The Crisscross and the Cup: Two Short 3-Twist Paper Moebius Bands

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Abstract

We introduce the crisscross and the cup, both of which are immersed 3-twist polygonal paper Moebius band of aspect ratio 3. We explain why these two objects are limits of smooth embedded paper Moebius bands having knotted boundary. We conjecture that any smooth embedded paper Moebius band with knotted boundary has aspect ratio greater than 3. The crisscross is planar but the cup is not.

1 Introduction

Informally, a *paper Moebius band* is what you get when you take a strip of paper, give it an odd number of twists in space, then tape the ends together. A formal definition is given e.g. in [**S1**]. A related concept is that of a *folded ribbon knot*. This is what you get when you take a paper strip, fold it up so that the ends meet, and then press it into the plane. A formal definition of a folded ribbon knot is given e.g. in [**DL**]. The difference between paper Moebius bands and folded ribbon knots is that the former are smooth surfaces in space and the latter are polygonal objects in the plane, with some additional combinatorial data akin to a knot crossing diagram.

We say that a paper Moebius band is *multi-twisted* if the boundary loop ∂M_λ is a non-trivial knot. One can make a similar definition for folded ribbon

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knots. The multi-tristed case corresponds to giving the strip of paper at least 3 twists.

This paper is a sequel to the papers [**S1**] and [**S2**]. In [**S1**], R.E.S. resolves the minimum aspect ratio question for paper Moebius bands, discussed in W. Wunderlich's 1962 paper [**W**] and then formally conjectured by B. Halpern and C. Weaver [**HW**] in 1977. (See [**T**] for an English translation of [**W**].) The so-called *triangular paper Moebius band*, whose aspect ratio is $\lambda = \sqrt{3}$, is the best one can do. This example has an unknotted boundary, a perfect equilateral triangle. The triangular Moebius band is not quite a paper Moebius band because it is neither smooth nor embedded, but it has an interpretation as a folded ribbon knot.

In [**S2**] R.E.S. establishes a similar result for paper cylinders having at least 2 twists. In this case the best one has aspect ratio 2 and folds 4 times around a right-isosceles triangle. This result, and some soft work involving smooth approximations, resolves [**DL**, Conjecture 39] in the case $n = 1$.

B.E.B. subsequently got interested in this work and decided to find the minimum aspect ratio for a multi-twisted paper Moebius band by a physical experiment: Make a loose 3-twist paper Moebius band and then carefully pull it tight. She found two surprising limits, which she calls the *crosscross* and the *cup*. Both of these have aspect ratio 3, and they seem to be optimal in the sense that they minimize aspect ratio amonst multi-twisted paper Moebius bands.

The crisscross has an interpretation as a folded ribbon (un)knot. The cup is somehow closer to a basketweave. In this note we will describe the crosscross carefully and then sketch how it may be approximated by smooth paper Moebius bands having knotted boundary. We will describe the cup in a bit less detail, and then discuss heuristically some of the mathematical implications of the cup.

One impressive thing about these objects is that their aspect ratio is much less than $3\sqrt{3}$, which is what one would get from a 3-twist paper Moebius band by wrapping 3-times around the triangular Moebius band in a fairly obvious way. Indeed, the famous hexaflexagon is just such a 3-fold wrapping. For a while R.E.S. thought that the hexaflexagon had minimum aspect ratio amongst multi-twist paper Moebius bands. This same speculation for folded ribbon knots is the content of [**DL**, Conjecture 22]. The crisscross and the cup demolish the hexaflexagon in terms of aspect ratio. Since the crisscross is also a folded ribbon knot, it gives a counterexample to [**DL**, Conjecture 22].

The existence of the crisscross (or the cup) and its knotted approximations establishes the following result.

Theorem 1.1 *Let λ^* denote the infimal aspect ratio of a multi-twisted paper Moebius band. Then $\lambda^* \leq 3$.*

Inspired by the conjecture made by Halpern and Weaver in 1977 we make the following conjecture.

Conjecture 1.2 *A smooth embedded multi-twisted paper Moebius band has aspect ratio greater than* 3.

