Symmetric Patterns of Geodesics and Automorphisms of Surface Groups

Richard Evan Schwartz *

July 4, 2001

Abstract

We prove a non-equivariant version of Mostow rigidity for symmetric patterns of geodesics in hyperbolic space. This result allows for a classification of pseudo-Anosov surface group automorphisms, based on the large scale geometric structure of their orbits.

1 Introduction

Let $\boldsymbol{H}^n$ denote hyperbolic n-space. By a *symmetric pattern of geodesics* in $\boldsymbol{H}^n$, we shall mean a countable collection Γ of geodesics such that

1. The symmetry group of Γ is a co-compact lattice in the hyperbolic isometry group $\mathrm{Isom}(\boldsymbol{H}^n)$
2. The stabilizer of each geodesic in Γ acts with compact quotient.
3. There are only finitely many geodesics of Γ, modulo the symmetry group.

Such a pattern arises, tautologically, as the lift of a finite union of closed geodesics in a compact hyperbolic orbifold. In this paper we shall determine whether or not a person with poor eyesight could ever mistake one symmetric pattern of geodesics for another.

*Supported by an N.S.F. Postdoctoral Fellowship.

1.1 The Main Result

For $j = 1, 2$, let S_j be any set, and let $\rho_j : S_j \times S_j \to [0, \infty)$ be any function. We say that a map $\phi : S_1 \to S_2$ is *uniformly proper* with respect to the pair (ρ_1, ρ_2) if there is a function $p : \boldsymbol{N} \to \boldsymbol{N}$ such that for all $s, s' \in S_1$,

1. $\rho_1(s, s') \leq n \;\Rightarrow\; \rho_2(\phi(s), \phi(s')) \leq p(n)$.
2. $\rho_2(\phi(s), \phi(s')) \leq n \;\Rightarrow\; \rho_1(s, s') \leq p(n)$.

There is a canonical function ρ on pairs (g, g') of geodesics. $\rho(g, g')$ is defined to be the infimal distance between a point on g and a point on g'. We say that a map from one set of geodesics to another is *uniformly proper* if it is uniformly proper with respect to ρ. We say that a subset G of geodesics has *bounded geometry* if every point of $\boldsymbol{H}^n$ is within some uniform distance of a geodesic in G. In §2-5 we will prove:

Theorem 1.1 *Let $n \geq 3$. Suppose that Γ_1 and Γ_2 are symmetric patterns of geodesics in $\boldsymbol{H}^n$. Suppose that $G_j \subset \Gamma_j$ has bounded geometry. Then any uniformly proper bijection from G_1 to G_2 is induced by a hyperbolic isometry.*

Two special cases of Theorem 1.1 are worth pointing out:

1. There cannot be a uniformly proper bijection between non-isometric symmetric patterns of geodesics.
2. A symmetric pattern of geodesics does not admit a uniformly proper bijection onto a proper subset of itself.

1.2 Some Comparisons

In §3 we will see that the hypothesis in Theorem 1.1 implies that there is a quasi-isometry of $\boldsymbol{H}^n$ which bijectively "pairs up" the geodesics in G_1 with the geodesics in the image $G_2 = \phi(G_1)$. However, this "pairing up" is not equivariant in any sense of the word. The proof of Theorem 1.1 seems to be a kind of non-equivariant version of Sullivan's linefield argument [**Su**].

While similar in spirit to both Mostow Rigidity [**M**] and Tukia's theorem on quasiconformal groups [**Tu**], Theorem 1.1 does not seem to follow, trivially or otherwise, from either of these results. In fact, at the end of §2 we will show that Theorem 1.1 implies Mostow rigidity for uniform hyperbolic lattices.

While clearly false for $\boldsymbol{H}^2$, Theorem 1.1 is true for all other negatively curved symmetric spaces, and also (when suitably modified) for all non-positively curved symmetric spaces with no rank one factors. The case of complex hyperbolic space is similar to that of real hyperbolic space. (The "transcription" of our argument can be carried out along the lines of what is done in [**S1**].) For the other symmetric spaces mentioned, the rigidity results in [**P**] and [**KL**] say that an *arbitrary* quasi-isometry is equivalent to an isometry. In all these cases, Theorem 1.1 is an immediate corollary.

I would have liked to prove Theorem 1.1 in variable negative curvature. I have no idea how to do this.

1.3 Automorphisms of Surface Groups

Theorem 1.1 has an application for the study of surface group automorphisms. Let Σ be a closed surface, having genus at least 2. We equip $\pi_1(\Sigma)$ with a word metric, d, in such a way that multiplication on the left is an isometry. In such a word metric, group automorphisms are (individually) bi-lipschitz maps.

Given an automorphisms $A : \pi_1(\Sigma) \to \pi_1(\Sigma)$, we define

$$\xi_A(x, y) = \inf_{n \in \boldsymbol{Z}} d(A^n(x), A^n(y)).$$

Note that $\xi_A(gx, gy) = \xi_A(x, y)$. Note also that ξ_{A^k} and ξ_A are Lipschitz equivalent functions. ξ_A in some sense captures the large scale structure of the orbits of A.

For $j = 1, 2$, let $A_j : \pi_1(\Sigma) \to \pi_1(\Sigma)$ be two different surface group automorphisms. We say that A_1 and A_2 are *coarse orbit equivalent* if there is a bijection $g : \pi_1(\Sigma) \to \pi_1(\Sigma)$ which is uniformly proper with respect to the pair (ξ_{A_j}, ξ_{A_2}). This notion of equivalence captures, in some sense, the meaning that two different group actions have the same large scale geometric structure.

As a particular example, we say that a group isomorphism $g : \pi_1(\Sigma) \to \pi_1(\Sigma)$ *commensurates* A_1 to A_2 if there are nonzero integers n_1 and n_2 so that

$$g \circ A_1^{n_1} \circ g^{-1} = A_2^{n_2}.$$

It is easy to see that such a map induces a coarse orbit equivalence between the two group actions. In this case, we will say that A_1 and A_2 are *commensurable*. Commensurability is extremely rare amongst surface group automorphisms.

The *mapping cylinder* of A is the semidirect product of $\boldsymbol{Z}$ and $\pi_1(\Sigma)$, twisted by the A action. We say that A is *hyperbolic* if the mapping cylinder is the fundamental group of a closed hyperbolic three-manifold. According to Thurston [**T1**] this happens whenever A is induced from a surface diffeomorphism which is isotopic to a *pseudo Anosov* map. This situation is, in some sense, generic. We say that two (hyperbolic) mapping cylinders are *commensurable* if they are commensurable, in the usual sense, as co-compact hyperbolic lattices.

In §6, we use Theorem 1.1 (and some other machinery) to prove:

Corollary 1.2 *Let S be a closed hyperbolic surface group. Let $A_1, A_2, \ldots$ be a sequence of hyperbolic automorphismns of S, all coarse orbit equivalent to each other. Then*

1. *The corresponding mapping cylinders are all commensurable.*
2. *There are only finitely many commensurability classes of surface automorphisms in the sequence.*

To illustrate the sharpness of the corollary, we give, in §7, some examples showing

1. There is an infinite sequence of hyperbolic automorphisms $A_i : S_i \to S_i$ which are coarse-orbit-equivalent but pairwise incommensurable. The groups S_i are all hyperbolic surface groups, but the genus of the surface tends to ∞ with i.
2. There exist two hyperbolic automorphisms $A_1, A_2 : S \to S$ which are coarse orbit equivalent but not commensurable.

Both examples are based on manifolds which fiber over the circle in inequivalent ways.

