Four Lines and a Rectangle

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1 Introduction

There has been a lot of interest over the years in the problem of inscribing polygons, especially triangles and quadrilaterals, in Jordan loops. See, for instance, [**AA**], [**ACFSST**], [**CDM**], [**Emch**], [**H**], [**Jer**], [**Mak1**], [**Mak2**], [**Ma1**], [**Ma2**], [**M**], [**N**], [**NW**], [**S1**], [**S2**], [**S3**], [**Shn**], [**St**], [**Ta**], [**Va**]. This interest probably stems from the famous Toeplitz Square Peg Problem, which goes back to 1911 and asks if every Jordan loop has an inscribed square.

The purpose of this paper is to present some configuration theorems about rectangles inscribed in 4-tuples of lines, and especially to highlight the connection to hyperbolic geometry. I discovered these results experimentally, using a Java program [**S4**] I wrote. The interested reader can download the program and see the results in action. The proofs I give in this paper are mostly computational, though occasionally a geometric idea makes an appearance.

To make the results as clean as possible, we will consider those 4-tuples $L = (L_1, L_2, L_3, L_4)$ of lines in the plane such that

- The intersection of the 4 lines is empty.
- No two of the lines are parallel or perpendicular. We set $L_{ij} = L_i \cap L_j$.
- The two *diagonals* $\delta_0 = \overline{L_{23}L_{41}}$ and $\delta_\infty = \overline{L_{12}L_{34}}$ are not perpendicular.

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We call such configurations *nice*.

We say that a rectangle R is *inscribed in* L if the vertices (R_1, R_2, R_3, R_4) go cyclically around R, either clockwise or counterclockwise, and are such that $R_i \in L_i$ for $i = 1, 2, 3, 4$. We also consider the diagonals δ_0 and δ_∞ as inscribed degenerate rectangles. In the former case, $R_2 = R_3$ and $R_4 = R_1$ and in the latter case $R_1 = R_2$ and $R_3 = R_4$. Let H_L denote the set of centers of rectangles inscribed in L.

Theorem 1.1 *The set H_L is a hyperbola, and each point of H_L is the center of a unique rectangle inscribed in L.*

Figure 1 shows Theorem 1.1 in action.

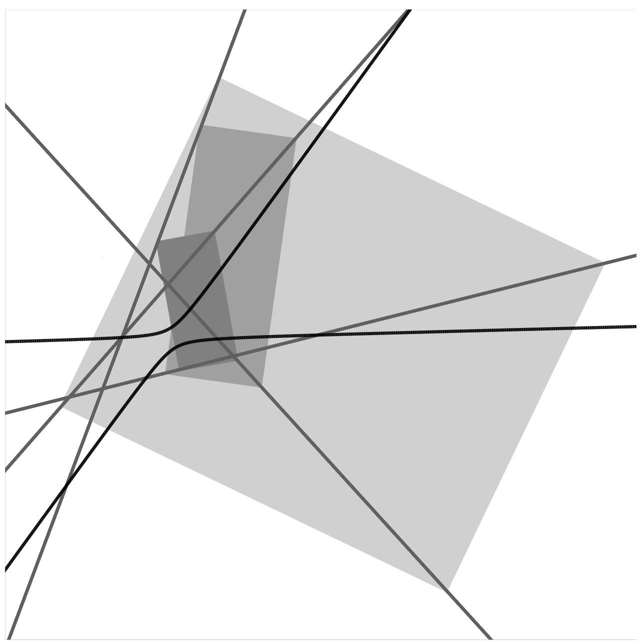

Figure 1: Theorem 1.1 in action.

We think of H_L as a conic section contained in the projective plane, $\boldsymbol{RP}^2$. One of the two components Ω_L of $\boldsymbol{RP}^2 - H_L$ is a convex set. We equip Ω_L with its projectively natural *Hilbert metric*, and this makes Ω_L isometric to the hyperbolic plane. (See §2.1 for a definition of the Hilbert metric.) Relatedly, we let $\boldsymbol{H}^2$ denote the upper half plane model of the hyperbolic plane.

We define

$$\rho(R) = \pm\frac{|R_3 - R_2|}{|R_2 - R_1|}, \qquad \sigma(R) = \text{slope}(\overline{R_1R_2}). \tag{1}$$

We call these two quantities the *aspect ratio* and the *slope*. The sign of ρ is $+1$ if R is clockwise ordered and -1 if R is counterclockwise ordered. The aspect ratios of δ_0 and δ_∞ respectively are 0 and ∞, and there is no sign. These formulas define canonical maps

$$\rho, \sigma : H_L \to \boldsymbol{R} \cup \infty, \tag{2}$$

Here $\rho(p)$ and $\sigma(p)$ respectively are the slope and aspect ratio of the unique rectangle inscribed in L and having p as the center.

Theorem 1.2 *Both maps ρ and σ are restrictions of hyperbolic isometries from Ω_L to $\boldsymbol{H}^2$.*

Note that the ideal boundary of Ω_L is the union of H_L with the two points where H_L intersects the line at infinity in $\boldsymbol{RP}^2$. Thus, $\rho(H_L)$ omits two points of $\boldsymbol{R}\cup\infty$. We call these omitted points ρ_1 and ρ_2. Likewise, there are two omitted values σ_1, σ_2 for the map σ.

