The Dehn-Sydler Theorem Explained

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1 Introduction

The Dehn-Sydler theorem says that two polyhedra in $\boldsymbol{R}^3$ are scissors congruent iff they have the same volume and Dehn invariant. Dehn [**D**] proved in 1901 that equality of the Dehn invariant is necessary for scissors congruence. 64 years passed, and then Sydler [**S**] proved that equality of the Dehn invariant (and volume) is sufficient. In [**J**], Jessen gives a simplified proof of Sydler's Theorem. The proof in [**J**] relies on two results from homological algebra that are proved in [**JKT**]. Also, to get the cleanest statement of the result, one needs to combine the result in [**J**] (about stable scissors congruence) with Zylev's Theorem, which is proved in [**Z**].

The purpose of these notes is to explain the proof of Sydler's theorem that is spread out in [**J**], [**JKT**], and [**Z**]. The paper [**J**] is beautiful and efficient, and I can hardly improve on it. I follow [**J**] very closely, except that I simplify wherever I can. The only place where I really depart from [**J**] is my sketch of the very difficult geometric Lemma 10.1, which is known as Sydler's Fundamental Lemma. To supplement these notes, I made an interactive java applet that renders Sydler's Fundamental Lemma transparent. The applet departs even further from Jessen's treatment.

[**JKT**] leaves a lot to be desired. The notation in [**JKT**] does not match the notation in [**J**] and the theorems proved in [**JKT**] are much more general than what is needed for [**J**]. In these notes, I'll re-write the proofs in [**JKT**] so that they exactly handle the cases needed for Sydler's Theorem. This makes the proofs easier.

It took me a long time to claw through the short paper [**Z**] (translated from Russian). I'll follow the logic of Zylev's paper closely, but improve the exposition.

2 Tensor Products

Let K be a field and let V and W be K-vector spaces. One defines the tensor product $V \otimes_K W$ as follows: Let X_1 denote the set of finite formal sums

$$\sum a_{ij}(v_i, w_j); \qquad a_{ij} \in K.$$

Here $(v_i, w_j) \in V \times W$. This is meant to be a formal sum. We will often write $V \otimes W$ in place of $V \otimes_K W$. As usual, $1(v, w)$ is shortened to (v, w) whenever it comes up. The various pairs (v_i, w_j) are not meant to be added together. X_1 is naturally a K-vector space. One has addition

$$\sum a_{ij}(v_i, w_j) + \sum b_{ij}(v_i, w_j) = \sum (a_{ij} + b_{ij})(v_i, w_j),$$

and scaling

$$k\Big(\sum a_{ij}(v_i, w_j)\Big) = \sum k a_{ij}(v_i, w_j).$$

In the case of addition, we allow some of the coefficients to be 0, so that the two summands involve the same pairs.

Let $X_2 \subset X_1$ be the span of the following elements.

- $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$.
- $(v, w_1 + w_2) - (v, w_1) - (v, w_2)$.
- $k(v, w) - (kv, w)$ and $k(v, w) - (v, kw)$.

Then

$$V \otimes W = X_1 / X_2,$$

the quotient vector space.

We have the special elements

$$v \otimes w = [(v, w)]. \tag{1}$$

In case $\{e_i\}$ and $\{e'_j\}$ are bases for V and W respectively, one can show fairly readily that $\{e_i \otimes e'_j\}$ is a basis for $V \otimes W$. In case L is a field extension of K, and V is additionally an L-vector space, then $V \otimes_K W$ is an L-vector space. The scaling law is defined by (linearly extending) the rule

$$\lambda(v \otimes w) = (\lambda v) \otimes w. \tag{2}$$

3 The Result

Say that a *dissection* of a polyhedron P is an expression $P = P_1 \cup ... \cup P_n$, where the polyhedra P_j and P_k have disjoint interiors for all $j \neq k$. Say that two polyhedra P and Q are *scissors congruent* if they have dissections $P = P_1 \cup ... \cup P_n$ and $Q = Q_1 \cup ... \cup Q_n$ such that P_k and Q_k are isometric for all k. In this case, we write $P \sim Q$. Our polyhedra are always closed subsets.

Both R and $\boldsymbol{R}/(\pi\boldsymbol{Q})$ are $\boldsymbol{Q}$-vector spaces. We define

$$\mathcal{W} = \boldsymbol{R} \otimes_{\boldsymbol{Q}} \boldsymbol{R}/(\pi\boldsymbol{Q}).$$

The Dehn invariant of a polyhedron P is given by

$$\Delta(P) = \sum_{i=1}^{n} l_i \otimes [\alpha_i] \in \mathcal{W}. \tag{3}$$

Here $[\alpha]$ is the equivalence class of α in $\boldsymbol{R}/(\pi\boldsymbol{Q})$. The sum takes place over all edges of P. Here l_i is the length of the ith edge, and α_i is the interior dihedral angle of that edge.

Theorem 3.1 (Dehn) *If* $P \sim Q$ *then* $\mathrm{vol}(P) = \mathrm{vol}(Q)$ *and* $\Delta(P) = \Delta(Q)$.

Proof: The statement about volume is obvious. One checks that the Dehn invariant is preserved by the dissection process, and also obviously under isometric motion. That is

$$\Delta(P) = \sum \Delta(P_i) = \sum \Delta(Q_i) = \Delta(Q). \tag{4}$$

The main point of the check is that the sum of the dihedral angles around an interior edge is 2π, which is 0 in $\boldsymbol{R}/(\pi\boldsymbol{Q})$. ♠

Here is the celebrated converse result.

Theorem 3.2 (Sydler) *If* $\mathrm{vol}(P) = \mathrm{vol}(Q)$ *and* $\Delta(P) = \Delta(Q)$ *then* $P \sim Q$.

In short

Theorem 3.3 (Dehn-Sydler) *Two polyhedra in* $\boldsymbol{R}^3$ *are scissors congruent iff they have the same volume and Dehn invariant.*

4 Zylev's Theorem

Zylev's Theorem, proved in [**Z**], works in great generality. Here we prove a special case. All sets are (possibly disconnected) polyhedra. We draw a schematic two dimensional picture, but the argument works in any dimension.

Theorem 4.1 *Let $A, B \subset F$. If $F - A \sim F - B$ then $A \sim B$.*

Proof: Subdividing $F - A$ and $F - B$ if necessary, we can find dissections

$$F = A \cup \bigcup_{k=1}^{n} P_k = B \cup \bigcup_{k=1}^{n} Q_k; \qquad \forall k: \ P_k \sim Q_k \text{ and } \mathrm{vol}(P_k) < \frac{\mathrm{vol}(A)}{2}$$

We call (A, B, F) a *good triple of order* n. Our proof goes by induction on n. If $n = 0$ it means that $F - A = F - B = \emptyset$, so that $A = B$. In this case, we are done. Consider the case $n > 0$.