1.4 Acknowledgments

Almost everything in this paper concerning surface diffeomorphisms owes its existence to results of Bill Thurston. I would also like to thank Alberto Candel for suggesting that the analogue of [**S2**, Action Rigidity Theorem] might

be true for hyperbolic surfaces. (The Action Rigidity Theorem is similar to Corollary 1.2, but applies to semisimple ergodic abelian torus actions.) I would also like to thank the referee of this paper for catching a mistake in the proof of Corollary 1.2, in an earlier version of this paper.

Finally, I would like to thank the University of Maryland for their hospitality during the writing of this paper.

2 The Proof in Broad Strokes

let $\boldsymbol{H}$ denote n dimensional hyperbolic space, for some $n \geq 3$. Let d denote the hyperbolic metric. We will use the upper half space model of $\boldsymbol{H}$. In this model, $\partial\boldsymbol{H}$ consists of the one point compactification of $n-1$ dimensional Euclidean space $\boldsymbol{E}$.

2.1 Ambient Extension

A *K-net* $N \subset \boldsymbol{H}$ is a countable collection of points such that no two points of N are within 1 of each other, and every point of $\boldsymbol{H}$ is within K points of N. A *K-quasi-isometry* of $\boldsymbol{H}$ is a K-bi-lipschitz bijection between two K-nets $N_1, N_2 \subset \boldsymbol{H}$. Since such maps are bijective, we will frequently confound them with their inverses. Hopefully this will not cause confusion.

Let $\phi : G_1 \to G_2$ be as in Theorem 1.1. We say that a quasi-isometry *q pairs* the geodesics in G_1 with the geodesics in G_2 *as ϕ does* if there is a function $p : \boldsymbol{N} \to \boldsymbol{N}$ having the following property. If q is defined on a point x, and $d(x,\gamma) \leq k$, for some $\gamma \in G_j$, then $d(q(x), \phi(\gamma)) \leq p(k)$. A necessary and sufficient condition for this is that ∂q and ϕ coincide on endpoints of the relevant geodesics.

In §3 we will prove

Lemma 2.1 (Extension Lemma) *There is a quasi-isometry q which pairs the geodesics in G_1 with those in G_2 as ϕ does.*

We will abbreviate the conclusion of the Lemma above by saying that that *q pairs G_1 with G_2*. We will set $h = \partial q$.

2.2 Eccentric Maps

Let T be a real linear map of $\boldsymbol{E}$. Let g_1, g_2 be two conformal maps of $\partial\boldsymbol{H}$. we say that the map

$$\mu = g_2 \circ T \circ g_1^{-1}$$

is an *eccentric map* provided that

1. μ preserves $\boldsymbol{E}$ and fixes 0.

2. μ is differentiable at 0.

3. μ is <u>not</u> a real linear map.

The essential feature of μ is that it is a non-linear real rational map, of uniformly bounded degree.

In §4 we will use a geometric limiting argument to prove:

Lemma 2.2 (Eccentricity Lemma) *Let G_j, Γ_j, h and q be as above. If h is <u>not</u> a conformal map, then there are symmetric patterns of geodesics Ω_j, bounded geometry subsets $W_j \subset \Omega_j$, and a quasi-isometry $\omega : \boldsymbol{H} \leftrightarrow \boldsymbol{H}$ such that*

1. *ω pairs the geodesics in W_1 to those in W_2.*

2. *$\mu = \partial\omega$ is an eccentric map.*

3. *The geodesic $\gamma = \overline{0\infty}$ belongs to both W_1 and W_2.*

The analysis used to prove the Eccentricity Lemma is based on the regularity results in [**M**] concerning quasiconformal maps.

In the next two sections, we will see that the conclusion of Eccentricity Lemma is impossible. In other words, h must be conformal, and the isometry extending h pairs up the geodesics of G_1 with those of G_2 as ϕ does.

2.3 Scattering of Points

Let Λ be an infinite cyclic group generated by a similarity of $\boldsymbol{E}$ which is not an isometry. The quotient

$$Q = Q(\Lambda) = \boldsymbol{E} - \{0\}/\Lambda$$

is diffeomorphic to the product of a circle and a codimension one sphere. Let $\pi : \boldsymbol{E} - \{0\} \rightarrow Q$ be the covering map. Also, let $F = F(\Lambda)$ be the fundamental domain for Λ which has the form

$$F = \{x \in \boldsymbol{E} |\ 1 \leq \|x\| < \lambda\}.$$

Here λ is the expansion factor for the generator of Λ.

Now suppose that we have two such groups Λ_1 and Λ_2. Suppose also that we are given:

1. An eccentric map μ.
2. A subset $\Sigma \subset Q_1$.
3. A subset $S \subset \boldsymbol{E}$

we define a subset $\Psi(\mu, \Sigma, S) \subset Q_2$ as follows:

$$\Psi(\mu, \Sigma, S) = \pi_2 \circ \mu(S \cap \pi_1^{-1}(\Sigma)).$$

In this definition, $\pi_1^{-1}(\Sigma)$ is the complete inverse image of Σ.

We say that a subset $S \subset \boldsymbol{E}$ is *adapted to* Λ if there is an infinite sequence of distinct contracting elements $T_1, T_2, T_3, ... \subset \Lambda$ such that $T_j(F) \subset S$. We say that a subset $\Sigma \subset Q$ is *δ-dense* if every point of F is within δ of some point of $\pi^{-1}(\Sigma) \cap F$.

In §5 we will prove:

Lemma 2.3 (Scattering Lemma) *Independent of μ there is a constant $\delta_0 > 0$ having the following property: If $S \subset \boldsymbol{E}$ is adapted to Λ_1 and $\Sigma \subset Q_1$ is δ_0-dense, then $\Psi(\mu, \Sigma, S) \subset Q_2$ is an infinite set.*

2.4 The Contradiction

Let Ω_j, W_j, ω, μ, and γ be as in the Eccentricity Lemma. For ease of notation, we suppress the subscript $j = 1, 2$, which appears on all the objects below.

Let Λ be the stabilizer subgroup $\gamma = \overline{0\infty}$, in the isometry group of Ω. By passing to an index two subgroup if necessary, we can assume that Λ is cyclic. Let Q, π, and F be as above. Let T be the generator for Λ, which we take to be a contraction.

For any interval $[a,b]$ of integers, we define

$$F[a,b] = \bigcup_{i=a}^{b} T^i(F).$$

For any positive integer k, we define ∂W^k to be the set of endpoints of geodesics in W which come within k of γ. Finally, we define

$$\sigma[a,b,k] = \partial W^k \cap F[a,b].$$

Let δ_0 be the constant in the Scattering Lemma. Since W has bounded geometry, there are integers d_0 and k_0 such that $\pi(\sigma[a,b,k]) \subset Q$ is δ_0-dense provided that $b - a \geq d_0$ and $k \geq k_0$. We set

$$F[a] = F[a, a+d_0]; \qquad \sigma[a] = \sigma[a, a+d_0, k_0].$$

We now bring back the subscript j. Modulo Λ_1 there are only finitely many possible points of $\sigma_1[a]$, as a varies in $\boldsymbol{Z}$. Hence, by the Pidgeonhole Principle, there is an infinite sequence of positive integers $i_1, i_2, i_3, \ldots$ such that the sets $\sigma_1[i_m]$ are all equivalent under Λ_1. Define

$$S_1 = \bigcup_{m=1}^{\infty} F_1[i_m].$$

and

$$\Sigma_1 = \pi_1(\partial W_1^{k_0} \cap S_1).$$

The sets S_1 and Σ_1 enjoy the following properties:

1. $\Sigma_1 \in Q_1$ is finite and δ_0-dense.

2. S_1 is adapted to Λ_1.

3. $S_1 \cap \pi_1^{-1}(\Sigma_1) \subset \partial W_1^{k_0}$.

We conclude, from the Scattering Lemma, that $\Psi = \Psi(\mu, \Sigma_1, S_1)$ is an infinite subset of Q_2.