Theorem 1.3 *$\sigma_1\sigma_2 = -1$ and*

$$\rho_1\rho_2 = -\frac{(m_2 - m_3)(m_4 - m_1)}{(m_1 - m_2)(m_3 - m_4)}, \qquad m_i = \text{slope}(L_i).$$

Theorem 1.3 has some geometric interpretations. First of all, the quantity $\rho_1\rho_2$, being the cross ratio of the slopes, is an affine invariant. If we move our lines by an affine transformation, the quantity $\rho_1\rho_2$ does not change. The fact that $\sigma_1\sigma_2 = -1$ has a geometric interpretation as well. When two slopes have this property, the corresponding rectangles have parallel sides, but in a twisted way: Side 1 of rectangle 1 is parallel to side 2 of rectangle 2. We call such rectangles *partners*. We call two points of H_L *partners* if the corresponding inscribed rectangles are partners.

Theorem 1.4 *There is a family of parallel lines in $\boldsymbol{R}^2$ such that each line in this family intersects H_L in two partner points. Conversely, any line which intersects H_L in two partner points lies in this family.*

Theorem 1.4 has an interesting geometric consequence. Partner points of H_L lie in different connected components of H_L. Thus, partner rectangles in H_L cannot be joined by a bounded path of rectangles all inscribed in L.

So far we have focused mainly on the centers of the inscribed rectangles. Here is a result about the vertices. We state our result in terms of the aspect ratio, but the result could also be stated in terms of the slope. Let $V_k(r) \in L_k$ denote the kth vertex of the rectangle of aspect ratio r inscribed in L.

Theorem 1.5 *The map V_k is the real part of a holomorphic double branched cover from $\boldsymbol{C} \cup \infty$ to the complex line in $\boldsymbol{CP}^2$ extending L_k. Hence V_k is generically 2-to-1 on $\boldsymbol{R} \cup \infty$ and there is an isometric involution I_k of $\boldsymbol{H}^2$ such that $V_k \circ I_k = V_k$. Finally, $I_1 \circ I_2 \circ I_3 \circ I_4$ is the identity map.*

Theorem 1.5 implies that generically each rectangle inscribed in L shares the kth vertex with one other rectangle inscribed in L. This fact combines with the final statement of Theorem 1.5 to prove a configuration theorem that is reminiscent of the Poncelet and Steiner porisms. Given an inscribed rectangle $R(k)$, let $R(k+1)$ be the other rectangle sharing the kth vertex with $R(k)$. This gives us a chain $R(1), R(2), ...$ of inscribed rectangles. The last statement implies that this chain repeats after 4 steps: $R(4+k) = R(k)$ for all k. Figure 2 shows this configuration theorem in action. We have shown a particularly nice instance; in general the 4 rectangles can overlap messily.

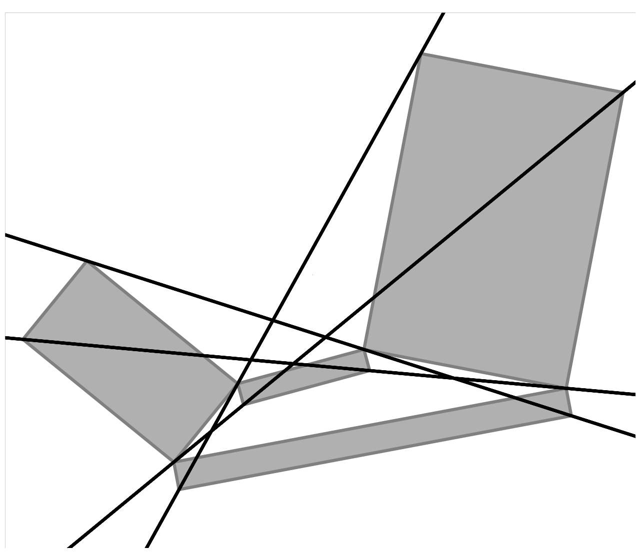

Figure 2: Theorem 1.5 in action.

This paper is organized as follows. In §2 we prove the five theorems listed above. Our proofs are mostly computational, and we use Mathematica [**W**] for many of the calculations. I don't have much geometric intuition for why the theorems are true, but the analytic reason seems clear. The constructions are simple enough so that, when complexified, they lead to very low degree holomorphic maps which are subject to the usual rigidity theorems from complex analysis.

In §3 we discuss additional features of the constructions above. Namely

- We use Theorem 1.4 to prove the simplest case of a conjecture we made in [**S1**] concerning continuous paths of rectangles inscribed in polygons.
- We consider a degenerate case the theorems above, in which the third niceness condition is dropped. In this case H_L turns out to be a pair of crossing lines.
- We explain how to encode most of the information from the theorems above as a map from the space of quadrilaterals into the projective tangent bundle of the Riemann sphere.

I would like to thank Arseniy Akopyan, Peter Doyle, Cole Hugelmeyer, and Sergei Tabachnikov for interesting discussions related to this work.

2 The Configuration Theorems

2.1 Projective Geometry

Here we recall a few basic facts about projective geometry.