The conjecture and the theorem would combine to show that $\lambda^* = 3$. In the category of folded ribbon knots, Conjecture 1.2 is the same as [**DL**, Conjecture 22] except that 3 replaces $3\sqrt{3}$.

The existence of the cup and the crisscross together rules out the possibility of a theorem like [**S1**, Triangular Limit Theorem]. Assuming that Conjecture 1.2 is true, a minimizing sequence of examples does not have a unique limit. It might converge to the crisscross and it might converge to the cup. How many possible limits are there? We think that probably there are just these two limits, but we are prepared for a surprise.

We would like to thank Elizabeth Denne, Ben Halpern, Curtis McMullen, Charles Weaver, and Sergei Tabachnikov for helpful discussions about topics related to this paper.

2 The Crisscross

2.1 Basic Description

The left side of Figure 1 shows the folding pattern for the crisscross. The middle picture shows an intermediate stage of folding. The right side shows the thing all folded up. The strip of paper is aqua on one side and magenta on the other.

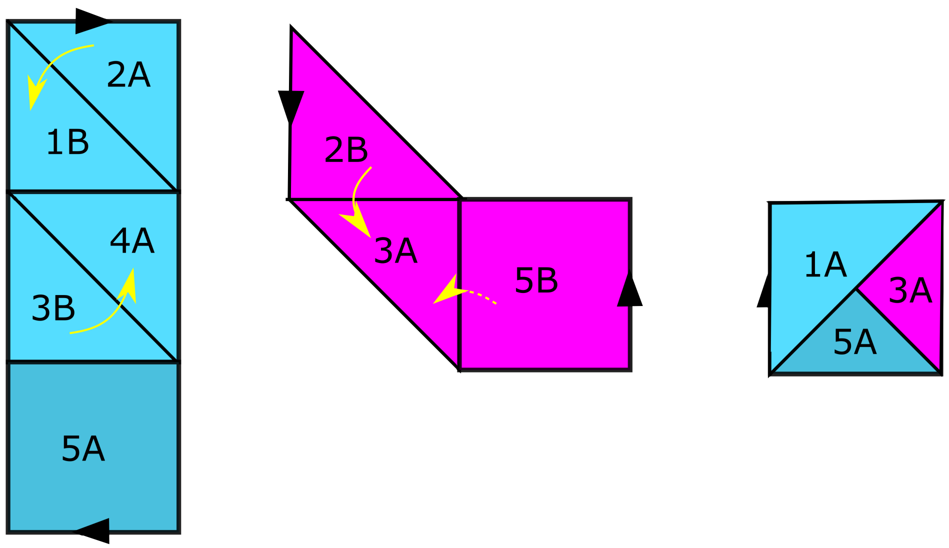

Figure 1: The crisscross folding pattern.

The yellow arrows indicate the successive folds. The solid yellow arrows indicate that the fold should be made in a "forward direction" with the crease receding away from the viewer. In the bottom fold on the left, the bottom square goes along for the ride. The one dotted arrow in the middle indicates that the fold should be made "around the back". The lettering has the following meaning: Imagine that the crisscross is siting on the table as on the right side of Figure 1. If you stick a pin though it, the pin will encounter pieces $1, 2, 3, 4, 5$ in order. The A-faces are facing up and the B-faces are facing down. The crisscross is taped on the left hand vertical side.

Figure 2 shows another view of the crosscross. This time we are separating out the 5 faces and indicating how the edges are glued together. The thick red sides indicate the boundary. The oriented blue segments piece together to make the midline of the crisscross.