On the other hand, since μ is the boundary map of a quasi-isometry ω which pairs the geodesics of W_1 with those of W_2, we see that there is a constant k_0' such that

$$\mu(\partial W_1^{k_0}) \subset \partial W_2^{k_0'}.$$

Combining this with statement (3) above, we see that

$$\mu(S_1 \cap \pi^{-1}(\Sigma_1)) \subset \partial W_2^{k_0'}.$$

Projecting, we see that

$$\Psi \subset \pi_2(\partial W_2^{k_0'}).$$

Recall that $W_2 \subset \Omega_2$. Modulo the isometry group of Ω_2, there are only finitely many geodesics of W_2 which come within k_0' units of the geodesic γ. Therefore, $\pi_2(\partial W_2^{k_0'})$ is a finite subset of Q_2. Hence, so is Ψ. This is a contradiction.

2.5 Mostow Rigidity

In this section, we will use Theorem 1.1 to prove that two diffeomorphic closed hyperbolic n-manifolds are isometric, if $n \geq 3$. This is a slightly more restrictive statement than is usually associated with Mostow rigidity. We make this restriction only for ease of exposition.

Suppose $f : M_1 \to M_2$ is a diffeomorphism from one hyperbolic n-manifold to another. Choose any closed geodesic $\gamma_1 \subset M_1$. The image $\gamma_2 = f(\gamma_1)$ is a closed curve in M_2.

Let Γ_1 be the lift of γ_1 to $\boldsymbol{H}$. Let Γ_2' denote the lift of γ_2 to $\boldsymbol{H}$. Let Γ_2 denote the family of geodesics obtained by replacing each curve in Γ_2' by the geodesic which has the same endpoints. Note that both Γ_1 and Γ_2 are symmetric patterns of geodesics.

The lift $\tilde{f} : \boldsymbol{H} \to \boldsymbol{H}$ is a bi-lipschitz mapping. Since every geodesic of Γ_2 is uniformly close to the correpsonding curve in Γ_2', the map $\tilde{f}$ induces a uniformly proper map from Γ_1 to Γ_2. From Theorem 1.1, the map $\tilde{f}$ is uniformly close to an isometry f_* of $\boldsymbol{H}$. Clearly f_* conjugates the fundamental group of M_1 to that of M_2. This is to say that M_1 and M_2 are isometric.

3 Extension Lemma

The purpose of this chapter is to prove Extension Lemma of §2.1. We will use the notation established in §2.1.

3.1 Subsets of Geodesics

We begin with a structural result about symmetric patterns of geodesics.

Lemma 3.1 *Let Γ be a symmetric pattern of geodesics. Then there is a function $h : \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{N}$ having the following property: Let $K \subset \Gamma$ be a subset consisting of at most k_1 geodesics. Suppose that $d(\alpha, \beta) \leq k_2$ for any two geodesics $\alpha, \beta \in K$. Then there is some ball of radius $h(k_1, k_2)$ which intersects every geodesic of K.*

Proof: By induction, it is sufficient to prove the Lemma for triples of geodesics–i.e. for $k_1 = 3$. Suppose that we have a sequence of counterexamples, $\alpha_k, \beta_k, \gamma_k$, which remain pairwise with in $n = k_2$ of each other. Since there are only finitely many lines modulo the symmetry group of Γ, we can assume $\alpha_k \equiv \alpha$. For now, we will drop the subscript from β and γ, even though it is implicit in what it said.

Let G_α be the stabilizer of α in the symmetry group of Γ. By passing to an index two subgroup if necessary, we may assume that G_α is a cyclic group. The generator of G_α has an attracting fixed point α_+ and a repelling fixed point α_-. These two points are the endpoints of α. Let F_α be any fundamental domain for α/G_α. Let F_β and F_γ respectively be translates of F_α so that $d(\beta, F_\beta) \leq n$ and $d(F_\gamma, \gamma) \leq n$.

We now mention the subscripts explicitly. Translating by suitable elements of G, we assume that F_α is fixed, $F_{\beta_k} \to \alpha_+$, and $F_{\gamma_k} \to \alpha_-$. Since Γ is a symmetric pattern of lines, there are only finitely many lines which have points within n of F_α. Call the set of these lines S_α. Likewise define S_{β_k} and S_{γ_k}. Note that $\beta_k \in S_{\beta_k}$ and $\gamma_k \in S_{\gamma_k}$, by definition. Let g be the generator of G_α. There are integers m_k, n_k such that

$$g^{m_k}(S_\alpha) = S_{\beta_k}; \quad g^{n_k}(S_\alpha) = S_{\gamma_k}.$$

Furthermore, $m_k \to +\infty$ and $n_k \to -\infty$.

From the finiteness of S_α, we see that the endpoints of geodesics in S_{β_k} converge to α_+ and the endpoints of geodesics in S_{γ_k} converge to α_-. Hence, for sufficiently large k, any geodesic in S_{β_k} will be more than n units from any geodesic in S_{γ_k}. $\square$

3.2 Two Basic Facts

In this section we state two easily proven facts about symmetric patterns of geodesics. We omit the proofs.

Fact 1: Let Γ be a symmetric pattern of geodesics. There is a positive lower bound for the distance between two non-intersecting geodesics in Γ. Furthermore, there is a positive lower bound to the angle between two intersecting geodesics in Γ.

fact 2: Let $\gamma_1, \gamma_2 \in \Gamma$ be two distinct geodesics. Let r be any positive number. Let B_1 and B_2 be two metric r-balls in $\boldsymbol{H}$, each of which intersects both γ_1 and γ_2. There is some uniform upper bound r' to the distance between B_1 and B_2.

3.3 Main Construction

Let G_j, Γ_j, and ϕ be as in Theorem 1.1. Since G_1 has bounded geometry, there is a positive number d having the following property: Every ball of radius d in $\boldsymbol{H}$ intersects at least two geodesics of G_1.

For any point $x \in \boldsymbol{H}$, there is some finite subset $K_x \subset G_1$ which intersects the ball of radius d about x. By definition, there are at least 2 geodesics in K_x. From Fact 1, there is a uniform upper bound on the number of geodesics in K_x. We define $q(x)$ to be the center of any ball of minimum radius which intersects all the geodesics of $\phi(K_x)$.

Since ϕ is uniformly proper, and there is a uniform upper bound to the cardinality of $\phi(K_x)$, there is by Lemma 3.1 a uniform upper bound to the minimum radius of the ball about $q(x)$ which intersects all geodesics in $\phi(K_x)$. Fact 2 now says that the choice of minimal ball in the definition of q is unique up to a uniformly small additive error.

Lemma 3.2 *q is a uniformly proper map, with respect to the hyperbolic metric on $\boldsymbol{H}$.*

Proof: Rather than introduce the auxilliary function used in the definition of uniform properness, we will give a heuristic argument which can easily be made numerical. If x and y are close together, then there is a uniform bound to the pairwise (minimal) distance between a geodesic in K_x and K_y. From Lemma 3.1 it follows that there is a uniform bound to the distance from a ball of minimum radius intersecting both $\phi(K_x)$ and $\phi(K_y)$. Hence $q(x)$ and $q(y)$ are uniformly close.