Basic Definition: Let $\boldsymbol{RP}^2$ denote the real projective plane, i.e., the space of lines through the origin in $\boldsymbol{R}^3$. One can also think of $\boldsymbol{RP}^2$ as the space of equivalence classes of nonzero vectors in $\boldsymbol{R}^3$, with two vectors being equivalent if they are scalar multiples of each other. The coordinates $[x : y : z]$ denote such an equivalence class. We identify $\boldsymbol{R}^2 \subset \boldsymbol{RP}^2$ with the *affine patch*

$$\{[x : y : 1] |\ x, y \in \boldsymbol{R}^2\}.$$

Projective Transformations and Cross Ratios: The *projective transformations* are diffeomorphisms of $\boldsymbol{RP}^2$ induced by the action of invertible 3×3 real matrices. The *cross ratio* of 4 collinear points A, B, C, D is given by

$$[A, B, C, D] = \frac{(A - C)(B - D)}{(A - B)(C - D)} \tag{3}$$

In this equation, we identify the line containing the points with $\boldsymbol{R}$ *via* a projective transformation. The answer is independent of the choice. By construction, the cross ratio of 4 points is projectively invariant:

$$[A, B, C, D] = [A', B', C', D']$$

if there is a projective transformation T such that $A' = T(A)$, etc.

Conic Sections: A *nondegenerate conic section* is the solution set of an irreducible homogeneous polynomial of degree 2, considered as a subset of $\boldsymbol{RP}^2$. A conic section intersect the affine patch in an ellipse, a hyperbola, or a parabola. Projective transformations transitively permute the nondegenerate conic sections. One beautiful thing about projective geometry is that there is just one nonsingular conic section up to projective transformations. To save words, we will just say *conic section* when we mean a nondegenerate conic section.

Hilbert Metric: We use the cross ratio to define the *Hilbert metric* on a convex domain $\Omega \subset \boldsymbol{RP}^2$. The distance between two points $B, C \in \Omega$ is

$$\text{dist}(B, C) = \log[A, B, C, D], \tag{4}$$

where $A, D \in \partial\Omega$ and A, B, C, D appear in order on a line segment contained in $\Omega \cup \partial\Omega$. This metric is projectively natural. Each conic section H bounds a convex domain Ω on exactly one side. We call Ω the *hyperbolic domain* bounded by H, and we equip Ω with the Hilbert metric. This is commonly called the *Klein model* of the hyperbolic plane. When H is the unit circle, Ω is the unit disk. In this case, Ω has a second natural metric which makes it isometric to $\boldsymbol{H}^2$, namely the *Poincare metric*. In this model, the geodesics are arcs of circles which meet the unit circle at right angles. There is an isometry between the two models which is the identity map on the unit circle.

Parametrizing Conic Sections: Given a 3×3 invertible matrix $A = A_{ij}$, we introduce the parametric curve

$$\Gamma_A(r) = [A_{00} + A_{01}r + A_{02}r^2 : A_{10} + A_{11}r + A_{12}r^2 : A_{20} + A_{21}r + A_{22}r^2] \tag{5}$$

Here r is the parameter. The condition $\det(A) \neq 0$ guarantees not all coordinates vanish at once, so that Γ_A makes sense as a curve in $\boldsymbol{RP}^2$.

Lemma 2.1 *Γ_A parametrizes a conic section and is the restriction of a hyperbolic isometry from $\boldsymbol{H}^2$ to the associated hyperbolic domain.*

Proof: For any invertible matrix M, we have $M(\Gamma_A) = \Gamma_{MA}$. The map M acts as a projective transformation permuting the conic sections and acting isometrically on their associated hyperbolic domains. So, the result is true for MA if and only if it is true for A. We choose M to that

$$MA = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

In this case

$$\Gamma_A(r) = [-1 + r^2 : -2r : 1 + r^2].$$

We recognize this map as stereographic projection from $\boldsymbol{R} \cup \infty$ to the unit circle. In complex coordinates, this map is given by $r \to (r - i)/(r + i)$. As is well known, stereographic projection is the restriction of a hyperbolic isometry from $\boldsymbol{H}^2$ to the unit disk, equipped with either hyperbolic metric. ♠

2.2 The Perpendicularity Test

We consider a configuration $L = (L_1, L_2, L_3, L_4)$ such that the intersection of the lines is empty and no two lines are parallel or perpendicular. In this section we give a criterion for when the diagonals δ_0 and δ_∞ are not perpendicular.

The equation for L_j is $V_j \cdot (x, y, 1) = 0$, where $V_j = (m_j, -1, b_j)$. We normalize by a similarity so that $m_1 = b_1 = 0$ and $b_2 = 0$ and $b_3 = 1$. The point $L_{ij} = L_i \cap L_j$ is the projectivization of the cross product $V_i \times V_j$. We compute

$$(L_{12} - L_{34}) \cdot (L_{23} - L_{41}) = \frac{\Delta}{(m_2 - m_3)(m_3 - m_4)m_4}, \tag{6}$$

$$\Delta = {b_4}^2(m_2 - m_3) + b_4(m_2m_3m_4 - m_2 + m_3 + m_4) + (-m_2{m_4}^2 - m_4). \tag{7}$$

So, the diagonals are perpendicular if and only if $\Delta = 0$.