Now we consider the induction step. The idea is to "eliminate" Q_n in some sense. The volume bound tells us that $A - Q_n$ has greater volume than Q_n. This means that we can fund disjoint $T_1, ..., T_n \subset A - Q_n$ such that

$$T_k \sim P_k \cap Q_n.$$

Figure 1 shows the case $n = 3$. Note that the sets T_k need not be connected. But, each T_k is a finite union of polyhedra. We have schematically drawn T_1 and T_2 just below P_1 and P_2 respectively, though in actuality the arrangement could be much more complicated. Now define

$$F' = F - Q_n; \qquad Q'_k = Q_k; \qquad B' = F' - \bigcup_{k=1}^{n=1} Q'_k;$$

$$P'_k = (P_k - Q_n) \cup T_k \sim Q'_k; \qquad A' = F' - \bigcup_{k=1}^{n-1} P'_k; \tag{5}$$

No relevant volume changes. Hence (A', B', F') is a good triple of order $n - 1$. We have $B' = B$, and by induction $A' \sim B'$. We just need to show $A' \sim A$. The idea is to introduce an intermediate region A'' and show that both $A'' \sim A'$ and $A'' \sim A$.

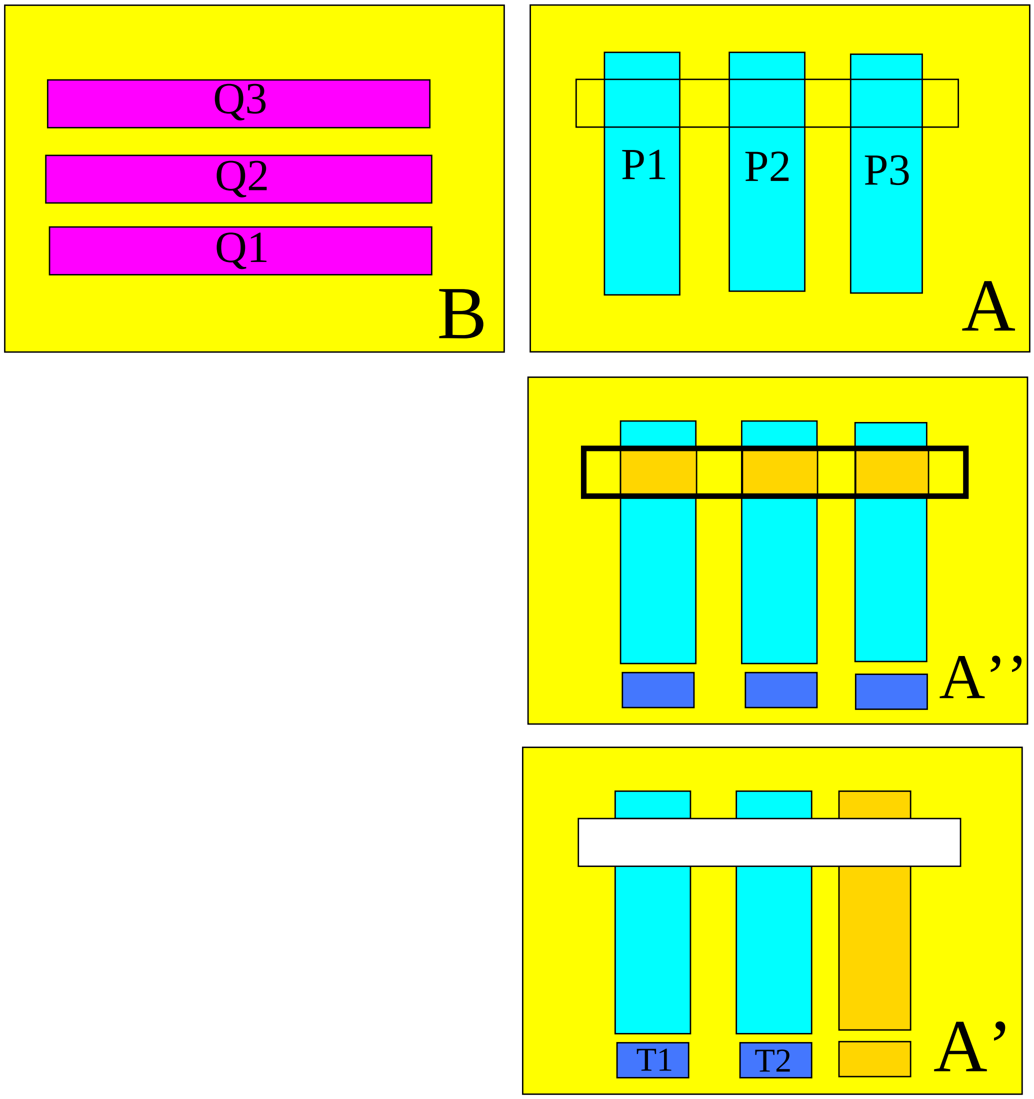

Figure 1: Induction Step

We define A'' as in Figure 1. We have colored this set yellow and gold. A'' is obtained from A by exchanging T_k with $Q_n \cap P_k$ for all k. Hence $A \sim A''$. We get A' from A'' by dissecting the yellow/gold Q_n and moving the pieces into $(P_n - Q_n) \cup T_n$. This is possible because $Q_n \sim (P_n - Q_n) \cup T_n$. Hence $A'' \sim A'$. Hence $A \sim A'$. ♠

5 Stable Scissors Congruence

Let $\mathcal{P}$ denote the abelian group of formal finite sums

$$a_1 P_1 + ... + a_n P_n; \qquad a_k \in \boldsymbol{Z} \tag{6}$$

where $P_1, ..., P_n$ are polyhedra in $\boldsymbol{R}^3$. The volume and Dehn invariant extend linearly to all of $\mathcal{P}$. Let $\mathcal{E}$ be the subgroup of $\mathcal{P}$ generated by the relations of the form

- $P - (P_1 + ... + P_n)$, where $P_1 \cup ... \cup P_n$ is a dissection of P.
- $P - I(P)$, where I is a Euclidean isometry.

We call P and Q *stable scissors congruent* (SSC) if $P \equiv Q \bmod \mathcal{E}$.