Suppose, on the other hand, that $q(x)$ and $q(y)$ are close. Let B_x and B_y be the balls of radius d which respectively intersect all geodesics of K_x and all geodesics of K_y. From Lemma 3.1, there is a uniformly small ball in $\boldsymbol{H}$ which intersects all the geodesics in $\phi(K_x) \cup \phi(K_y)$. Since ϕ is uniformly proper, Lemma 3.1 says that there is a uniformly small ball B which intersects all the geodesics $K_x \cup K_y$. Fact 2 says that B must be uniformly close to both B_x and B_y. Hence x and y are uniformly close. □

Since G_2 has bounded geometry, $q(\boldsymbol{H})$ is dense in $\boldsymbol{H}$ up to an additive constant. Thus, we may define a "near inverse" map q^{-1}. This map is also uniformly proper, and the composition of q and q^{-1}, in either order, is uniformly close to the identity. Since $\boldsymbol{H}$ is a path metric space, these two conditions imply that q restricts to a quasi-isometry.

q has a continuous and bijective extension $h = \partial q$ to $\partial \boldsymbol{H}$. It is a standard fact that q takes geodesics into uniformly thin tubular neighborhoods of geodesics. The action of both ϕ and q on geodesics is therefore controlled by the action of h on endpoints. These statements imply the Extension Lemma.

4 Eccentricity Lemma

The purpose of this chapter is to prove the Eccentricity Lemma of §1.2. We will use the notation established in §1.2. The material here is an adaptation of [**S1**, Ch. 6] to the present setting.

4.1 Hausdorff Topology

The *Hausdorff distance* between two compact subsets $K_1, K_2 \subset \boldsymbol{H}$ is defined to be the minimum value $\delta = \delta(K_1, K_2)$ such that every point of K_j is within δ of a point of K_{j+1}. (Indices are taken mod 2.) A sequence of closed subsets $S_1, S_2, ... \subset \boldsymbol{H}$ is said to converge to $S \subset \boldsymbol{H}$ in the *Hausdorff topology* if, for every compact set $K \subset \boldsymbol{H}$, the sequence $\{\delta(S_n \cap K, S \cap K)\}$ converges to 0.

Let $q : \boldsymbol{H} \to \boldsymbol{H}$ be a quasi-isometry, defined on nets $N_1, N_2 \subset \boldsymbol{H}$. Let $Gr(q) \subset \boldsymbol{H} \times \boldsymbol{H}$ denote the graph of q. Note that $Gr(q)$ is a net in $\boldsymbol{H} \times \boldsymbol{H}$.

In what follows, the positive integer K is fixed. Let q_n be a K-quasi-isometry defined relative to K-nets $N_{n,1}$ and $N_{n,2}$. We say that $q_1, q_2, ...$ *converges* to q provided that:

1. $N_{n,j}$ converges to a K-net N_j in the Hausdorff topology.

2. $Gr(q_n)$ converges to $Gr(q)$ in the Hausdorff topology.

$q : N_1 \to N_2$ will automatically be K-bi-lipschitz.

Lemma 4.1 *Let $\{q_n\}$ be a sequence of K-quasi-isometries of* $\mathbf{H}$*. Let $h_n = \partial q_n$ be the extension of q_n. Suppose that:*

1. $h_n(0) = 0$.

2. $h_n(\mathbf{E}) = \mathbf{E}$.

3. *h_n converges uniformly on compacta to a homeomorphism $h : \mathbf{E} \to \mathbf{E}$.*

Then the maps q_n converge on a subsequence to a quasi-isometry q. Furthermore $\partial q = h$.

Proof: Let 0 be any chosen origin of hyperbolic space. We will first show that the set $\{q_n(0)\}$ is bounded. Let γ_1 and γ_2 be two distinct geodesics through 0. Then the quasi-geodesics $q_n(\gamma_j)$ remain within uniformly thin tubular neighborhoods of geodesics $\delta_{n,1}$ and $\delta_{n,2}$. By hypothesis, the endpoints of $\delta_{n,1}$ and and $\delta_{n,2}$ converge to four distinct points of $\partial \boldsymbol{H}$. Furthermore, the point $q_n(0)$ must lie close to both $\delta_{n,1}$ and $\delta_{n,2}$. This implies that $\{q_n(0)\}$ is bounded. Statement 1 now follows from a routine diagonalization argument.

By thinning out the sequence, we can assume that q_n converges to a quasi-isometry q_∞. Let $h_\infty = \partial q_\infty$. Let p be any point in $\mathbf{E}$. Let γ be any geodesic, one of whose endpoints is p. Let δ_n denote the geodesic whose tubular neighborhood contains $q_n(\gamma)$. Then δ_n converges to some geodesic δ_∞, in the Hausdorff topology. Hence the endpoints of δ_n converge to those of δ_∞. This means that $h_n(p)$ converges to $h_\infty(p)$. Hence $h(p) = h_\infty(p)$. Since p is arbitrary, we get Statement 2. $\square$

Lemma 4.2 *Let I_n be a sequence of hyperbolic isometries. Let Γ be a symmetric pattern of geodesics. Let $G \subset \Gamma$ be a bounded geometry subset. Let $G_n = I_n(G)$ and $\Gamma_n = I_n(\Gamma)$. Then, on a subsequence, Γ_n converges to a symmetric pattern of geodesics, Γ', and G_n converges to a bounded geometry subset $G' \subset \Gamma'$.*

Proof: The convergence of Γ_n, on a subsequence, is obvious from the fact that Γ has co-compact symmetry group. The subsets $I_n(G)$ have uniformlly bounded geometry, independent of n. A routine diagonalization argument shows that we may extract a convergent subsequence. $\square$

4.2 Differentiability Principle

We say that a map $D : \boldsymbol{E} \to \boldsymbol{E}$ is a *pure dilation* if has the form $D(v) = \lambda v$ for some $\lambda > 1$. We say that a sequence $D_1, D_2, ...$ of pure dilations is *unbounded* if the expansion constants $\lambda_1, \lambda_2, ...$ tend to ∞.

Suppose that $f : \boldsymbol{E} \to \boldsymbol{E}$ is a homeomorphism which fixes the origin, and which is differentiable at the origin. Then the differential $df(0)$ is a linear transformation of the tangent space $T_0(\mathbf{E})$.

For any unbounded sequence $D_1, D_2, ...$ of pure dilations, consider the sequence:

$$f_n = D_n \circ f \circ D_n^{-1}.$$

It is a standard fact from several variable calculus that f_n converges, uniformly on compacta, to a linear transformation f_∞, and that $f_\infty = df(0)$, under the canonical identification of $T_0(\mathbf{E})$ with $\mathbf{E}$. We will call this the *Differentiability Principle*.

4.3 Main Construction

Let G_j, Γ_j, q and h be as in the Extension Lemma. Then h is a quasi-conformal map of $\partial \boldsymbol{H}$. Composing with isometries, we may assume that $h(\infty) = \infty$. Thus h is a quasi-conformal self map of $\boldsymbol{E}$. Such maps are almost everywhere differentiable. [**M**, Th. 9.1]. Let $dh(x)$ denote the differential of h at x. If h is <u>not</u> conformal, then there is a positive measure subset $S \subset \boldsymbol{E}$ where $dh(x)$ is <u>not</u> a conformal map, for $x \in S$. [**M**, Lemma 12.2].

Translating, we may assume that $x = 0$, that $h(0) = 0$, and that $dh(0)$ exists but is not conformal.

Choose an unbounded sequence $D_1, D_2, \ldots$ of pure dilations. There are hyperbolic isometries $\hat{D}_1, \hat{D}_n, \ldots$ such that $\partial \hat{D}_n = D_n$. Consider the objects:

1. $q_r = \hat{D}_r \circ q \circ \hat{D}_r^{-1}$
2. $\Gamma_{r,j} = \hat{D}_r(\Gamma_j)$
3. $G_{j,r} = \hat{D}_r(G_j)$.
4. $h_r = \partial q_r = D_r \circ h \circ D_r^{-1}$.

From the differentiability principle, the maps h_r converges to the linear map $h' = dh(0)$, as $r \to \infty$. By Lemma 4.2, $\Gamma_{r,j}$ converges, on a subsequence, to a symmetric pattern of geodesics Γ'_j. Likewise, $G_{j,r}$ converges to a bounded geometry subset $G'_j \subset \Gamma'_j$. By Lemma 4.1, the maps q_r converge, on a thinner subsequence, to a quasi-isometry q' with $\partial q' = h' = dh(0)$. Clearly, q' pairs G'_1 with G'_2.