Note that Δ is quadratic in b_4, so one can explicitly solve the equation $\Delta = 0$ in terms of b_4. The distriminant of the quadratic equation is

$$D = (m_2^2 + m_3^2 + m_4^2 + 2m_2m_4 + 2m_2m_3^2m_4 + 4m_2^2m_4^2 + m_2^2m_3^2m_4^2) -$$
$$2m_3(m_2 + m_4)(1 + m_2m_4). \tag{8}$$

Lemma 2.2 *$D > 0$.*

Proof: Let $S \subset \boldsymbol{R}^3$ denote the set (m_2, m_3, m_4) where $m_2 \neq 0$ and $m_4 \neq 0$ and $m_2m_4 + 1 \neq 0$. Solving $D = 0$ for m_3 yields

$$m_3 = \frac{(m_2 + m_4) \pm 2\ i\ m_2m_4}{m_2m_4 + 1}. \tag{9}$$

Hence there are no real solutions in S. Since every point in S can be connected by a continuous path in S to a point where $m_3 = 0$, it suffices to prove that $D > 0$ when $m_3 = 0$. In this case, we have the simpler formula $D = (m_2 + m_4)^2 + (2m_2m_4)^2 > 0$. ♠

So, if we fix L_1, L_2, L_3 and translate L_4, there are exactly two positions where the diagonals are perpendicular, By symmetry, the same result holds for any of the other lines as well.

We call Δ the *perpendicularity test* and D the *positive discriminant*.

2.3 Four Coincident Lines

As a warm-up, we study rectangles inscribed in a 4-tuple $L = (L_1, L_2, L_3, L_4)$ of lines which all contain the origin. We insist that no two lines are parallel or perpendicular. The set of rectangles inscribed in L is invariant under scalar multiplication. That is, if R is inscribed in L, then so is λR, the rectangle obtained by scaling all the vertices of R by the same nonzero real number λ.

Lemma 2.3 *No rectangle inscribed in L has its vertex or center at the origin, and two rectangles inscribed in L and having their centers on the same line through the origin are scalar multiples of each other.*

Proof: Let R be a rectangle inscribed in L. If R is centered at the origin then the lines of L coincide in pairs. If R has a vertex at the origin then two lines of L are perpendicular. If R_1 and R_2 have centers on the same line through the origin then there is some real number λ such that R_1 and the scalar multiple λR_2 have the same center, c. If $R_1 \neq \lambda R_2$ then R_1 and λR_2 determine at least 2 lines L_k and L_{k+2} of L. Reflection in c swaps these two lines, forcing L_k and L_{k+1} to be parallel. This is a contradiction. ♠

Lemma 2.4 *The set of centers of L is a pair of unequal crossing lines (minus the origin), and each point on these crossing lines is the center of a unique rectangle inscribed in L.*

Proof: We rotate so that L_1 is the x-axis. In view of the previous result, it suffices to prove that there are exactly 2 rectangles inscribed in L having $(1, 0)$ as a vertex. Let m_j be the slope of the line L_j. The quantities m_2, m_3, m_4 are nonzero and distinct, and $1 + m_i m_j \neq 0$ for $i \neq j$.

A point $(x_j, y_j) \in L_j$ satisfies $y_j = m_j x_j$. Define $T(x, y) = (-y, x)$. Note that (x, y) and $T(x, y)$ are perpendicular vectors. Let ρ be the aspect ratio of the rectangle we seek. We have $R_1 = (1, 0)$ and $R_2 = (x_2, m_2 x_2)$, and

$$R_3 = R_2 + \rho T(R_2 - R_1), \qquad R_4 = R_1 + \rho T(R_2 - R_1). \tag{10}$$

Solving for $R_3 \in L_3$ and $R_4 \in L_4$ gives $x_2 = (A \pm \sqrt{D})/B$ where D is the positive discriminant and

$$A = m_2 - m_3 - m_4 - m_2 m_3 m_4, \qquad B = 2(m_2 - m_3)(1 + m_2 m_4). \tag{11}$$

Since $B \neq 0$ and $D > 0$ there are exactly 2 solutions, as desired. ♠

2.4 A Matrix Equation

From now on, unless otherwise stated, L is a nice configuration. We normalize as in §2.2. We also recall that

$$T(x,y) = (-y,x). \tag{12}$$

We seek a rectangle (R_1, R_2, R_3, R_4) inscribed in L having aspect ratio r. We have the same equations for the vertices of R as in Lemma 2.4. The two equations $R_3 \in L_3$ and $R_4 \in L_4$ lead to two equations in two unknowns:

$$M \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -b_4 \end{bmatrix}, \qquad M = \begin{bmatrix} r & (m_3 - m_2) - r(1 + m_2 m_3) \\ r + m_4 & -r(1 + m_2 m_4) \end{bmatrix} \tag{13}$$

This has a unique solution provided that $\det(M) \neq 0$. The equation $\det(M)$ is quadratic in r. The discriminant of this quadratic equation is D, the positive discriminant from §2.3. Hence there are always 2 values where $\det(M) = 0$.