Lemma 5.1 *$P \sim Q$ if and only if P and Q are SSC.*

Proof: It is pretty obvious that scissors congruence implies SSC. For the interesting half of the result, suppose P, Q are SSC. Spreading the sum in $\mathcal{P}$ so that all coefficients are ± 1, we get

$$P - Q = \sum (R_i - \sum R_{ij}) - \sum (S_i - \sum S_{ij}) + \sum \Big(T_i - U_i\Big). \tag{7}$$

The first two terms on the right collect all the dissection relations and the third term on the right collects all the isometry relations. Hence

$$P + \sum R_i + \sum S_{ij} + \sum T_i = Q + \sum S_i + \sum R_{ij} + \sum U_i.$$

Suitably translating the polyhedra on each of this equation, we find polyhedra P' and Q' such that

$$P \cap P' = Q \cap Q' = \emptyset; \qquad P' \sim Q'; \qquad P \cup P' \sim Q \cup Q'.$$

Let $A = P$ and $F = P \cup P'$. Let θ be the piecewise isometry that carries $Q \cup Q'$ to $P \cup P'$. Let $B = \theta(Q)$. Then $B \sim Q$ and θ defines a scissors congruence between $F - B$ and Q'. Hence $F - A = P' \sim Q' \sim F - B$. By Zylev's Theorem, $A \sim B$. Hence $P \sim Q$. ♠

In light of Lemma 5.1, the following result finishes the proof of the Dehn-Sydler Theorem.

Theorem 5.2 *If $\mathrm{vol}(P) = \mathrm{vol}(Q)$ and $\Delta(P) = \Delta(Q)$, then P, Q are SSC.*

6 Introducing Prisms

A *simple prism* is any polyhedron affinely equivalent to the product

$$T \times I \subset \boldsymbol{R}^2 \times \boldsymbol{R} = \boldsymbol{R}^3.$$

Here T is a triangle and I is an interval.

Lemma 6.1 *A simple prism is SSC to* $[0,1] \times [0,1] \times [0,v]$. *Here* v *is the volume of the prism.*

Proof: Here is a sketch of the argument. Stack an infinite number of copies of the prism on top of each other. The result is a triangular column. A $\boldsymbol{Z}$-subgroup of translations acts on this column. A set of the form $T \times I$ is a fundamental domain for the action, and one sees easily that $T \times I$ is SSC to the original prism. Use 2-dimensional cut-and-paste arguments to show that $T \times I$ is equivalent to a block of the form $I_1 \times I_2 \times I$. Again use 2 dimensional arguments to show that $I_1 \times I_2 \times I$ is equivalent to $[0,1] \times [0,1] \times [0,v]$. ♠

Let $\mathcal{F}$ denote the subgroup of $\mathcal{P}$ generated by $\mathcal{E}$ and by the simple prisms. By Dehn's Theorem (or direct calculation) the Dehn invariant of any simple prism is 0. Also, we have already mentioned that Δ vanishes on $\mathcal{E}$. Hence, Δ vanishes identically on $\mathcal{F}$. Thus, Δ induces a map

$$\delta : \mathcal{V} \to \mathcal{W}; \qquad \mathcal{V} = \mathcal{P}/\mathcal{F}. \tag{8}$$

Lemma 6.2 *If* δ *is an injection, then Theorem 5.2 is true.*

Proof: Suppose that $P, Q \in \mathcal{E}$ have the same volume and Dehn invariant. Let $\langle P \rangle$ and $\langle Q \rangle$ denote the equivalence classes in $\mathcal{V}$. Since δ is an injection, we have $\langle P \rangle = \langle Q \rangle$. That is, $\langle P - Q \rangle = 0$. But then, we can write

$$P - Q = R + S - S'$$

where $R \in \mathcal{E}$ and S, S' are positive sums of simple prisms. Since $P - Q$ and R both have volume 0, the sums S and S' have the same volume. But then, by Lemma 6.1, we have $S - S' \in \mathcal{E}$. Hence $P - Q \in \mathcal{E}$. Hence $P = Q \bmod \mathcal{E}$. ♠

7 Vector Space Structure

As we already mentioned (for general extension fields) the operation given by $r(a \otimes b) = (ra) \otimes b$ makes $\mathcal{W}$ into a real vector space. Here we explain how $\mathcal{V}$ is a real vector space. There is an obvious scaling operation on $\mathcal{P}$. Given $\lambda \in \boldsymbol{R}$ and a polyhedron P one defines λP to be the result of scaling P by a factor of λ about the center of mass of P. This scaling operaton then extends to all of $\mathcal{P}$ by linearity. The scaling operation makes sense on $\mathcal{V}$, and the following things are trivial to verify:

- $\lambda(V_1 + V_2) = \lambda V_1 + \lambda V_2$.
- $\lambda_1\lambda_2(V) = \lambda_1(\lambda_2(V))$.

What is more subtle, we must also show that

$$(\lambda_1 + \lambda_2)v = \lambda_1 v + \lambda_2 v; \qquad \forall v \in \mathcal{V}. \tag{9}$$

Any polyhedron can be dissected into tetrahedra. Hence $\mathcal{P}$ is generated by the tetrahedra. To establish Equation 9 it suffices to to show, for an arbitrary tetrahedron T, that $(\lambda_1 + \lambda_2)T$ can be dissected into two prisms, and translates of $\lambda_1 T$ and $\lambda_2 T$.

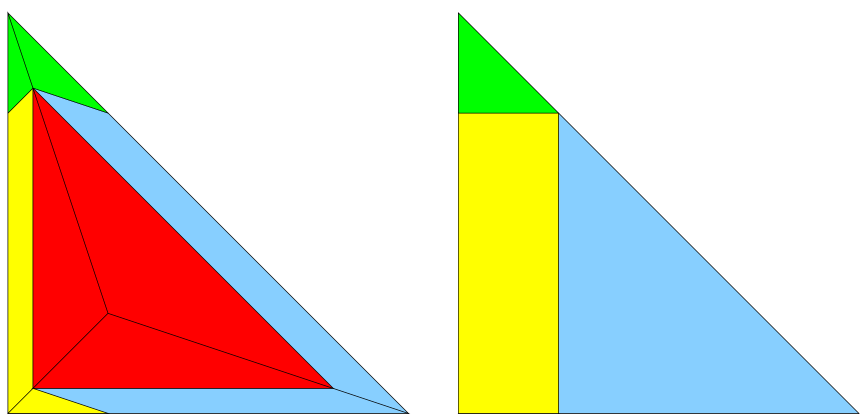

Figure 3: Projection to the plane

Start with $(\lambda_1 + \lambda_2)T$. Stick $\lambda_1 T$ (red) into one corner and $\lambda_2 T$ (green) into another corner. Then observe that what is left is the union of 2 prisms (yellow, blue). Figure 3 shows the view of the top 3 faces on the left, and the view of the bottom face on the right.

Now we observe that $\delta : \mathcal{V} \to \mathcal{W}$ is a real linear map, relative to these vector space structures.