Let γ_j be a geodesic of G'_j which has both endpoints in $\boldsymbol{E}$. We may choose these geodesics so that h' takes the endpoints of γ_1 to those of γ_2. Let g_j be an isometry which takes the endpoints of γ_j to 0 and ∞. We may choose g_j in such a way that h' takes $g_1^{-1}(\infty)$ to $g_2^{-1}(\infty)$. The following objects satisfy the conclusion of the Eccentricity Lemma:

1. $W_j = g_j(G'_j)$.
2. $\Omega_j = g_j(\Gamma'_j)$.
3. $\omega = g_2 \circ q' \circ g_1^{-1}$.
4. $\mu = g_2 \circ h' \circ g_1^{-1}$.

The only slightly non-obvious point is that μ is non-linear. Here is a quick proof[1]: We write $g_2^{-1} \circ \mu = h' \circ g_1$. Suppose μ is linear. Then, for a generic hyperplane Π, we have that $g_1(\Pi)$ is a round codimension-one sphere, and so is $g_2^{-1} \circ \mu(\Pi)$. In particular, there is some codimension-one round sphere S such that $h'(S)$ is also a codimension-one round sphere. This implies that h is a similarity, in contradiction to our assumption.

[1]Supplied by the referee.

5 Scattering Lemma

The purpose of this chapter is to prove the Scattering Lemma of §1.3. We will use the notation developed in §1.3.

5.1 Zariski Density

Let $\Sigma \subset U$ be two subsets of $\boldsymbol{E}$. We say that Σ is δ-*dense* in U if every point of U is within δ of some point of Σ.

Let $\phi = p/q$ be a real rational function on $\boldsymbol{E}$. Here $p = p(x_1, ..., x_n)$ and $q = q(x_1, ..., x_n)$ are polynomials. We say that $\phi \in F(k_1, k_2)$ if

1. p and q have degree at most k_1.
2. The sets $\{p = 0\}$ and $\{q = 0\}$ have dimension zero and cardinality at most k_2.

The following technical result is quite analogous to the statement that two polynomials agree everywhere if they agree on sufficiently many points.

Lemma 5.1 *Let U be any open set, and let k_1, k_2 be given. There is some constant $\delta = \delta(k_1, k_2, U) > 0$ having the following property: If $\Sigma \subset U$ is δ-dense, then any two elements of $F(k_1, k_2)$ which agree on Σ agree everywhere.*

Proof: Let $n = \dim(\boldsymbol{E})$. Let $v_1, ..., v_N$ be the monomials on $\boldsymbol{E}$ of degree at most k_1, listed in any order. Define $\psi : \boldsymbol{E} \to \boldsymbol{R}^N$ by:

$$\psi(x) = (v_1(x), ..., v_N(x)).$$

It is clear that $\psi(U)$ spans $\boldsymbol{R}^N$. Otherwise, we could construct some nonzero polynomial which vanished on U but which did not identically vanish. Since ψ is uniformly continuous on the closure of U, the linear span of $\psi(\Sigma)$ is all of $\boldsymbol{R}^N$, for any sufficiently dense subset $\Sigma \subset U$.

Now consider our two functions $\phi_1, \phi_2 \in F(k_1, k_2)$. We write $\phi_j = p_j/q_j$. By deleting at most $2k_2$ points, we can assume that the denominator q_j never vanishes on Σ. Thus, for every point $x \in \Sigma$, we have $p_j(x) = 0$. We write:

$$p_j = \sum_{k=1}^{N} a_{j,k} v_k; \qquad A_j = (a_{j,1}, ..., a_{j,N}).$$

Thus, for each $x \in \Sigma$, we have

$$\psi(x) \cdot (A_1 - A_2) = 0.$$

Since $\psi(\Sigma)$ spans $\boldsymbol{R}^N$, we see that $A_1 = A_2$. We finish the proof by applying the same argument to $1/\phi_j = q_j/p_j$. $\square$

Corollary 5.2 *Let $U \subset \boldsymbol{E}$ be any open subset. Then there is a constant $\delta = \delta(U) > 0$ having the following property. If two eccentric linear maps agree on a δ-dense subset of U, then they agree everywhere.*

Proof: Let I denote inversion in the unit sphere of $\boldsymbol{E}$. I is a quadratic rational map of $\boldsymbol{E}$. It is not hard to see that every eccentric map may be written in the form

$$\tau_1 \circ (I \circ T \circ I) \circ \tau_2,$$

where T is linear, and τ_j is an isometry of $\boldsymbol{E}$. An easy computation shows that the coordinate functions of any eccentric map belong to (say) $F(100, 100)$. We now apply the Lemma above coordinatewise. $\square$

5.2 Flexible Differentiability Principle

Suppose that $h : \boldsymbol{E} \to \boldsymbol{E}$ is a homeomorphism, such that $h(0) = 0$ and $dh(0)$ exists. Let T_1 and T_2 be two contracting similarities of $\boldsymbol{E}$, both of which fix 0. These maps need not be dilations; some rotational component is allowed. For each pair k, k' of positive integers we define the map

$$h[k, k'] = T_2^{-k'} \circ h \circ T_1^k.$$

Lemma 5.3 *Suppose that $K_1, K_2 \subset \boldsymbol{E}$ are compact subsets. Suppose that $(k_1, k_1'), (k_2, k_2'), \ldots$ is a sequence of pairs such that*

1. *$k_n \to \infty$.*

2. *$h[k_n, k_n'](K_1) \cap K_2 \neq \emptyset$.*

Then, on some subsequence, $h[k_n, k_n']$ converges, uniformly on compacta, to a linear map.

Proof: There is a similarity s_n, a pure dilation D_n, and a pure rotation r_n, such that

$$h[k_n, k_n'] = s_n \circ (D_n \circ h \circ D_n^{-1}) \circ r_n.$$

Moreover, the sequence $D_1, D_2, ...$ is unbounded. From the Differentiability Principle of §4, the maps $D_n \circ h \circ D_n^{-1}$ converge, uniformly on compacta, to $dh(0)$. Hence, the condition on the image of K_1 implies that the sequence s_n lies in a compact subset of maps. The same of course is true for the rotations r_n. Hence, on a subsequence, we get the desired convergence. $\square$

We shall call this the *Flexible Differentiability Principle.*

5.3 The Moment of Truth

Let μ, $Q_j = Q(\Lambda_j)$ and $F_j = F(\Lambda_j)$ be as in the Scattering Lemma. Let T_j be the generator of Λ_j, which we take to be a contraction. let $\delta_0 = \delta(U)$ be the positive constant in the conclusion of Lemma 5.2.

Suppose that $\Sigma \subset Q_1$ is and δ_0-dense subset, and suppose that $S \subset \boldsymbol{E}$ is any subset which is adapted to Λ_1. Our goal is to prove that the set $\Psi(\mu, \Sigma, S) \subset Q_2$ is infinite.

Define

$$\Sigma_0 = \pi_1^{-1}(\Sigma) \cap F_1.$$

By hypothesis, there is an infinite sequence of positive integers $k_1, k_2, k_3, ...$ such that

$$T_1^{k_m}(F_1) \subset S.$$

This is to say, in particular, that

$$T_1^{k_m}(\Sigma_0) \subset \pi_1^{-1}(\Sigma) \cap S.$$

For each such number k_m, choose any positive integer k_m' such that

$$T_2^{-k_m'} \circ \mu \circ T_1^{k_m}(\Sigma_0)$$

has nonempty intersection with $\overline{F}_2$. Let μ_m be this composition. Define

$$V = \bigcup_{m=1}^{\infty} \mu_m(\Sigma_0).$$

Here is the moment of Truth:

Lemma 5.4 *V contains a bounded infinite set.*

Proof: Since μ is eccentric, so is μ_m. Since Σ_0 and $\overline{F}_2$ are both compact, the Flexible Differentiability Principle implies that the maps μ_m converge, on a subsequence, to a linear map. By deleting elements, we may assume that our whole sequence converges. Thus, V is bounded.

