on R_ϵ which lies on the same edge of γ. Let us focus on the picture near one endpoint A_1 of A. One of the edges γ_{11} of γ incident to A_1 lies the left of the other edge γ_{12} of γ_1 incident to A_1. The vertex of Q_ϵ on γ_{11} lies to the left of the vertex of Q_ϵ on γ_{12} because both lie on the disk of radius ϵ centered at A_1. The same goes for the corresponding vertices of R_ϵ because these vertices are the intersection of a nearly horizontal line with γ_{11} and γ_{12} respectively. What this means is that during the straight line interpolation from Q_ϵ to R_ϵ, the short side near A_1 never becomes vertical. The same goes for the picture near the other endpoint A_2 of A. But then we can interpolate between Q_ϵ and R_ϵ by a continuous path of inscribed embedded quadrilaterals. This implies that Q_ϵ and R_ϵ are inscribed in the same way. ♠

4 Proof of the Main Result

4.1 The Generic Case

Now we prove Theorem 1.1 for a generic polygon γ. We keep the notation from the last chapter. By Lemma 3.1, there are $\Delta(\gamma)/2$ arc components in $\overline{I}(\gamma)$.

Case 1: Let α be a null arc of $\overline{I}(\gamma)$. The region $\Omega(\alpha)$ has signed area 0. Hence $Z(\alpha)$ has a self-intersection. This self-intersection corresponds to a pair of isometric gracefully inscribed rectangles, and this adds 1 to $M(\gamma)$.

Case 2: Let α be a hyperbolic arc of $\overline{I}(\gamma)$ whose shape curve is not embedded. Then we get the same conclusion as in Case 1.

Case 3: Finally, consider the d hyperbolic arcs on our list which have embedded shape loops. If α_1 and α_2 are two such arcs, then the regions Ω_1 and Ω_2 both have the same area and both contain all points in the positive quadrant sufficiently close to the origin. Hence their boundaries intersect somewhere in the positive quadrant. The intersection point corresponds to a coincidence involving a rectangle associated to α_1 and a rectangle associated to α_2. Call this the *intersection property*.

We label so that $\alpha_1, ..., \alpha_d$ are the hyperbolic arcs having embedded shape loops. We argue by induction that these d arcs contribute at least $d-1$ to the count for $M(\gamma)$. If $d = 1$ then there is nothing to prove. By induction, rectangle coincidences associated to the arcs $\alpha_1, ..., \alpha_{d-1}$ contribute $d-2$ to the count for $M(\gamma)$.

By the intersection property, α_d intersects each of the other arcs, and $\overline{I}(\gamma)$ is a manifold, there is at least one new rectangle involved in our count, namely one that corresponds to a point on $Z(\alpha_d)$ that is also on some of the shape loop. The corresponding rectangle adds 1 to the count for $M(\gamma)$, one way or another. So, all in all, we add $d-1$ to the count for $M(\gamma)$ by considering the rectangle coincidences associated to $\alpha_1, ..., \alpha_d$.

Our count shows that $M(\gamma) \geq N - 1$, where N is the number of components in $G(\gamma)$. Since $N = \Delta(\gamma)/2$, we get $M(\gamma) \geq \Delta(\gamma)/2 - 1$. This proves Theorem 1.1 in the generic case.

4.2 Taking a Limit

Now we prove Theorem 1.1 for an arbitrary polygon γ. We first assume that all the diameters of γ are stable, and then deal with unstable diameters at the end.

Let N be the number of sides of γ. Let $\{\gamma_n\}$ be a generic sequence of N-gons converging to γ. As we mentioned in the introduction, a stable diameter persists under perturbation. Hence, any diameter A of γ is the limit of a sequence $\{A_n\}$ where A_n is a stable diameter of γ_n.

For each n, the space $\overline{I}(\gamma_n)$ is a finite union of arcs and loops. The arc components join the diameters of γ_n in pairs. Passing to a subsequence we can assume that the (combinatorial) way that these diameters are paired is independent of n. Call two diameters of γ *matched* if the corresponding stable diameters of γ_n are paired by an arc component of $\overline{I}(\gamma_n)$.

Taking a limit with respect to the Hausdorff topology on closed subsets of $\boldsymbol{R}^8$, we see that there is a closed connected subset $C^*(A_1, A_2) \subset \overline{I}(\gamma)$ which joins the matched diameters A_1 and A_2 of γ. What makes this convergence work is that there is a uniform lower bound to the diameter of any rectangle inscribed in γ_n, independent of n.

The space $\overline{I}(\gamma)$ is a finite union of spaces of the form $G(L)$, where L is some quadruple of lines associated to γ. The individual spaces $G(L)$ are either analytic arcs, as in the generic cases analyzed above, or analytic manifolds of dimension 0 or 2. These pieces come together in a finite-to-one manner at points of $\overline{I}(\gamma)$ corresponding to inscribed rectangles which have more than one vertex in common with γ. Given this structure, the existence of the set $C^*(A_1, A_2)$ implies that A_1 and A_2 can be connected by a piecewise analytic arc $C(A_1, A_2) \subset \overline{I}(\gamma)$. We call $C(A_1, A_2)$ a *matching arc*.

If two matching arcs intersect more than once, we can splice these arcs at an intersection point and produce two new arcs which still pair up diameters but which intersect in fewer points. Figure 4.1 shows this operation. This operation changes the way that the diameters of γ are paired, but this does not bother us.