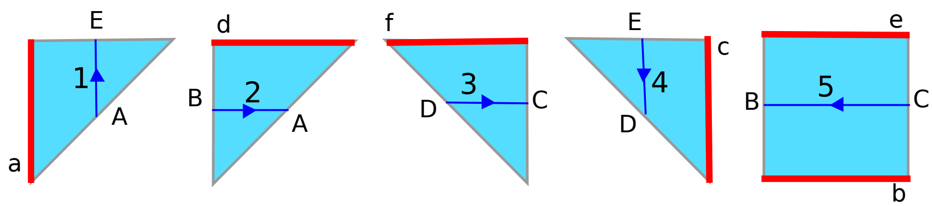

Figure 2: Crisscross gluing pattern

Figure 2 uses two systems of letters. The letters A, B, C, D, E indicate the side pairings. The letters a, b, c, d, e, f indicate the way the boundary goes around the crisscross. We orient each of the sides labeled a, b, c, d, e, f towards the letter. Thus the tail of each edge is at the unlettered vertex and the head is at the lettered vertex. The edge labeled a runs into the edge labeled b, with the head of a going to the tail of b. And so on, all the way around.

2.2 Smooth Approximation

In this section we show that the crisscross can be smoothly approximated as closely as we like. There are two ways to look at this problem. One way to look at it is to observe that we made the crisscross by making a 3-twist Moebius band and pulling it tight. With a bit of faith in the ability of mathematics to model physical phenomena, one could imagine that the intermediate stages are the very approximations we seek.

The above approach is somewhat vague, so here we describe a more precise method. The method goes back at least to the 1930 paper by M. Sadowski [**Sa**]. (See [**HF**] for an English translation.) In Sadowski's paper the examples are made by splicing together pieces of cylinders with flat polygons, resulting in C^1 examples with discontinuous mean curvature. A very similar method, used by Halpern and Weaver [**HW**], uses more general smooth surfaces in place of pieces of cylinders. This argument is described informally in [**FT**]. We use the same argument, more or less, in [**S2**]. For convenience we repeat the argument in [**S2**] almost *verbatim*.

Using smooth bump functions one can easily make a U-shaped curve. This curve agrees with line parallel line segments at either end and then curves around to join these line segments. Call this curve U. The product $U \times [a, b]$ is an isometrically embedded rectangle. Next we take the polygons in Figure 2 and stack them on top of each other, separated by a very small distance. We now join the newly created edges of this stack of triangles by the U-shaped rectangles.

If we make these U-shaped rectangles slightly thinner than the sides of the polygons, they will not overlap each other. The key observation here is that we never encounter 4 edges laying directly above each other and having an interlaced gluing pattern. Indeed, there are exactly 2 edges lying vertically above each other in each relevant plane.

The new object will be an embedded smooth paper Moebius band whose boundary is not quite totally geodesic. To get a totally geodesic example, we just trim off the rough edges. This gives you a smooth embedded twisted cylinder with slightly larger aspect ratio. With this procedure you can make λ as close to 3 as you like.

2.3 Knottedness of the Boundary

We give 3 ways to think about the knottedness of the boundary of the approximations to the crisscross.

Method 1: The informal method of approximating the crisscross by pulling tight a loose example, then the knotting is automatic. The initial boundary is a trefoil knot and then when we pull tight we are moving the boundary by an isotopy and so it remains a trefoil knot.

Method 2: Another way to analyze the smooth approximations coming from the more formal construction is just to make one, then to tape some yarn along the edges, then detach the yarn, and then observe that the result is actually knotted. We did this on the crisscross we built, which one can think of as a very close relative of the smooth approximations constructed above.

Here we describe the boundary of an approximation to the crisscross based on the gluings in Figure 2. Figure 3 shows the path very nearly taken by the boundaries. The numbers indicate the face containing the edges. The little magenta segments indicate very nearly vertical segments which join a piece on one layer to a piece on another. The crossings are dictated by the numbering.

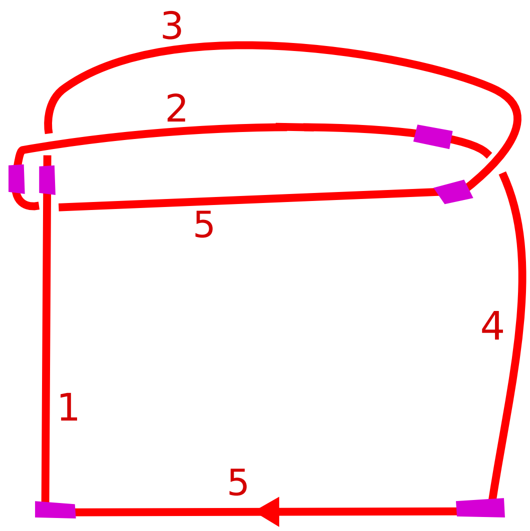

Figure 3: Knottedness of the boundary

In the actual crisscross the arcs which look horizontal are horizontal and the arcs which look vertical are vertical. The red boundary makes a perfect square, but with some backtracking.