8 A Reformulation of the Result

An *orthoscheme* is any tetrahedron isometric to the convex hull of the points

$$(0,0,0); \qquad (x,0,0); \qquad (x,y,0); \qquad (x,y,z).$$

All the faces of an orthoscheme are right-angled triangles, and 3 of the dihedral angles are $\pi/2$. The Dehn invariant of an orthoscheme is the 3-term relation in Lemma 9.2 below. Any tetrahedron T can be dissected into 24 orthoschemes. Each orthoscheme in this dissection is the convex hull of the center of the inscribed sphere, the projection of this center to some face of T, the projection of the center to some edge of T, and a vertex of T. The orthoschemes in this dissection come in 12 pairs. Each two in a pair are mirror reflections of each other. In particular, any tetrahedron is scissors congruent to its mirror reflection.

Say that a *good function* is a map $\phi : \boldsymbol{R} \to \mathcal{V}$ such that

- (P1) $\phi(\pi) = 0$.
- (P2) $\phi(a+b) = \phi(a) + \phi(b)$ for all $a, b \in \boldsymbol{R}$.
- (P3) For any orthoscheme T, we have $\langle T \rangle = \sum_{i=1}^{6} l_i \phi(\alpha_i)$.

The sum takes place over edges of T.

Lemma 8.1 *The existence of a good function implies Theorem 5.2.*

Proof: Being an abelian homomorphism from $\boldsymbol{R}$ to $\mathcal{V}$, the map ϕ is actually $\boldsymbol{Q}$-linear. In particular, ϕ vanishes on $\pi\boldsymbol{Q}$. Hence ϕ induces a well defined $\boldsymbol{Q}$-linear map $\boldsymbol{R}/(\pi\boldsymbol{Q})$ to $\mathcal{V}$. Now we define $\Phi : \mathcal{W} \to \mathcal{V}$ by linearly extending the rule $\Phi(x \otimes y) = x\phi(y)$. One checks easily that this is well-defined.

Comparing P3 above to the definition of the Dehn invariant, we have

$$\Phi \circ \delta(\langle T \rangle) = \Phi \circ \Delta(T) = \langle T \rangle \tag{10}$$

for any orthoscheme T. Hence $\Phi \circ \delta$ is the identity map on (classes of) orthoschemes. An arbitrary polyhedron can be dissected into tetrahedra, and hence into orthoschemes. Hence $\mathcal{V}$ is generated by the orthoschemes. Hence, $\Phi \circ \delta$ is the identity on $\mathcal{V}$. This is only possible if δ is injective. But this implies Theorem 5.2. ♠

9 Constructing the Good Function

Let
$$c' = \sqrt{(1/c) - 1}. \tag{11}$$
For $a, b \in (0,1)$, let $T(a,b)$ denote the orthoscheme whose vertices are
$$(0,0,0); \qquad (a',0,0); \qquad (a',a'b',0); \qquad (a',a'b',b'). \tag{12}$$
In the next section we will produce a function $h : (0,1) \to \mathcal{V}$ such that

- (H1) $\langle T(a,b)\rangle = h(a) + h(b) - h(ab)$.
- (H2) If $a + b = 1$ then $ah(a) + bh(b) = 0$.

We call h the *homological function*. One might say that h is the input from the homological algebra component of the proof. Assume for now that h exists. Define
$$\phi\left(\frac{n\pi}{2}\right) = 0; \qquad \text{and otherwise} \qquad \phi(\alpha) = \tan(\alpha)h(\sin^2(\alpha)). \tag{13}$$
Note that
$$\phi(\pi) = 0; \qquad \phi(\pi - \alpha) = -\phi(\alpha); \qquad \phi(\alpha + n\pi) = \phi(\alpha). \tag{14}$$
The first equation is just property P1.

Lemma 9.1 *Suppose that $\alpha + \beta = \pi/2$. Then $\phi(\alpha) + \phi(\beta) = 0$.*

Proof: Let $a = \sin^2(\alpha)$ and $b = \sin^2(\beta)$. Note that $h(a) = \phi(\alpha)\cot(\alpha)$, and similarly for b. We have $a + b = \sin^2(\alpha) + \sin^2(\beta) = 1$. Hence,
$$0 = ah(a) + bh(b) = \sin^2(\alpha)\phi(\alpha)\cot(\alpha) + \sin^2(\beta)\phi(\beta)\cot(\beta) =$$
$$\cos(\alpha)\sin(\alpha)\phi(\alpha) + \cos(\beta)\sin(\beta)\phi(\beta) = X(\phi(\alpha) + \phi(\beta)).$$
Here we have set
$$X = \cos(\alpha)\sin(\alpha) = \frac{1}{2}\sin(2\alpha) =^* \frac{1}{2}\sin(2\beta) = \cos(\beta)\sin(\beta).$$
The starred equality comes from the fact that $2\alpha + 2\beta = \pi$. Cancelling the X gives our result. ♠

We pause to explain the logic of the argument. Before showing that ϕ satisfies P2, we will show that ϕ satisfies P3. Then, using property P1, property P3, Equation 14, and the conclusion of Lemma 9.1, we will establish P2.

Lemma 9.2 *Let $\alpha, \beta, \alpha * \beta \in (0, \pi/2)$ be such that*

$$\sin^2(\alpha) = a; \qquad \sin^2(\beta) = b; \qquad \sin^2(\alpha * \beta) = ab.$$

Then

$$\Delta(T(a,b)) = \cot(\alpha) \otimes \alpha + \cot(\beta) \otimes \beta + \cot(\alpha * \beta) \otimes (\pi/2 - \alpha * \beta). \quad (15)$$

Proof: We've already mentioned that $T(a,b)$ has 3 right dihedral angles. It is just a matter of trigonometry to compute that the other lengths are $\cot(\alpha)$, $\cot(\beta)$, and $\cot(\alpha * \beta)$,and that the corresponding dihedral angles are α, β, and $\pi/2 - \alpha * \beta$. ♠

Lemma 9.3 *ϕ satisfies P3.*

Proof: Recalling the formula in Lemma 9.2, we have

$$h(a) = \cot(\alpha)\phi(\alpha); \qquad h(b) = \cot(\beta)\phi(\beta);$$

We also have

$$h(ab) = \cot(\alpha * \beta)\phi(\alpha * \beta) = -\cot(\alpha * \beta)\phi(\pi/2 - \alpha * \beta).$$

The last equality comes from Lemma 9.1. Putting everything together, we have

$$\langle T(a,b) \rangle = h(a) + h(b) - h(ab) =$$

$$\cot(\alpha)\phi(\alpha) + \cot(\beta)\phi(\beta) + \cot(\alpha * \beta)\phi(\pi/2 - \alpha * \beta) = \sum_{i=1}^{6} l_i \phi(\alpha_i).$$

The other three terms in the last sum vanish, because the corresponding dihedral angles are right angles. ♠

Lemma 9.4 *Let $\alpha, \beta, \gamma \in (0, \pi/2)$ be such that $\alpha + \beta + \gamma = \pi$. Then it is possible to dissect a rectangular solid into 6 orthoschemes, all sharing a common diagonal of the solid, such that the dihedral angles (in pairs) associated to this diagonal are α, β, γ.*

Proof: One can orthogonally project an orthoscheme into the plane so that the longest diagonal maps to a point. The image of the orthoscheme is a triangle. At the same time, one can orthogonally project a rectangular solid R into the plane so that one of the diagonals maps to a point. The resulting figure is a hexagon with opposite sides parallel, and with each main diagonal parallel to a pair of opposite sides.