Since none of the maps μ_m is linear, but the limit is, there are infinitely many distinct maps in our sequence. If V was a finite set, then there would only be finitely many choices for the image $\mu_m(\Sigma_0)$. On the other hand, by our choice of constants, two eccentric maps which agree on Σ_0 coincide. Hence V is infinite. $\square$

Observe that

$$\mu_m(\Sigma_0) \subset T_2^{-k'_m}\mu(S \cap \pi_1^{-1}(\Sigma)).$$

Summing over m, we have:

$$V \subset \bigcup_{\lambda \in \Lambda_2} \lambda \circ \mu(S \cap \pi_1^{-1}(\Sigma)).$$

Projecting, we have

$$(*) \qquad \pi_2(V) \subset \Psi(\mu, \Sigma, S).$$

This follows from the fact that $\pi_2(\lambda(J)) = \pi_2(J)$ for any subset $J \subset \boldsymbol{E}$ and any $\lambda \in \Lambda_2$.

Since V contains a bounded infinite set, $\pi_2(V)$ is an infinite set. From (*), we see that $\Psi(\mu, \Sigma, S)$ is also an infinite set, as desired.

6 Surface Group Automorphisms

6.1 The Auxilliary Pattern

Let $S = \pi_1(\Sigma)$ be a hyperbolic surface group, with identity element e. We will equip S with a left-invariant word metric d_S. Let M be the semidirect product of $\boldsymbol{Z}$ and S. The group law is:

$$[n_1, s_1] \cdot [n_2, s_2] = [n_1 + n_2, s_1 A^{n_1}(s_2)].$$

We will equip M with a left invariant word metric d_M. We identify $\boldsymbol{Z}$ with the subgroup generated by $[1, e]$.

By hypothesis, M has a faithful representation into $\mathrm{Isom}(\boldsymbol{H}^3)$. We will call this representation R. Let γ be the unigue geodesic stabilized by the element $R[1,e]$. Let Γ be the pattern of geodesics given by the orbit $RM(\gamma)$.

Let $I : S \to M$ be given by $I(s) = [0,s]$. It is easy to see that I is uniformly proper with respect to the two word metrics. Let $G = R \circ I$. The group G acts freely and transitively on Γ. Define $\beta : S \to \Gamma$ by the formula

$$\beta(s) = R[0,s](\gamma).$$

By construction, β is a bijection from S to Γ.

Let $\xi = \xi_A$ be the A-invariant symmetric function defined, in the introduction, on $S \times S$. Let ρ be the symmetric function on $\Gamma \times \Gamma$ describing the distance between pairs of geodesics.

The following technical result is at the heart of the relationship between the Main Theorem and Corollary 1.2.

Lemma 6.1 (Comparison Lemma) *The bijection $\beta : S \to \Gamma$ is uniformly proper with respect to the pair (ξ, ρ).*

Proof: Consider the bijection $\underline{I}$, from elements of S to right cosets of $\boldsymbol{Z}$ in M:

$$\underline{I}(s) = [0,s]\boldsymbol{Z}.$$

The group $R\boldsymbol{Z}$ stabilizes the geodesic γ, by definition. Consider the bijection ψ, from right cosets of $\boldsymbol{Z}$ into Γ, given by

$$\psi(a\boldsymbol{Z}) = Ra(\gamma).$$

Since $R\boldsymbol{Z}$ stabilizes γ, the map ψ is well defined. We compute

$$\psi \circ \underline{I}(s) = \psi([0,s]\boldsymbol{Z}) = R[0,s](\gamma) = \beta(s).$$

Hence

$$\beta = \psi \circ \underline{I}.$$

Define the symmetric function $\delta(a\boldsymbol{Z}, b\boldsymbol{Z})$ to be the minimum d_M distance from a representative of $a\boldsymbol{Z}$ to a representative of $b\boldsymbol{Z}$.

Sub-Lemma 6.2 *ψ is uniformly proper with respect to (δ, ρ).*

Proof: Let $x \in \gamma$ be any chosen basepoint. Consider the map $\Psi : M \to \boldsymbol{H}^3$ given by $\Psi(m) = Rm(x)$. It is well known that Ψ is a quasi-isometry with respect to the pair $(d_M, d_{\boldsymbol{H}})$. Here $d_{\boldsymbol{H}}$ is hyperbolic distance. Our sublemma folllows immediately from this and from the definitions. □

Sub-Lemma 6.3 *$\underline{I}$ is uniformly proper with respect to (ξ, δ).*

Proof: Below, the constants $C_1, C_2, ...$ have the desired dependence. Suppose that $\xi(s,t) \leq C_1$. By definition, there is an integer n for which

$$d_A(A^n(s), A^n(t)) \leq C_1.$$

Since I is uniformly proper with respect to (d_A, d_M), we have

$$d_M([0, A^n(s)], [0, A^n(t)]) \leq C_2.$$

A routine computation shows that

$$[-n, e] \cdot [0, A^n(v)] = [-n, v] \in [0, v]\boldsymbol{Z}.$$

This is true for $v = s, t$. Since left multiplication is an isometry, we have that

$$d_M([-n, s], [-n, t]) \leq C_2.$$

By definition, this implies that $\delta([0, s]\boldsymbol{Z}, [0, t]\boldsymbol{Z}) \leq C_2$.

At this point, we reset our constants. For the converse, suppose that $\delta([0, s]\boldsymbol{Z}, [0, t]\boldsymbol{Z}) \leq C_1$. Then, by definition, there are integers n_s, n_t such that

$$d_M([-n_s, s], [-n_t, t]) \leq C_1.$$

From the structure of the semidirect product, we conclude that

$$|n_s - n_t| \leq C_3.$$

Therfore, by the triangle inequality.

$$d_M([-n, s], [-n, t]) \leq C_4.$$

Here we have set $n = n_s$. Since left multiplication is an isometry, we have

$$d_M([0, A^n(s)], [0, A^n(t)] \leq C_4.$$

Since I is uniformly proper with respect to the two word metrics, we have

$$d_A(A^n(s), A^n(t)) \leq C_5.$$

By definition, this says that $\xi(s,t) \leq C_5$. $\square$

Combining the two Sub-Lemmas, and recalling that $\beta = \psi \circ \underline{I}$, we see that β is uniformly proper with respect to (ξ, ρ), as desired. $\square$

6.2 Commensurability of Mapping Cylinders

In this section, we prove the first statement of Corollary 1.2. Suppose that A_1 and A_2 are hyperbolic automorphisms of S. Suppose that $g : S \to S$ is a bijection which is uniformly proper with respect to (ξ_1, ξ_2).

Consider the associated map

$$\phi = \beta_2 \circ g \circ \beta_1^{-1}.$$

ϕ is a bijection from Γ_1 to Γ_2. By the Comparison Lemma, ϕ is uniformly proper with respect to the pair (ρ, ρ). Hence, by our Main Theorem, ϕ is a hyperbolic isometry. In particular, the patterns Γ_1 and Γ_2 are isometric. Hence, the lattices $Aut(\Gamma_1)$ and $Aut(\Gamma_2)$ are conjugate.

Now, the group $R_j M_j$ has finite index in $Aut(\Gamma_j)$. Thus, $R_1 M_1$ and $R_2 M_2$ have finite index in conjugate groups. This is to say that M_1 and M_2 are commensurable as lattices. This is Statement 1 of Corollary 1.2.