Lemma 2.5 *If r is a value where $\det(M) = 0$, there are no rectangles inscribed in L having aspect ratio r.*

Proof: In case r is a value where $\det(M) = 0$, the two rows of M are multiples of each other. We only get inscribed rectangles of aspect ratio r in this case if the second row of M is b_4 times the first row. This happens if and only if

$$r = \frac{m_4}{b_4 - 1} = \frac{b_4 m_3 - b_4 m_2}{-1 + b_4 + m_2 m_3 - m_2 m_4}. \tag{14}$$

If $b_4 = 1$ there are no solutions. In general, solving this equation for b_4 leads to the same equation as solving $\Delta = 0$ for b_4. Here Δ is the perpendicularity test. Since $\Delta \neq 0$ for nice configurations, there are no additional rectangles inscribed in L. ♠

Remark: When $\Delta = 0$, the case of perpendicular diagonals, we draw a different conclusion from Lemma 2.5. In this case, there is an entire line's worth of inscribed rectangles having aspect ratio r.

2.5 Proofs of the Results

In this section we prove Theorems 1.1-1.5 advertised in the introduction.

For r such that $\det(M) \neq 0$, the solution to Equation 13 is given by

$$x_1 = \frac{n_{10} + n_{11}r}{d_0 + d_1 r + d_2 r^2}, \qquad x_2 = \frac{n_{20} + n_{21}r}{d_0 + d_1 r + d_2 r^2}, \tag{15}$$

- $n_{10} = b_4(m_3 - m_2)$,
- $n_{11} = 1 - b_4 - b_4 m_2 m_3 + m_2 m_4$,
- $n_{20} = m_4$,
- $n_{21} = 1 - b_4$,
- $d_0 = m_4(m_2 - m_3)$,
- $d_1 = m_2 m_3 m_4 + m_2 - m_3 + m_4$,
- $d_2 = m_2(m_3 - m_4)$.

We remark that the discriminant of the equation $d_0 + d_1 r + d_2 r^2 = 0$ is, once again, the positive discriminant D from §2.3.

Lemma 2.6 *H_L is a nondegenerate conic section.*

Proof: Recall that H_L is the curve of inscribed centers. In view of the work in the previous section, we get all points on H_L by analyzing Equation 15. Using the equations above for R_3 and R_4, we find that H_L is the curve Γ_A, where A is the matrix

$$\begin{matrix} b_4(m_3 - m_2) + m_4 & 2 - b_4(m_2 m_3 + 2) & (b_4 - 1)m_2 \\ m_2 m_4 & m_2 - b_4 m_3 + m_4 & m_2(b_4 m_3 - m_4) \\ 2(m_2 - m_3)m_4 & 2(m_3 m_4 m_2 + m_2 - m_3 + m_4) & 2m_2(m_3 - m_4) \end{matrix} \tag{16}$$

The point $H_A(r)$ is the center of the rectangle of aspect ratio r. We compute

$$\det(A) = 2m_2 m_3(m_2 - m_4)\Delta,$$

where Δ is the perpendicularity test. Thus $\det(A) \neq 0$. By Lemma 2.1, the set H_L is a conic section. ♠

Lemma 2.7 *H_L not an ellipse or a parabola.*

Proof: Referring to Equation 15, the equation $d_0 + d_1 r + d_1 r^2$ has 2 real roots r_1 and r_2. because $D > 0$. Since $n_{20} = m_4 \neq 0$, we see that there are (two) values r for which the point x_2 is arbitrarily large. Since x_2 is one of the vertices of an inscribed rectangle, and since no two lines of L are parallel, the distance from x_2 to the other lines tends to ∞ with the size of x_2. Hence, the corresponding rectangles have unboundedly large diameter. If these large rectangles had bounded centers then we could take a rescaled limit and find a quadruple of lines as in §2.1 which had an inscribed rectangle centered at the origin. This would force two pairs of lines in L to be parallel, a contradiction.

We have just shown that H_L contains points unboundedly far from the origin. Hence H_L is not an ellipse. Suppose that H_L is a parabola. From the analysis above, and continuity, we can find two rectangles $R_{n,1}$ and $R_{n,2}$ inscribed in L such that both rectangles have diameter n, and aspect ratios converging to the values $r_1 \neq r_2$. If H_L is a parabola. can take a rescaled limit and arrive at a quadruple of lines as in §2.1 which has 2 distinct inscribed rectangles with the same center. But then the lines of L would have to be parallel in pairs, a contradiction. ♠

Now we know that H_L is a hyperbola. If two rectangles are inscribed in L and have the same center, then reflection in this center permutes the lines, forcing L to have some parallel lines. Hence every point of H_L is the center of a unique inscribed rectangle. This completes the proof of Theorem 1.1.

Lemma 2.8 *Theorems 1.2 and 1.3 are true for the map ρ.*

Note that the map Γ_A considered in the previous section is the inverse of the map ρ from Theorem 1.2. By Lemma 2.1, the map Γ_A is the restriction of a hyperbolic isometry from $\boldsymbol{H}^2$ to Ω_L. Hence ρ is the restriction of an isometry from Ω_L to $\boldsymbol{H}^2$.