Method 3: Our last method is completely algorithmic in the sense that it does not require any visualization or physical manipulation. It just requires an analysis of Figure 2. In Figure 4 below we have copied down our knotted red loop, but for this method we do not need to know anything about this loop except the numbering of the strands. The information inside the grey disks, which we got by some model-making in Method 2, is irrelevant for Method 3.

The red loop in Figure 4 is as in Figure 3. The blue loop is the midline of the crisscross. After orienting both the red and blue loops we can figure out which passes over which, at each crossing, using the numbering of the edges. Following this, we can assign a (+) or a (−) to each local crossing according to the rules given by the 16 local models running around the outside of the picture. (One of the groups of 4 comes from the Wikipedia page on linking number and the other groups are obtained from the first one by rotations.) Computing all the linking numbers and using the formula for linking number (total sign divided by 2) we find that the linking number between the red/magenta and blue loops is −3. This tells us that we have 3-twist Moebius bands, and these have knotted boundaries.

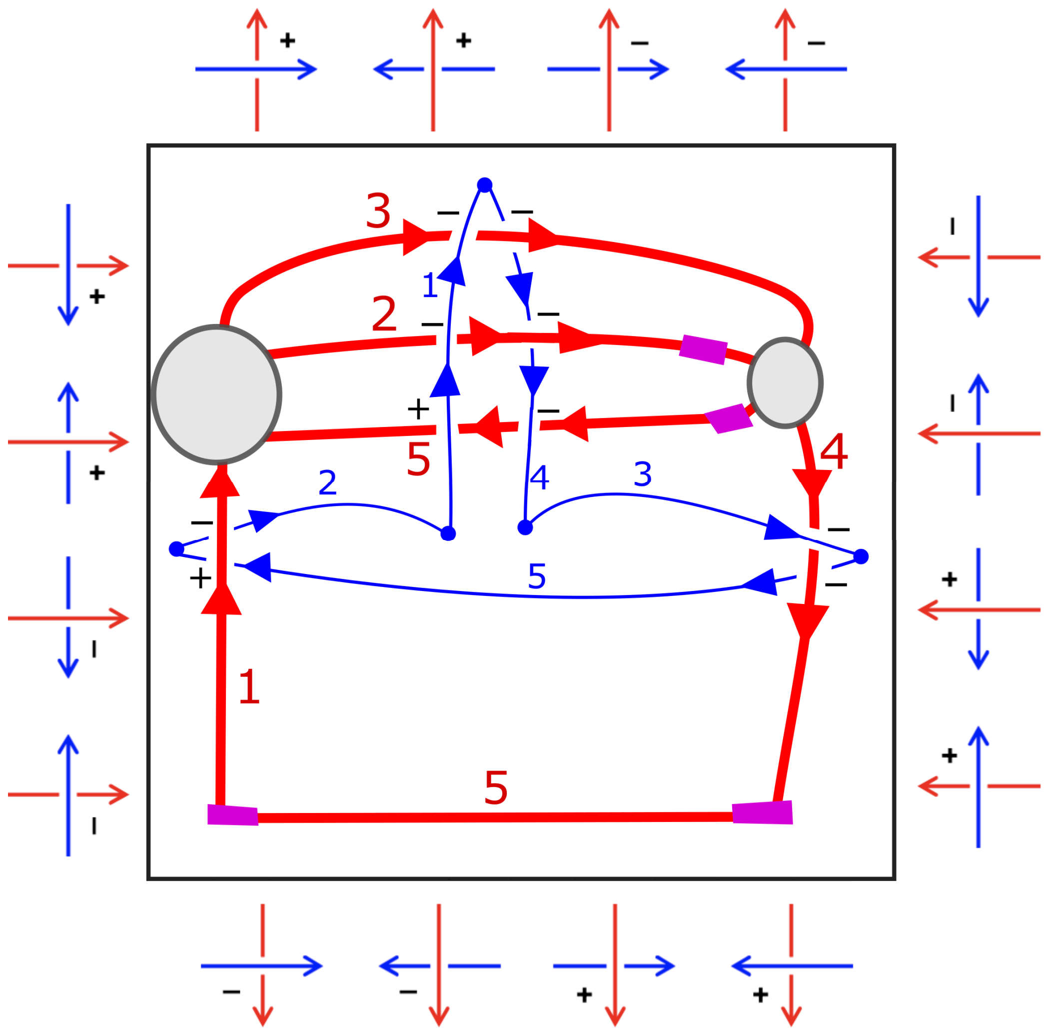

Figure 4: The linking of the boundary and the midline