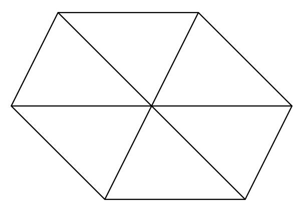

Figure 4: Projection to the plane

The 6 triangles in Figure 4 are all projections of the orthoschemes from Lemma 9.4. The angles of these triangles, around the central vertex, correspond to the dihedral angles around the diagonal of interest. Adjusting R, we can make the three angles whatever we like, subject to the given constraints. ♠

Let R be the rectangular solid from Lemma 9.4. We have $\langle R\rangle = 0$ in $\mathcal{V}$. We scale R so that its diagonals have length 1. We apply P3 to the 6 orthoschemes involved to get:

$$0 = \langle R\rangle = \Big(2\phi(\alpha) + 2\phi(\beta) + 2\phi(\gamma)\Big) + \sum_{k=1}^{6} l_k\Big(\phi(\theta_{k1}) + \phi(\theta_{k2})\Big). \tag{16}$$

The first summand comes from the angles around the diagonal of interest to us. The remaining nonzero terms are grouped into 6 pairs, corresponding to the edges of R that project to the "spokes" of the hexagon above. Each pair of angles adds to $\pi/2$. Applying Lemma 9.1 and summing over these 6 edges, we find that the last sum in Equation 16 vanishes. Hence

$$\phi(\alpha) + \phi(\beta) + \phi(\gamma) = 0. \tag{17}$$

This works for any $\alpha, \beta, \gamma \in (0, \pi/2)$ that sum to π. Combining this information with Equation 14, and Lemma 9.1, we get P2.

10 Constructing the Homological Function

Let $F : (0,1)^2 \to \mathcal{V}$ be given by

$$F(a,b) = \langle T(a,b) \rangle \in \mathcal{V}. \tag{18}$$

Since $T(a,b)$ and $T(b,a)$ are isometric, we have $F(a,b) = F(b,a)$. The following input from geometry is called Sydler's Fundamental Lemma.

Lemma 10.1 *For any $a,b,c \in (0,1)$ these two elements are equal in $\mathcal{V}$:*

$$T(a,b) + T(ab,c); \qquad T(a,c) + T(ac,b)$$

Now we invoke the first result from [**JKT**]:

Theorem 10.2 *Let V be a real vector space. Let $F : (0,1)^2 \to V$ be a function satisfying*

- *$F(a,b) = F(b,a)$ for all a,b.*
- *$F(a,b) + F(ab,c) = F(a,c) + F(ac,b)$ for all a,b,c.*

Then there is some function $f : (0,1) \to V$ such that

$$F(a,b) = f(a) + f(b) - f(ab).$$

Our function F satisfies these relations, so there is some $f : (0,1) \to \mathcal{V}$ satisfying H1.

Note that $f - g$ also satisfies H1 if $g : \boldsymbol{R}_+ \to \mathcal{V}$ is a *homomorphism*:

$$g(ab) = g(a) + g(b). \tag{19}$$

To get property H2, the idea is to subtract off a homomorphism that encodes some additional geometry of orthoschemes.

Let $G : \boldsymbol{R}_+^2 \to \mathcal{V}$ be the function

$$G(a,b) = af\Big(\frac{a}{a+b}\Big) + bf\Big(\frac{b}{a+b}\Big) \tag{20}$$

We have $G(a,b) = G(b,a)$ and $G(\lambda a, \lambda b) = \lambda G(a,b)$ and

$$a + b = 1 \qquad \Longrightarrow \qquad G(a,b) = af(a) + bf(b). \tag{21}$$

Lemma 10.3 *$G(a,b)+G(a+b,c) = G(a,c)+G(a+c,b)$ for all $a,b,c \in \boldsymbol{R}_+$.*

When we expand out this relation in terms of f and groups terms, we see that it boils down to the identity

$$aF\Big(\frac{a+b}{a+b+c}, \frac{a}{a+b}\Big) + bF\Big(\frac{a+b}{a+b+c}, \frac{b}{a+b}\Big) =$$
$$aF\Big(\frac{a+c}{a+b+c}, \frac{a}{a+c}\Big) + cF\Big(\frac{a+c}{a+b+c}, \frac{c}{a+c}\Big).$$

This identity comes down to a certain dissection problem. Say that the *corner* $C(x,y,z)$ is the tetrahedron with vertices

$$O=(0,0,0); \qquad X=(x,0,0); \qquad Y=(0,y,0); \qquad Z=(0,0,z).$$

A corner can be divided into two orthoschemes in three different ways. For example one lets Π_X be the plane containing OX and also the point $P_X \in \overline{YZ}$ that $\overline{O_P X}$ is perpendicular to $\overline{YZ}$. Then Π_X divides the $C(x,y,z)$ into two orthoschemes. One gets two other such divisions by permuting the letters. Equating any two of these subdivisions in $\mathcal{V}$ gives a relation. When

$$x=(bc)^{1/2}; \qquad y=(ac)^{1/2}; \qquad z=(ab)^{1/2}$$

and we use Π_X and Π_Y, we get the above relation after some computation. ♠

Now we invoke the second result from [**JKT**].

Theorem 10.4 *Let $G: \boldsymbol{R}_+^2 \to V$ be a function satisfying*

- *$G(a,b) = G(b,a)$ for all a,b.*
- *$G(\lambda a, \lambda b) = \lambda G(a,b)$ for all λ, a, b.*
- *$G(a,b) + G(a+b,c) = G(a,c) + G(a+c,b)$.*

Then there is a homomorphism $g: (0,\infty) \to V$ such that

$$a+b=1 \qquad \Longrightarrow \qquad G(a,b) = ag(a)+bg(b). \tag{22}$$

The function $h = f - g$ clearly satisfies H2 as well as H1.

11 Sketch of Lemma 10.1

To be sure, Jessen checks that $T(a,b)+T(ab,c)$ and $T(a,c)+T(ac,b)$ have the same volume and Dehn invariant. The volume calculation is elementary. The Dehn invariant calculation follows from Lemma 9.2.