6.3 Finiteness of Classes

In this section, we prove Statement 2 of Corollary 1.2. Let $A_1, A_2, ...$ be a sequence of hyperbolic surface automorphisms of S, all of which are coarse orbit equivalent. To prove Statement 2, we just have to find a pair of distinct indices (k, l), for which A_k and A_l are commensurable.

From the proof in the previous section, we may normalize our representations $R_1, R_2, ...$ so that the isometric patterns $\Gamma_1, \Gamma_2, ...$ all coincide. Let Γ denote this common pattern.

Consider the sequence of groups $G_j = R_j \circ I_j(S)$. The groups G_j are all copies of S, sitting in $Aut(\Gamma)$. At this point, we quote the following

Theorem 6.4 (Finiteness Theorem) *Let S be a closed hyperbolic surface group. Let Λ be a co-compact 3-dimensional hyperbolic lattice. Then, modulo conjugacy, there are only finitely many isomorphic images of S in Λ.*

Proof: In case Λ is torsion-free, this is exactly [**T2**, Cor. 8.8.6a]. For an extremely general case, which in particular covers the case above, see [**RS**, Theorem 7.1]. $\square$

From the Finiteness Theorem, we may find a pair of distinct indices (a, b) such that the groups G_a and G_b are conjugate in $Aut(\Gamma)$. Conjugating R_a by this element, we can assume that $G_a = G_b$.

Since G_a acts transitively on Γ, we may further conjugate R_a so that the geodesic γ_a (stabilized by $R_a[1, e]_a$) coincides with the geodesic γ_b (stabilized by $R_b[1, e]_b$.) Let γ denote this common geodesic. Consider the bijection

$$h = \beta_b^{-1} \circ \beta_a.$$

Lemma 6.5 *h is a group isomorphism.*

Proof: Clearly, h is bijective. Let $s \in S$. The element $h(s)$ has the property that

$$R_b \circ I_b(h(s))(\gamma) = R_a \circ I_a(s)(\gamma).$$

Since $G_a = G_b$, and since both these groups act freely on Γ, we conclude that

$$R_b \circ I_b(h(s)) = R_a \circ I_a(s).$$

Since $R_b \circ I_b$ and $R_a \circ I_a$ have the same image, and since $s \in S$ is arbitrary, we may write

$$h = (R_b \circ I_b)^{-1} \circ (R_a \circ H_a).$$

Thus h is an isomorphism. $\square$

To Complete the proof of Corollary 1.2, we prove

Lemma 6.6 *h commensurates A_a to A_b.*

Proof: From the Comparison Lemma, h is uniformly proper with respect to the pair (ξ_a, ξ_b). Conjugating A_a by h, we may assume that h is the *identity*. We just have to prove the following statement: If the identity is uniformly proper with respect to (ξ_a, ξ_b), then A_a and A_b are commensurable.

Choose any finite generating set $W = \{w_1, ..., w_n\}$ for S. Choose any element $w \in W$, and consider the orbit $A_a(w), A_a^2(w), ...$ Since the identity map is uniformly proper with respect to the pair (ξ_a, ξ_b), there are integers $k_1, k_2, ...$ and some fixed constant C such that

$$\omega^j = A_b^{k_j} \circ A_a^j(w)$$

is within C of the identity. Since there are only finitely many points within C of the identity, the sequence $\omega^1, \omega^2, ...$ must repeat infinitely often. Taking subsequences, and applying this argument to each element in W, we produce distinct integers p and q such that $A_a^{k_p} A_b^p$ and $A_a^{k_q} A_b^q$ agree on all of W. This easily implies that A_a and A_b are commensurable. $\square$

7 Some Examples

In this chapter we will give an example of an infinite sequence of hyperbolic automorphisms $A_i : S_i \to S_i$, which are coarse orbit equivalent but pairwise incommensurable. Similarly, we give an example of two hyperbolic automorphisms $A_1, A_2 : S \to S$ which are coarse-orbit-equivalent but incommensurable. The examples are based on hyperbolic manifolds which fiber over the circle in inequivalent ways.

7.1 Forms and Fibrations

Let M be a closed hyperbolic 3-manifold, with chosen origin $0 \in M$. Let $H_1(M)$ denote first singular homology on M. Every element in $H_1(M)$ can be represented by a closed geodesic in M, though not uniquely.

A closed 1-form ω is *integral* provided that it has integral periods on all elements of $H_1(M)$. Further, ω is *nonsingular* if it never vanishes. A vector field V is *adapted* to ω if $\omega_p(V_p) = 1$ at every point $p \in M$. Every nonsingular form admits an adapted vector field, though not a unique one. Say that the pair (ω, V) is *good* if

1. ω is closed, integral, and nonsingular.

2. V is adapted to ω.

3. The integral curve α of V through 0 is closed.

4. ω has period one on α.

Let γ be a closed geodesic. We say that ω is *primitive* on γ if ω has period 1 on γ.

Lemma 7.1 *Suppose that ω is closed, integral, and nonsingular. Suppose that ω is primitive on a closed geodesic γ. Then there is a good pair (ω, V) such that the associated pattern is isometric to the lift of γ to $\boldsymbol{H}^3$.*

Proof: First, choose any vector field V which is adapted to ω. If T is another vector field, tangent to the fibers of the fibration–i.e. annihilated by ω–then $V+T$ is still adapted to ω. It is easy, but tedious, to choose T in such a way that the trajectory through 0 is closed, and that the trajectory through 0 is freely homotopic to γ. $\square$

Integration of ω gives rise to a fibration $\phi = \phi_\omega : M \to \boldsymbol{R}/\boldsymbol{Z}$. We normalize so that $\phi(0) = 0$. Let $\Sigma = \phi^{-1}(0)$. Naturally, we use 0 as the basepoint of Σ. The field V defines a flow, whose time-1 map is a a basepoint preserving self-diffeomorphism $f : \Sigma \to \Sigma$. The induced map $\pi_1(f) : \pi_1(\Sigma) \to \pi_1(\Sigma)$ is a hyperbolic surface automorphism. We call this map the *associated automorphism*. The trajectory α is in the same free homotopy class as some closed geodesic γ. Let Γ be the lift of γ to the universal cover $\boldsymbol{H}^3$. We call Γ the *associated pattern*.

Lemma 7.2 (Equivalence Criterion) *Suppose (ω_1, V_1) and (ω_2, V_2) are good pairs. Then the associated automorphisms are coarse orbit equivalent if the associated patterns are isomorphic.*

Proof: Tracing through the construction in §6.1, and recalling the Comparison Lemma, we see that there is a bijection $\beta_j : \pi_1(\Sigma_j) \to \Gamma_j$, which is uniformly proper with respect to the pair (ξ_j, ρ). If Γ_1 and Γ_2 are isometric, then the composition $\beta_2^{-1} \circ \beta_1$ is a coarse orbit equivalence. $\square$

Lemma 7.3 (Incommensurability Criterion) *Suppose that* (ω_1, V_1) *and* (ω_2, V_2) *are good pairs. Suppose that the associated automorphisms are commensurable. Then there is a self-isometry of* M *which carries the cohomology class of* ω_1 *to that of* ω_2

Proof: For the moment, we supress indices. There is an n-fold cyclic cover $\tilde{M}(\omega, n)$, defined as follows: Let $\Omega(M)$ denote the space of paths of M, based at 0. Identify two paths if they have the same endpoints, and if the integral of ω over the first path agrees mod n with the integral over the second path. The quotient space is $\tilde{M}(\omega, n)$.