Referring to the discussion in §2.5, the omitted aspect ratios ρ_1 and ρ_2 are the roots of the equation $d_0 + d_1 r + d_2 r^2$. The product is therefore d_0/d_2, and (remembering that $m_1 = 0$) this leads to the expression in Theorem 1.3. Since the expression is invariant under rotations, the general formula works even when $m_1 \neq 0$. ♠

Lemma 2.9 *Theorems 1.2 and 1.3 are true for σ.*

Proof: We compute explicitly that there is a 2×2 matrix $\mu = \{\mu_{ij}\}$ such that

$$\sigma = \mu(r) = \frac{\mu_{11}r + \mu_{12}}{\mu_{21}r + \mu_{22}}, \tag{17}$$

where

$$\mu_{11} = m_2 - m_2 b_4, \qquad \mu_{12} = m_2 m_4,$$

$$\mu_{21} = m_2 m_3 b_4 - m_2 m_4, \qquad \mu_{22} = m_2 b_4 - m_3 b_4 + m_4.$$

Solving the equation $\det(\mu) = 0$ leads to the same equation as solving $\Delta = 0$. Hence μ is nonsingular. Hence μ acts as a linear fractional transformation of $\boldsymbol{R} \cup \infty$. (Depending on the sign of $\det(\mu)$, the map μ is either an isometry of $\boldsymbol{H}^2$ or an isometric map from $\boldsymbol{H}^2$ to the lower half plane model of $\boldsymbol{H}^2$.) By construction $\sigma = \mu \circ \rho$ or $\sigma = \overline{\mu} \circ \rho$, depending on the sign of $\det(\mu)$. So, the truth of Theorem 1.2 for ρ implies the truth of Theorem 1.2 for σ. Finally, an explicit calculation shows that $\sigma_1\sigma_2 = -1$ when $\sigma_j = \mu(r_j)$. ♠

Proof of Theorem 1.4: The map σ maps partner points p_1, p_2 to points s_1, s_2 satisfying $s_1 s_2 = -1$. But then the hyperbolic geodesic with endpoints s_1, s_2 contains the point i. Hence σ maps the Ω_L-geodesic $\overline{p_1 p_2} \cap \Omega_L$ to a hyperbolic geodesic through i. But then all the geodesics in Ω_L connecting partner points contain a common point, namely $\sigma^{-1}(i)$. This point lies at ∞ in the projective plane because i lies on the geodesic through the omitted slopes σ_1, σ_2 and these correspond to the two infinite points of H_L. Hence all the lines connecting partner points are parallel to a single direction. Conversely, any line parallel to this direction intersets H_L twice and is mapped by σ to a geodesic through i. Hence such a line joins partner points. ♠

Proof of Theorem 1.5: To check that V_k is the real part of a holomorphic double branched cover, it suffices to prove that $f = \pi_1 \circ V_k$ is the real part of a holomorphic double branched cover, because the second coordinate depends linearly on the first. Here π_1 is projection onto the first coordinate. From Equations 10 and 15 we see that

$$f(r) = \frac{a_k + b_k r + c_k r^2}{d_0 + d_1 r + d_2 r^2}$$

where a_k, b_k, c_k are rational expressions in $m_1, m_2, m_3, m_4, b_1, b_2, b_3, b_4$, the parameters for the lines. When complexified, V_k is a holomorphic double branched cover from $\boldsymbol{C} \cup \infty$ to $\boldsymbol{C} \cup \infty$.

Now we study the involutions I_1, I_2, I_3, I_4. This time we do not normalize so that $m_1 = b_1 = 0$. We find explicitly that

$$I_1(r) = \frac{Z_{02}r + Z_{01}}{Z_{21}r + Z_{20}}, \qquad Z_{ij} = \det \begin{bmatrix} B_i & B_j \\ M_i & M_j \end{bmatrix}. \tag{18}$$

Here B_0, B_1, B_2 respectively equal

$$b_{14}m_{23}, \quad b_{12} + b_{34} + b_{14}m_3m_2 + b_{31}m_2m_4 + b_{12}m_3m_4, \quad b_{12}m_{34},$$

and M_0, M_1, M_2 respectively equal

$$m_{14}m_{23}, \quad m_{12} + m_{34} + m_{14}m_3m_2 + m_{31}m_2m_4 + m_{12}m_3m_4, \quad m_{12}m_{34},$$

We have set $b_{ij} = b_i - b_j$ and $m_{ij} = m_i - m_j$. To get the remaining maps, we define $I_1^{(j)}$ to be the map above with respect to the cycled line configuration $L^{(j)} = (L_{1+j}, L_{2+j}, L_{3+j}, L_{4+j})$. This just means that we cyclically permute the variables above and recompute. We have

$$I_2(r) = \frac{1}{I_1^{(1)}(1/r)}, \qquad I_3(r) = I_1^{(2)}(r), \qquad I_4(r) = \frac{1}{I_1^{(3)}(1/r)}. \tag{19}$$

These maps all act as hyperbolic isometries. Using Mathematica, we check symbolically that $I_1I_2I_3I_4$ fixes the points $-1, 0, 1$ and hence is the identity. ♠