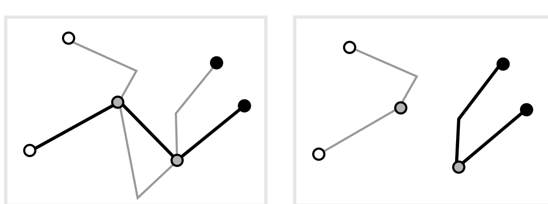

Figure 4.1: Splicing two matching arcs

We splice until any two matching arcs intersect at most once. If two hyperbolic arcs intersect. we can splice them so that, instead, we have 2 null arcs intersecting. This operation does not change the number of intersections. Finally, our union of arcs contains a topological loop, we can splice so as to remove this loop, as shown in Figure 4.2 for the case when the loop is made from 3 sides.

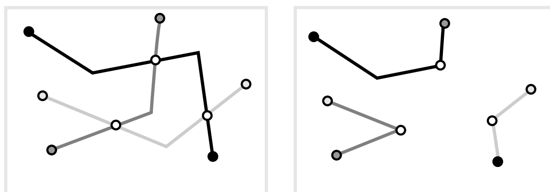

Figure 4.2: Removing a 3-cycle by splicing

So, after a finite number of splices we arrive at a the following situation.

- Each pair of arcs intersects at most once.
- Hyperbolic arcs do not intersect.
- The union of all arcs is simply connected.

Suppose that there are d hyperbolic arcs whose shape graphs are embedded. Since these arcs are pairwise disjoint, the same argument as for Case 3 above shows that points on the union of these arcs contribute at least $d-1$ to the total $M(\gamma)$. Suppose by induction that we have a collection $\mathcal{A}$ of $k \geq d$ arcs, including the d just considered, and that points on $\mathcal{A}$ contribute at least $k-1$ to the total value of $M(\gamma)$. Let α be an arc not on $\mathcal{A}$. Either α is null or α is hyperbolic with a self-intersecting shape graph. As in Case 1 or Case 2 above, there are 2 distinct points of α corresponding to another rectangle coincidnce. These points cannot both belong to arcs of $\mathcal{A}$ because then $\alpha \cup \mathcal{A}$ would not be simply connected. Hence points on $\alpha \cup \mathcal{A}$ contribute $k+1$ to the total count. By induction, then, we have $M(\gamma) \geq \Delta(\gamma)/2 - 1$. This proves Theorem 1.1 for any polygon with all stable diameters.

Suppose now that γ is a general polygon. As above, every stable diameter of γ is the limit of diameters of the sequence $\{\gamma_n\}$ but perhaps some unstable diameters of γ are such limits as well. We pass to a subsequence as above, so that the combinatorics of the pairing on γ_n is unchanged. It might happen that a pair of matched diameters of γ_n both shrink to the same unstable diameter of γ in the limit. In this case, there may be no rectangles in $I(\gamma)$ which are limits of the corresponding rectangles of $I(\gamma_n)$. All the corresponding rectangles in the path may shrink to the diameter itself.

However, suppose that $\{\delta_n\}$ and $\{\delta'_n\}$ are matched diameters of $\{\gamma_n\}$ and δ_n converges to a stable arc of γ. Then there is a diameter δ' of γ, distinct from δ, such that some connected arc of $I(\gamma)$ connects δ to δ'. The stability of δ keeps δ'_n far away from δ_n, so to speak. Let us call a diameter of γ *semistable* if it is either stable or connected to a stable diameter by an arc of $I(\gamma)$. There are at least $\Delta(\gamma)$ semistable arcs. Using semistable arcs in place of stable ones, our proof above goes through word for word.

4.3 Discussion

The one part of our paper that is far from self-contained is the quote of Theorem 2.3. Here we discuss what to do without it. As we already remarked, without Theorem 2.3 we can still say that the region $\Omega(\alpha)$ in Theorem 1.1 has signed area kA_γ for some integer k with $|k| \leq 6$. This leaves 6 nonzero possibilities up to sign. We sort the components of $\overline{I}(\gamma)$ having an associated shape Ω of nonzero signed area into 6 kinds. Running the results above separately and then putting everything together, we would get $\Delta(\gamma)/2 - 6$ rather than $\Delta(\gamma)/2 - 1$ in Theorem 1.1. For complicated polygons this is pretty close to the original.

Alternatively, we could restrict our attention solely to gracefully inscribed rectangles. Letting $\Delta_+(\gamma)$ denote the number of graceful diameters, and $M_+(\gamma)$ the number of rectangle coincidences amongst gracefully inscribed rectangles, the proof above (without Theorem 2.3 gives $M_+(\gamma) \geq \Delta_+(\gamma)/2-1$

Now we discuss a more precise version of Theorem 1.1. Let $\Delta_-(\gamma)$ and $\Delta_0(\gamma)$ to be the number of antigraceful and ungraceful diameters respectively. Let $M_-(\gamma)$ and $M_0(\gamma)$ be the number of rectangle coincidences amongst antigraceful and ungraceful rectangles respectively. The component of $\overline{I}(\gamma)$ corresponding to different combinatorial types do not intersect in any case, so the argument above actually shows that $M_*(\gamma) \geq \Delta_*(\gamma)/2 - 1$ for each $* \in \{+, -, 0\}$.

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