As a prelude to the main construction, Jessen identifies a certain family of polyhedra which I'll call *pseudoprisms*. A pseudoprism $OPQRS$ satisfies $|QS| = 2|PR|$ and projects orthogonally to an isosceles triangle, as shown in the top left corner of Figure 5. By chopping off a suitable tetrahedron from a pseudoprism and gluing it in a different way, one obtains a prism. Figure 5 shows two views of a pseudoprism and the corresponding prism.

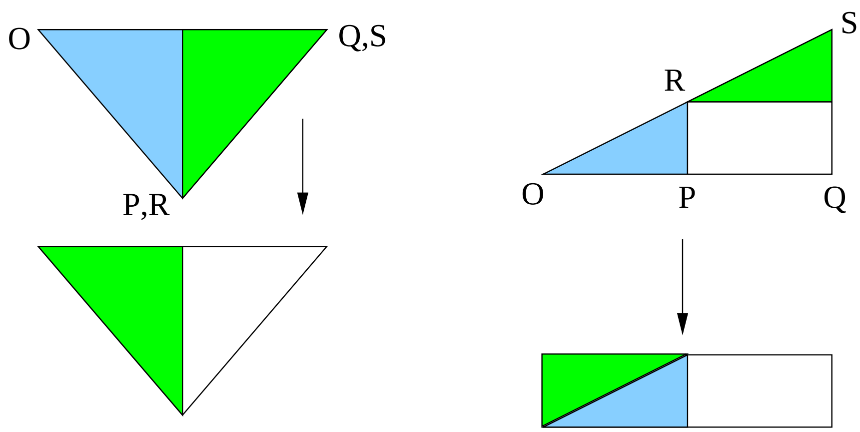

Figure 5: Pictures of a pseudoprism

Now we come to the main construction. Things are arranged so that $T(a,b)$ is given vertices $ABCD$ and $T(a,c)$ is given vertices $ABEF$, and

- BCD and BEF lie in a plane Π_0.
- ABC and ABF lie in a plane Π_1.
- ABD and ABE lie in a plane Π_2.
- Π_1 and Π_2 are orthogonal to Π_0.

(Note: Jessen considers the case when BC and EF do not cross. The crossing case is similar.) Figure 6 shows the orthogonal projections to each of these planes. The blue-labelled points lie in the relevant plane, and the black-labelled points lie to one side. Some points get more than one label. Only the projection to Π_0 shows all the points.

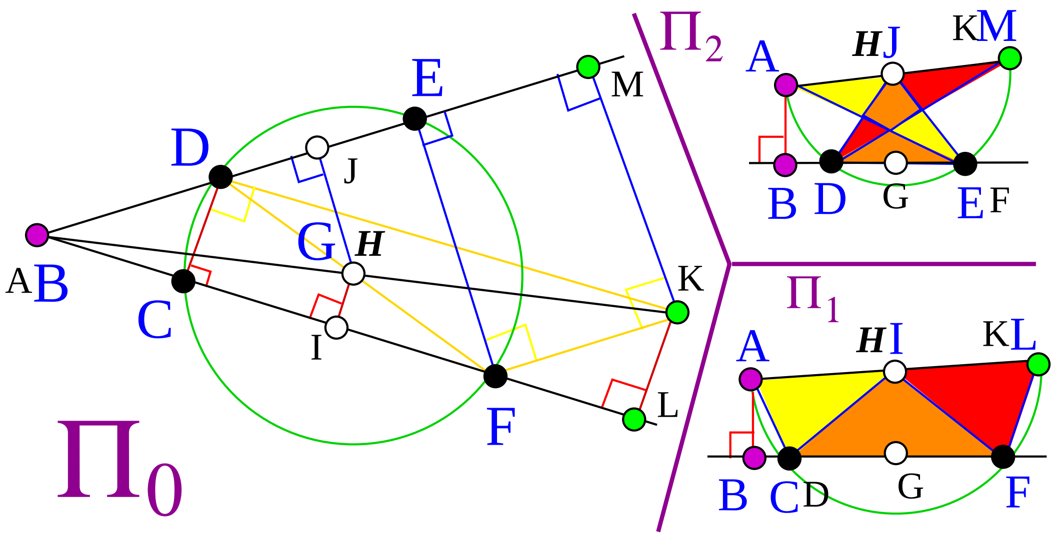

Figure 6: Projections

Again,

$$T(a,b) = ABCD; \qquad T(a,c) = ABEF.$$

The points in this figure obey certain distance relations:

- $ACDEFKLM$ are all equidistant from H.
- G, I, J are the projections of H respectively to Π_0, Π_1, Π_2.
- $CDEF, ACFL, ADEM$ are equidistant from G, I, J respectively.
- DGF, AHK, AIL, AJM are each evenly spaced and collinear.
- $|MK| = |EF|$ and $|KL| = |CD|$
- angle(AMD) = angle(AED) and angle(ALF) = angle(ACB)

These relations, especially the last two, imply

$$T(ab,c) = ADMK; \qquad T(ac,b) = AFLK.$$

Now let P be the polyhedron with vertics $ABDFHIJ$. By comparing the boundaries of the various polyhedra involved, one shows that

$$P - (AICHD) - (FICHD) + (DJMHK) = T(a,b) + T(ab,c)$$

$$P + (DJEHF) - (AJEHF) + (FILHK) = T(a,c) + T(ac,b)$$

where the 6 polyhedra (...) are the pseudoprisms in Figure 6.

12 Proof of Theorem 10.2

12.1 A Technical Lemma

Lemma 12.1 *Let $F : \boldsymbol{R}_+^2 \to V$ be a function satisfying*

- $F(a,b) = F(b,a)$ *for all* a,b.
- $F(a,b) + F(ab,c) = F(a,c) + F(ac,b)$ *for all* a,b,c.
- $F(a,1) = F(a,1/a) = 0$ *for all* a.

Then $F(a,b) = f(a) + f(b) - f(ab)$ *for some* $f : \boldsymbol{R}_+ \to V$.