Remark: The quantities M_0, M_1, M_2 above are the same as what we would get for d_0, d_1, d_2 (up to sign) were we to recompute these quantities without normalizing so that $m_1 = b_1 = 0$. For the record, we also give the formula for the matrix μ from Equation 17 without the condition that $b_1 = m_1 = 0$.

$$\mu_{11} = m_{12}b_{34} - b_{12}m_{34}, \quad \mu_{12} = m_1m_2b_{34} - m_1m_3b_{24} + m_2m_4b_{13} - m_3m_4b_{12}$$

$$\mu_{21} = m_2m_3b_{14} - m_1m_3b_{24} + m_1m_4b_{23} - m_2m_4b_{13}, \quad \mu_{22} = b_{14}m_{23} - m_{14}b_{23}$$

3 Further Results

3.1 The Saddle Conjecture

Now let γ be a polygon in the plane. Let A be a chord of γ. That is, A is a segment having both endpoints $A_1, A_2 \in \gamma$. Let B_j be the line perpendicular to A at A_j. Let γ_j be the union of edges of γ, either one or two, which contain A_j. We call A a *diameter* if γ_j lies one one side of B_j for $j = 1, 2$. This means that γ_j does not intersect both open half spaces of $\boldsymbol{R}^2 - B_j$.

Some of the diameters are extrema for distance function $d : \gamma \times \gamma \to \boldsymbol{R}$, and we call these the *extreme diameters*. Some diameters are not extreme diameters, and we call the additional diameters *saddles*. Typically, one can lengthen a saddle by varying one endpoint and shortened it by varying the other. Figure 3 suggests how a path of rectangles in an equilateral triangle interpolates between a max and a saddle.

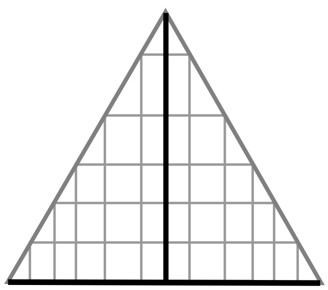

Figure 3: Inscribed rectangles interpolating between a max and a saddle.

When $I(\gamma)$ is a 1-dimensional manifold, we call an arc component A of $I(\gamma)$ a *proper arc* if the aspect ratios of rectangles corresponding to points in A tend to 0 or ∞ as one exits the ends of A. We also insist that there are a pair of diameters of γ on which these degenerating rectangles accumulate. The example in Figure 3 shows a proper arc of $I(\gamma)$ when γ is an equilateral triangle.

In [**S1** we proved the following result.

Theorem 3.1 *For an open dense set of polygons γ, the space $I(\gamma)$ is a piecewise smooth 1-manifold consisting of loops and proper arcs.*

After looking at hundreds of examples, I conjectured that generically the proper arcs of $I(\gamma)$ always connect extreme diameters to saddles. We emphasize that this result is only true generically. When γ is a square, there is certainly a path of inscribed rectangles connecting the two diagonals. Here I will use Theorem 1.4 to prove this conjecture in the simplest case.

Theorem 3.2 *Let Q be a generic convex quadrilateral. Then the arc components of $I(Q)$ must connect extreme diameters to saddles.*

Proof: The quadrilateral Q has no minimal diameters and 2 or 3 maximal diameters. The two diagonals are maximal diamaters and possibly a side is another maximal diameter. There are the same number of saddles as maximal diamaters, and these are obtained by dropping perpendiculars from vertices of Q to other sides of Q.

Let L be the quadruple of lines extending the sides of Q, taken in counterclockwise cyclic order. We first claim that $I(Q)$ cannot contain a path of rectangles connecting the diagonals of Q such that all the rectangles are also inscribed in L. To prove our claim, let ρ_1 and ρ_2 be the omitted aspect ratios, as in Theorem 1.4. We have $\rho_1\rho_2 < 0$, because the slopes m_1, m_2, m_3, m_4 are cyclically ordered on $\boldsymbol{R} \cup \infty$. But then and continuous path in $\boldsymbol{R} \cup \infty$ contains some ρ_j. Geometrically, this means that there is no continuous path of rectangles inscribed in L which connects the two diagonals. This proves the claim.

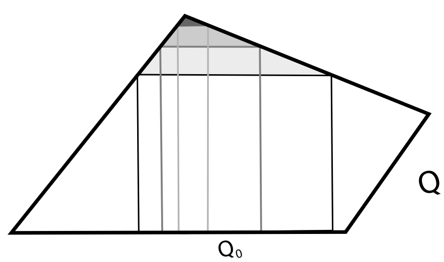

Figure 4: Some rectangles that are inscribed in Q but not L_Q.

It remains to consider the case when our arc of $I(\gamma)$ contains rectangles which have a pair of vertices contained in the same side Q_0 of Q. Such a path always has a saddle at one end. Intuitively, if we rotate the side as in Figure 4, then the two bottom vertices squeeze together and the top two vertices seek out the highest point. So, a maximal diameter is always paired with a saddle.