Having more than one method for a relatively simple example might seem like overkill, but it seems that Method 3 is a nice way to algorithmically deduce the twisting number of more general folded paper Moebius bands based on gluing diagrams like Figure 2. One should compare the method in [**DL**] for folded ribbon knots; it seems very similar.

3 The Cup

3.1 Basic Description

As we mentioned in the introduction, B.E.B. has a much more symmetric model for a 3-twist polygonal paper Moebius band of aspect ratio 3. In this alternate version, which we call the *cup*, the image is a union of 3 right-isosceles triangles which make 3 faces of a tetrahedron whose fourth face is an equilateral triangle.

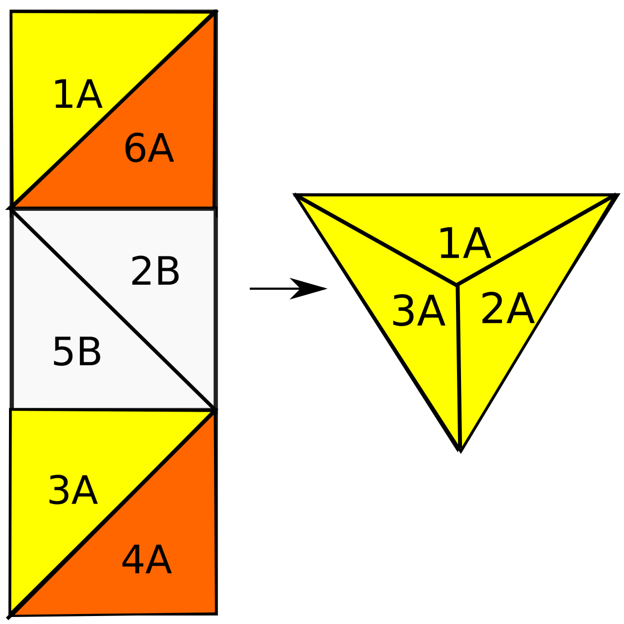

Figure 5: The folding pattern for the cup.

The left side of Figure 5 shows the folding pattern for the cup. The right side shows what you would see if you were looking into the cup. The lettering and coloring is a bit different. The A-faces are colored orange and yellow and the B-faces are colored white. When this is folded up, the cup is yellow on the "inside" and orange on the "outside". The white faces are all pressed together and would not be visible if you held the cup in your hands. The faces $1A, 2A, 3A$ are on the "inside" of the cup and the faces $4A, 5A, 6A$ are on the "outside". Again, the cup does not lie flat in the plane and so the right side of Figure 5 is a planar projection and hence geometrically distorted.

We leave the details of the construction to the reader who likes to cut out strips of paper and fold them up. Here is an example we made from wax paper and washi tape.