Let $W = \boldsymbol{R}_+ \times V$. Equip W with the operation

$$(a,x) + (b,y) = (ab, x + y + F(a,b)). \tag{23}$$

Lemma 12.2 *W is an abelian group.*

Since $F(a,b) = F(b,a)$, the operation is commutative. The operation is associative because the two quantities below are equal.

$$(a,x) + \Big((b,y) + (c,z)\Big) = (a + b + c, x + y + z, F(b,c) + F(bc,a)).$$

$$\Big((a,x) + (b,y)\Big) + (c,z) = (a + b + c, x + y + z, F(b,a) + F(ba,c)).$$

$(1,0)$ is the zero element because $F(1,a) = 0$, and $(a,x)^{-1} = (1/a, x)$. ♠

Lemma 12.3 *Let $\pi : W \to \boldsymbol{R}_+$ be projection. Suppose there is a subgroup $S \subset W$ such that $\pi : S \to \boldsymbol{R}_+$ is a bijection. Then Lemma 12.1 is true.*

Proof: Let $\pi_V : W \to V$ be projection. Define $f(a) = -\pi_V \circ \pi^{-1}(a)$. Let $x = f(a)$ and $y = f(b)$. Since π is a homomorphism, bijective on S, the map $\pi^{-1} : \boldsymbol{R}^+ \to S$ is a group isomorphism. The following calculation finishes the proof.

$$-f(ab) = \pi_V \circ \pi^{-1}(ab) = \pi_V \circ \Big((a,x) + (b,y)\Big) =$$

$$\pi_V \circ (ab, x + y + F(a,b)) = -f(a) - f(b) + F(a,b). \quad ♠$$

To finish the proof of Lemma 12.1, we need to produce the subgroup S. Let $A = \boldsymbol{R}_+$ for this proof. Here the group law is multiplication. The proof goes by transfinite induction. Note that $\pi(1,0) = 1$. So, we start with the two trivial subgroups:

$$S_0 = \{(1,0)\}; \qquad A_0 = \{1\}.$$

Now we consider the induction step. Let $S_1 \subset W$ be a subgroup of W and let $A_1 = \pi(S_1)$. Choose $a \in A - A_1$. Let $A_2 = \langle A_1, a\rangle$ be the subgroup generated by A_1 and a. Let

$$S_2 = \langle S_1, (a,x)\rangle.$$

For any choice of x, the sets A_2 and S_2 are subgroups and $\pi(S_2) = A_2$. We just have to choose x so that $\pi : S_2 \to A_2$ is injective. Any element of $\pi^{-1}(0)$ has the form

$$[S_1] + n(a,x) = [S_1] + \Big(a^n, nx + \sum_{i=1}^{n-1} F(a,a^i)\Big). \tag{24}$$

Here $[S_1]$ denotes some element of S_1. Note also that $a^n = -\pi([S_1]) \in A_1$. If we knew that $n(a,x) \in S_1$ then we could conclude that the above element equals 0_W, because $\pi : S_1 \to A_1$ is injective. So, we are reduced to the following problem: Choose x such that

$$a^n \in A_1 \qquad \Longrightarrow \qquad n(a,x) \in S_1. \tag{25}$$

If the left side of Equation 25 never happens, then any choice of x will work. Otherwise, since S_1 is a subgroup, just have to solve the problem for the minimum positive n such that $a^n \in A_1$. For this value of n, there is some unique $v = v_n \in V$ such that

$$(a^n, v) \in S_1. \tag{26}$$

We can solve for x:

$$x = \frac{1}{n}\Big(v - \sum_{i=1}^{n-1} F(a,a^i)\Big) \tag{27}$$

This choice of x does the job.

12.2 The Extension

Suppose that $F : (0,1) \to V$ satisfies the hypotheses of Theorem 10.2. To apply Lemma 12.1, we just have to show that F extends to some $\widehat{F} : \boldsymbol{R}_+ \to V$ satisfying the hypotheses of Lemma 12.1. Then we let $\widehat{f} : \boldsymbol{R}_+ \to V$ be the function from Lemma 12.1, and we let f be the restriction of $\widehat{f}$ to $(0,1)$.

Let

$$T(a,b) = (ab, 1/a). \tag{28}$$

T and its powers define an order 6 group action on $\boldsymbol{R}^2_+$, and the domain of definition of F, namely $\{(a,b) |\ a < 1, b < 1\}$, is a fundamental domain. We define $\widehat{F}$ in such a way that $\widehat{F} \circ T = -\widehat{F}$. Concretely, we introduce the symbol $(\pm,\pm,\pm)$ according to the sign of $(1-a, 1-b, 1-ab)$. Our extension is as follows:

- $+++$: $\widehat{F}(a,b) = +F(a,b)$.
- $-++$: $\widehat{F}(a,b) = -F(ab, 1/a)$.
- $-+-$: $\widehat{F}(a,b) = +F(b, 1/(ab))$.
- $---$: $\widehat{F}(a,b) = -F(1/a, 1/b)$.
- $+--$: $\widehat{F}(a,b) = +F(1/(ab), a)$.
- $+-+$: $\widehat{F}(a,b) = -F(ab, 1/b)$.

A case by case analysis shows that the second hypotheses of Lemma 12.1 is also satisfied. For instance, suppose that the triple (a,b,ab) and (a,c,ac) are of types $(-,+,+)$ and $(-,+,-)$ and $abc < 1$ Then (ab,c) is of type $(+,+,+)$ and (ac,b) is of type $(-,+,+)$. Then

$$\widehat{F}(a,b) + \widehat{F}(ab,c) = -F(ab,1/a) + F(ab,c);$$

$$\widehat{F}(a,c) + \widehat{F}(ac,b) = F(c,1/(ac)) - F(abc,1/(ac)).$$

Using symmetry, we see that the two relevant quantities are equal iff

$$F(c,ab) + F(cab,1/(ac)) = F(c,1/(ac)) + F(1/a,ab).$$

Setting $\alpha = c$ and $\beta = ab$ and $\gamma = 1/(ac)$, we find that the last expression is the same as $F(\alpha,\beta) + F(\alpha\beta,\gamma) = F(\alpha,\gamma) + F(\alpha\gamma,\beta)$, which is true given the properties of F. The other cases are similar. In [**JKT**], a more conceptual (and difficult) argument is also given, which eliminates the case-by-case checking.

13 Proof of Theorem 10.4

13.1 A Variant

The following variant of Theorem 10.4 is proved in [**JKT**].

Theorem 13.1 *Let $G : (0,\infty)^2 \to \boldsymbol{R}$ be a function satisfying the hypotheses of Theorem 10.4. Then there is a function $\gamma : (0,\infty) \to V$ such that*

$$G(a,b) = \gamma(a+b) - \gamma(a) - \gamma(b); \qquad \gamma(ab) = b\gamma(a) + a\gamma(b). \tag{29}$$

Let's deduce Theorem 10.4 from Theorem 13.1. Let $g(x) = -\gamma(x)/x$. We compute

$$g(ab) = -\frac{\gamma(ab)}{ab} = -\frac{b\gamma(a)}{ab} - \frac{a\gamma(b)}{ab} = -\frac{\gamma(a)}{a} - \frac{\gamma(b)}{b} = g(a) + g(b).$$

In general, we have

$$G(a,b) = ag(a) + bg(b) - (a+b)g(a+b).$$

Note that $g(1) = 0$. So, when $a+b=1$, we have $G(a,b) = ag(a) + bg(b)$. This shows that Theorem 13.1 implies Theorem 10.4.