Since there are the same number of saddles as extreme diameters, and no two extreme diagonals are paired together, each extreme diameter is paired with a saddle and *vice versa*. ♠

3.2 A Degenerate Case of the Theorems

Here we discuss a degenerate case of our results. Suppose that we have a configuration $L = (L_1, L_2, L_3, L_4)$ of lines which satisfies the first 3 niceness conditions but not the fourth one. This means that the diagonals δ_0 and δ_∞ are perpendicular.

Theorem 3.3 *The set H_L is a pair of unequal crossing lines H_σ and H_ρ. The well-defined maps σ and ρ are constant on H_σ and H_ρ respectively, and projective transformations on H_ρ and H_σ respectively. In particular, H_σ consists of centers of rectangles having constant slope and H_ρ consists of centers of rectangles having constant aspect ratio.*

We omit the proof, though we mention that the result follows from a careful analysis of what happens to the equations in the previous chapter when $D = \Delta = 0$. The analogue of Lemma 2.5 gives rise to H_ρ and the analogue of Lemma 2.6 gives rise to H_σ. An alternative method of proof would be to take a limit from the nice case.

What is going on geometrically is that L is the limit of a sequence L_n of nice configurations. The hyperbolas $\{H_{L_n}\}$ converge to H_L, pinching together in the limit. The hyperbola H_{L_n} is very nearly partitioned into two pieces. On one of the pieces ρ is very nearly constant and on the other piece σ is very nearly constant. In the limit, each piece becomes one of the two lines. The map $\rho_n \circ \sigma_n^{-1}$ is a hyperbolic isometry for all n, but in the limit the translation length of this isometry tends to ∞. The limit maps all of $\boldsymbol{H}^2$ to a single ideal point. Likewise, the limit of the map $\sigma_n \circ \rho_n^{-1}$ maps all of $\boldsymbol{H}^2$ to a single ideal point.

3.3 The Projective Tangent Bundle

We can encode most of the information contained in our configuration theorems into a map from the space of quadruples of lines into $\mathcal{P}$, the projective tangent bundle of $\boldsymbol{C} \cup \infty$. To each nice quadruple L we associate the pair $\Psi(L) = (p, \ell)$ where

$$p = \mu^{-1}(i). \tag{20}$$

and ℓ is the circle (or line) in $\boldsymbol{C} \cup \infty$ which extends the geodesic whose endpoints are ρ_1 and ρ_2. Here μ is the map from Equation 16. Geometrically, the extension of ρ to $\boldsymbol{H}^2$ maps one of p or $\overline{p}$ (whichever lies in $\boldsymbol{H}^2$) to the common point of the parallel lines from Theorem 1.4.

Let us justify our claim that $p \in \ell$. The condition that $\sigma_1\sigma_2 = -1$ means that i lies in the geodesic with endpoints $\sigma_1\sigma_2$. Since μ maps i, σ_1, σ_2 to p, ρ_1, ρ_2, we see that p really does lie in ℓ. In other words, the domain of Ψ really is $\mathcal{P}$. To put things more classically, we could say that the element of the projective tangent bundle is the point p and the tangent line at $T_p(\boldsymbol{C} \cup \infty)$ to ℓ. This tangent line uniquely determines ℓ and conversely.

Now we mention, mostly without proof, some properties of Ψ. Assuming that $\Psi(L) = (p, \ell)$ we denote p as $\Psi_p(L)$ and ℓ as $\Psi_\ell(L)$. The purpose of this notation is to discuss the functions Ψ_p and Ψ_ℓ separately.

Similarity Invariance: If L and L' are two quadruples related by a similarity, then $\Psi(L) = \Psi(L')$. This is pretty obvious if there is no rotation involved. In case there is rotation involved, the key observation is that $\mu' = \mu \circ \beta$ where β is some hyperbolic isometry that fixes i.

The Angular Property: To each configuration L, we let $\theta(L) \in [0, \pi/2]$ be the small angle between the diagonals of L. We have already seen that H_L is a hyperbola if and only if $\theta(L) < \pi/2$. Let V_θ denote the pair of lines in $\boldsymbol{C}$ which make the angle θ with the y-axis. It turns out that $\Psi_p(L) \in V_\theta$, where $\theta = \theta(L)$.

The Diagonal Property: We call two configurations L and L' *diagonally related* if δ_0' is a translate of δ_0 and δ_∞' is a translate of δ_∞'. The point

$$q = i\frac{L_{12} - L_{34}}{L_{23} - L_{41}} \tag{21}$$

is the same for all configurations in $\mathcal{D}(L)$. It turns out that $q \in \Psi_\ell(L')$ for all L' in $\mathcal{D}(L)$. Note that Equations 20 and 21 give us a practical way to compute $\Psi(L)$.

The Swivel Property: We fix some index k and let $\mathcal{L}$ denote the space of quadruples we get by holding 3 of the lines of L fixed and varying the kth line. A point of $\mathcal{L}$ represents a quadruple of lines, and a line in $\mathcal{L}$ represents all the quadruples where the kth line belongs to a pencil of lines all through a fixed point. Now we think of Ψ as a map on $\mathcal{L}$. There is a point β, depending on the 3 fixed lines, such that $\beta \in \Psi_l(\lambda)$ for all $\lambda \in \mathcal{L}$. Also, Ψ_p maps any line in $\mathcal{L}$ to a circle (or straight line) through β.

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