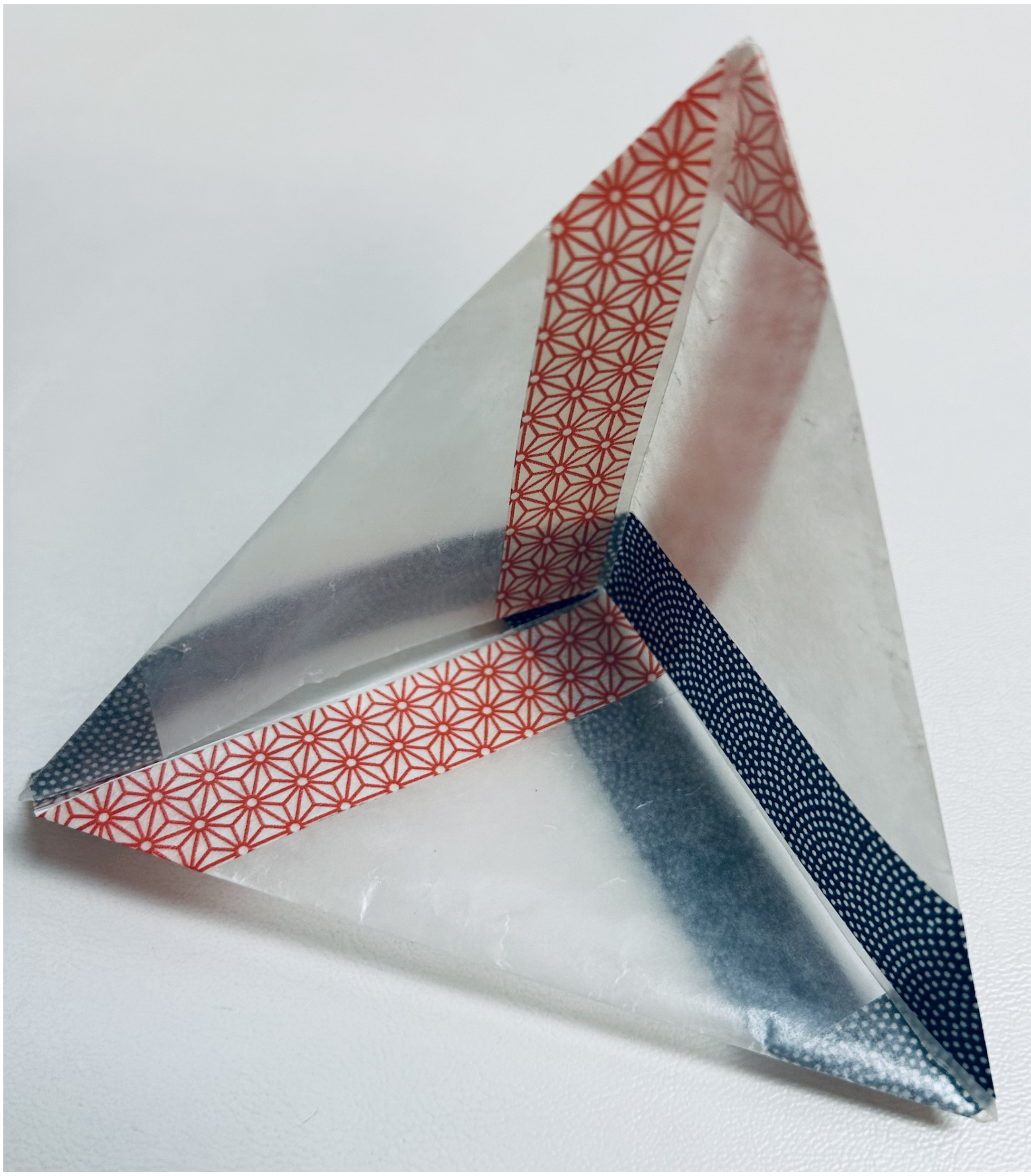

Figure 6: A cup made from wax paper and washi tape.

3.2 Smooth Approximations

We approximate the cup by smooth paper Moebius bands much in the same way as we did for the crisscross. We separate out the 6 triangular faces from each other and then attach the appropriate smooth folds. The folding pattern is very interesting. First of all, there are 3 folds which go around the rim of the cup. Then there are 3 more folds which make a Y-pattern and meet at the central point of the cup.

It is interesting to observe how the boundary interacts with the picture. The boundary consists of 6 unit segments, 3 of which run along the inside of the cup in a Y-pattern and 3 of which run along the outside of the cup in a Y-pattern. The whole picture has 3-fold rotational symmetry, so if you run your finger around either on the inside or the outside of the cup you will encounter the boundary in a kind of pinwheel fashion.