13.2 Another Technical Lemma

Lemma 13.2 *Let $G : \boldsymbol{R}^2 \to V$ be a function satisfying the hypotheses of Theorem 10.4, and also $G(a,0) = G(a,-a) = 0$ for all a. Then there exists a map $\gamma : \boldsymbol{R} \to V$ such that*

$$G(a,b) = \gamma(a+b) - \gamma(a) - \gamma(b); \qquad \gamma(ab) = b\gamma(a) + a\gamma(b).$$

Now we turn to the proof of Lemma 13.2. On $W = \boldsymbol{R} \times V$ define

$$(a,x)+(b,y) = (a+b, x+y+G(x,y)); \qquad (a,x)(b,y) = (ab, bx+ay). \tag{30}$$

Lemma 13.3 *W is a commutative ring with 1.*

Proof: The same argument, with $(\boldsymbol{R},+)$ in place of $(\boldsymbol{R}_+,\times)$ shows that W, equipped with the first operation, is an abelian group. The 0 element is $(0,0)$. A short calculation shows that the multiplication law is commutative and associative, and that $(1,0)$ is the "one" of the ring. Finally, the condition $G(\lambda a, \lambda b) = \lambda G(a,b)$ translates into the distributive law. ♠

Lemma 13.4 *Let $\pi : W \to \boldsymbol{R}$ be projection. Suppose that there is a subring $S \subset W$ such that $\pi : S \to \boldsymbol{R}$ is a bijection. Then Lemma 13.2 is true.*

Proof: Let $\pi_V : W \to V$ be projection. Define $\gamma(a) = \pi_V \circ \pi^{-1}(a)$. The same argument as in Lemma 12.3 shows that $G(a,b) = \gamma(a+b) - \gamma(a) - \gamma(b)$. This is the first condition. For the second condition, let $x = \gamma(a)$ and $y = \gamma(b)$. Since π is a ring homomorphism, bijective on S, the map $\pi^{-1} : \boldsymbol{R} \to S$ is a ring isomorphism. We have

$$\gamma(ab) = \pi_V \circ \pi^{-1}(ab) = \pi_V(\pi^{-1}(a) \times \pi^{-1}(b)) =$$

$$\pi_V\Big((a,x)(b,y)\Big) = \pi_V(ab, ay + bx) = ay + bx = a\gamma(b) + b\gamma(a).$$

This does it. ♠

To finish the proof, we have to produce the subring $S \subset W$ such that $\pi : S \to \boldsymbol{R}$ is a bijection. For this proof, we set $A = \boldsymbol{R}$. We set $S_0 = \boldsymbol{Z}\mathbf{1}$, where $\mathbf{1}$ is the unity of the ring W. As the same time, we set $A_0 = \boldsymbol{Z} \subset A$. So far, S_0 and A_0 are subrings and $\pi : S_0 \to A_0$ is a bijection. We will again argue by transfinite induction.

Suppose by induction that we have a subring $S_1 \subset S$ and a corresponding subring $A_1 \subset A$, such that $\pi : S_1 \to A_1$ is an isomorphism. Choose some $a \in A - A_1$. Let

$$A_2 = A[a]; \qquad S_2 = A[(a,x)] \tag{31}$$

where $x \in V$ is yet to be specified. No matter how we choose x, the map $\phi : S_2 \to A_2$ is a surjective ring map. We want to choose x so that ϕ is also injective.

Let T be a dummy variable. The rings A_1 and S_1 are isomorphic. Hence, the polynomial rings $A_1[T]$ and $S_1[T]$ are isomorphic. Let $\overline{P}$ denote the polynomial in $S_1[T]$ corresponding to the polynomial $P \in A_1[T]$. By construction $\pi(\overline{P}) = P$. We just need to make our choice of x such that

$$P(a) = 0 \qquad \Longrightarrow \qquad \overline{P}(a,x) = (0,0). \tag{32}$$

If a is transcendental over A_1, then the above equation is vacuous, and any choice of x will work.

Suppose that a is algebraic over A_1 and that Q is polynomial of minimal degree such that $Q(a) = 0$.

Lemma 13.5 *There exists $x \in V$ such that $\overline{Q}(a,x) = (0,0)$.*

Proof: Since $G(a,0) = 0$, we have $(a,x) = (0,x) + (a,0)$. Note also that

$$(0,x)^k = (0,x)(0,x) \times (0,x)^{k-2} = (0,0); \qquad k = 2,3,4...$$

We have

$$\overline{Q}(a,x) =^1 \overline{Q}(a,0) + \overline{Q}'(a,0)(0,x) =^2 (0,z) + (b,y)(0,x) = (0,z+bx). \quad (33)$$

Equality 1 comes from expanding $\overline{Q}$ in a Taylor series about $(a,0)$ and noting that all higher order terms vanish. Equality 2 comes from $Q(a) = 0$ and from the fact that $Q'(a) = b \neq 0$. We have $b \neq 0$ because Q' is nontrivial and has lower degree than Q. Since $b \neq 0$, we can choose x such that $\overline{Q}(a,x) = 0$. ♠

The following result finishes the induction step.

Lemma 13.6 *Let x be as above. If $P(a) = 0$ then $\overline{P}(a,x) = (0,0)$.*

Proof: Using long division of polynomials, and the minimal degree of Q, we can find some nonzero $d \in A_1$ such that $dP = QR$. Letting

$$\overline{d} = \pi^{-1}(d) = (d,u) \in S_1,$$

we have

$$(0,dz) = (d,u)(0,z) = \overline{d}\ \overline{P}(a,x) = \overline{Q}(a,x)\overline{R}(a,x) = (0,0).$$

Hence $z = 0$. Hence $\overline{P}(a,x) = (0,0)$. ♠

13.3 Another Extension Lemma

The extension that promotes Lemma 13.2 to Theorem 10.4 is exactly the same as we already did, except that we extend from $(\boldsymbol{R}_+, +)$ to $(\boldsymbol{R}, +)$ instead of extending from $((0,1), \times)$ to $(\boldsymbol{R}_+, \times)$. In short, we work with the group law formed by addition rather than with the group law formed by multiplication. Once this change is made, the extension is identical. For example, the $(+--)$ case here reads

$$a > 0,\ \ b < 0,\ \ a+b < 0; \qquad \Longrightarrow \qquad \widehat{G}(a,b) = +F(-a-b,a).$$

The only new step here here is that we also must verify that

$$\widehat{G}(\lambda a, \lambda b) = \lambda \widehat{G}(a,b),$$

but this is immediate from the definitions.

14 References

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