One can make the smooth approximations with 3-fold symmetry. Figure 6 shows that the boundary of such an approximation looks like.

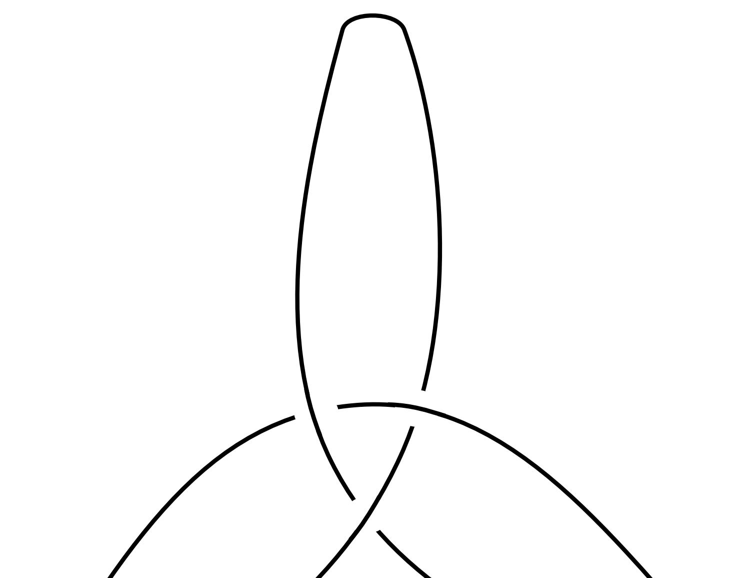

Figure 7: The boundary of a nearby smooth approximation.

3.3 Discussion

The existence of the cup means that the kinds of arguments made in [**S1**] and [**S2**], by themselves, are unlikely to resolve Conjecture 1.2. The idea in [**S1**] is to first show that a paper Moebius band has a pair of coplanar and perpendicular bends, called a *T-pattern*. After this, the idea is to consider the picture in the plane of the T-pattern and establish an optimization result. The idea in [**S2**] is to find a nice planar projection of a twisted paper cylinder and then do a similar kind of optimization trick. Both these approaches rely on planar ideas.

It is worth pointing out that [**S1**, Triangular Limit Theorem] implies the existence of some $\eta > 0$ such that any knotted paper Moebius band has aspect ratio at least $\sqrt{3} + \eta$. The point is that any paper Moebius band having aspect ratio very close to $\sqrt{3}$ is also very close to the triangular Moebius band. This would force the boundary of the paper Moebius band to be unknotted, a contradiction.

More directly, a multi-twisted paper Moebius band Ω has a T-pattern. If the aspect ratio of Ω is very near $\sqrt{3}$ then the convex hull ∇ of this T-pattern is close to an equilateral triangle of perimeter $2\sqrt{3}$. Since $\partial\Omega$ is knotted, the projection of $\partial\Omega$ could not just follow along $\partial\nabla$. This gives us the extra length and hence the better bound.

We did not try to find η, but we can say that such a scheme will never prove that $\sqrt{3} + \eta = 3$. Here is the problem: As a byproduct of such an argument we would also prove that any near minimizer for the aspect ratio is also nearly planar. But there are near minimizers which approximate the cup, and these are far from planar. This is a contradiction.

Another approach to Conjecture 1.2 would be to show that an arbitrary paper Moebius band could be deformed, through isometric embeddings, into a folded ribbon knot. Call such a paper Moebius band *flattenable*. Conjecture 1.2 seems easier in the category of folded ribbon knots, though probably still quite hard. If every paper Moebius band was flattenable, the folded ribbon knot case would imply the general case.

The cup is a rigid object. Presumably, nearby smooth approximations are also not flattenable. If they were, then we could probably extract a limit and contradict the rigidity of the cup. (We have not thought through the details of this.) In any case, one can ask: When is a paper Moebius band flattenable? We don't know any conditions which imply the answer one way or the other.

4 References

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