Suppose that the two maps $\pi_1(f_1)$ and $\pi_2(f_2)$ are commensurable. Then there is a basepoint preserving diffeomorphism $\delta : \Sigma_1 \to \Sigma_2$ which, up to basepoint preserving isotopy, conjugates $f_1^{n_1}$ to $f_2^{n_2}$. Standard arguments allow us to alter V_2 in such a way δ actually conjugates $f_1^{n_1}$ to $f_2^{n_2}$. (Note that altering V_2 alters f_2.)

The map δ can be extended, using the trajectories of V_j, to give a diffeomorphism

$$\tilde{\Delta} : \tilde{M}(\omega_1, n_1) \to \tilde{M}(\omega_2, n_2)$$

Which carries $\tilde{\omega}_1$ to $\tilde{\omega}_2$. Here $\tilde{\omega}_j$ is the lift of ω to $\tilde{M}(\omega_j, n_j)$.

For convenience, we set $\tilde{M}_j = \tilde{M}(\omega_j, n_j)$. By mostow rigidity, $\tilde{\Delta}$ is isotopic to an isometry $\tilde{I} : \tilde{M}_1 \to \tilde{M}_2$. $\tilde{I}$ carries the class of $\tilde{\omega}_1$ to that of $\tilde{\omega}_2$. From volume considerations, $n_1 = n_2$. Furthermore, $\tilde{I}$ conjugates the covering isometry group of $\tilde{M}_1$ to that of $\tilde{M}_2$. Finally, the class of $\tilde{\omega}_j$ is fixed by the elements of the covering group of $\tilde{M}_j$. Putting all this together, we see that $\tilde{I}$ induces an isometry $I : M \to M$ which carries the class of ω_1 to that of ω_2. $\square$

Below, we will produce a geodesic γ, and an infinite sequence $\omega_1, \omega_2 ...$ of closed, integral, nonsingular forms which are primitive on γ. These forms will have the following properties:

1. The genus of the fiber Σ_j defined by the fibration ϕ_j tends to ∞.
2. For some pair of indices i, j, the fibers Σ_i and Σ_j have the same genus. However, there is no self-isometry of M which carries the class of ω_i to the class of ω_j.

From the three Lemmas above, our sequence provides the two examples advertised at the beginning of this chapter. It is well known that such a situation can be arranged, using the tools in [—bf T3]. For convenience, we work this out explicitly.

7.2 Thurston's Norm on Homology

Let $H^1(M)$ denote the first de Rham cohomology group of M. Let $H^1(M, \boldsymbol{Z})$ denote the classes of integral forms. $H^1(M, \boldsymbol{Z})$ is isomorphic to $\boldsymbol{Z}^d$, for some integer d.

A *cone* in $H^1(M)$ is a set which is invariant under dilations. To save words later, we will stipulate that the origin does not belong to a cone. An *integral plane* in $H^1(M)$ is an affine subspace which intersects $H^1(M, \boldsymbol{Z})$ in a cocompact lattice. It is well known that two integral planes intersect in an integral plane. Here is a paraphrasal of some of the main results in [**T3**].

Lemma 7.4 (Cone Lemma) *There is a convex open cone $C \subset H^1(M)$ such that every point of $H^1(M, \boldsymbol{Z}) \cap C$ is represented by some nonsingular form. Moreover, C is foliated by convex pieces of codimension one parallel hyperplanes $\{H_t\}_{t>0}$. Points of $C \cap H^1(M, \boldsymbol{Z})$ which lie on the same leaf H_t all define fibrations having the same genus fiber. The hyperplane through the origin and parallel to H_t is integral.*

Proof: According to [**T3**, Th. 1], there is a norm on $H_2(M)$, whose unit sphere is an integral polyhedron. This norm induces, by duality, a norm N on $H^1(M)$, whose unit sphere is an integral polyhedron P. In case $\omega \in H^1(M, \boldsymbol{Z})$ is nonsingular, $-N(\omega)$ is equal to the Euler characteristic of the fiber in the fibration defined by ω.

By [**T3**, Th. 5], the set of classes in $H^1(M, \boldsymbol{Z})$ which have nonsingular representations forms a certain cone in $H^1(M)$. This cone intersects the unit sphere in top-dimensional faces. If we take a suitable sub-cone C, we can guarantee that C intersects P in a single top-dimensional face. Hence, the restriction of N to C agrees with a linear norm N'. The level sets of N' are exactly the foliating hyperplanes H_t. Since the relavent face of the unit sphere of N is contained in an integral hyperplane, the parallel hyperplane through the origin is also integral. $\square$

The isometry group of M acts as a finite group on $H^1(M)$. Hence, we may choose C so small that any nontrivial motion of $H^1(M)$ induced by an isometry moves C completely off itself.

Say that a subset Y of a metric space X is *fat* if Y contains arbitrarily large metric balls. Say that a closed geodesic $\gamma \in M$ is *primitive* if there is some integral 1-form which has period 1 on γ. Equivalently, γ is primitive if it represents a primitive element in $H_1(M)$.

Lemma 7.5 *There is a primitive closed geodesic γ having the following property: The integral hyperplane in $H^1(M)$ annihilating γ intersects C in a fat set.*

Proof: $H_1(M)$ is dual to the lattice $H^1(M, \boldsymbol{Z})$. Rays in $H_1(M) \otimes \boldsymbol{R}$ through primitive elements in this lattice are dense. Hence, by duality, annihilators of primitive closed geodesics in $H^1(M, \boldsymbol{Z})$ are dense in the space of codimension one subspaces of $H^1(M)$. The Lemma follows immediately from this. $\square$

We set $C_{\boldsymbol{Z}} = C \cap H^1(M, \boldsymbol{Z})$. We choose our manifold M so that the rank of $H^1(M)$ is at least 3. Let γ be any primitive closed geodesic guaranteed by Lemma 7.5. Let $\Pi \in H^1(M)$ be the annihilator of γ. Since Π is integral, the intersection $\Pi \cap H^1(M, \boldsymbol{Z})$ is a a co-compact lattice of Π. Since $\Pi \cap C$ is fat, there is an infinite unbounded sequence $\tau_1, \tau_2, ... \in C_{\boldsymbol{Z}}$ which has the following properties:

1. τ_n anniahilates γ.

2. The metric ball of radius n about τ_n is contained in C.

Let σ be any element of $H^1(M, \boldsymbol{Z})$ which has period 1 on γ. The elements $\omega_n = \sigma + \tau_n$ will belong to $C_{\boldsymbol{Z}}$ for sufficiently large n. Also, ω_n is primitive on γ. By The Cone Lemma, the genus of the fibration determined by a nonsingular form representing ω_n tends to ∞ with n. The forms ω_n provide the forms for our first example.

The plane $\Pi + \sigma$ is integral. Hence, the intersection

$$X_t = (\Pi + \sigma) \cap \hat{H}_t$$

is integral whenever this intersection contains a point of $H^1(M, \boldsymbol{Z})$. There is an unbounded increasing sequence $n_1, n_2, ...$ such that X_{n_j} is integral. Moreover, since X_{n_i} and X_{n_j} differ by a translation, there is a single number K such that every point of X_{n_j} is within K of a point of $H^1(M, \boldsymbol{Z}) \cap X_{n_j}$.

On the other hand, the intersection $X_{n_j} \cap C$ contains larger and larger metric balls (in X_{n_j}) as $j \to \infty$. Therefore, we can include in our sequence above two elements τ_i and τ_j such that $\omega_i, \omega_j \in C_{\boldsymbol{Z}}$ belong to the same hyperplane H_t. From the Cone Lemma, they define fibrations having the same genus fiber.

8 References

[**KL**] B. Kleiner and B.Leeb, *Rigidity of Quasi-Isometries for Symmetric Spaces of higher Rank*, preprint, 1995.

[**M**] G. D. Mostow, *Quasiconformal Mappings in n-Space and the Rigidity of Hyperbolic Space Forms*, Publ. I.H.E.S., vol. 33, 1968.

[**P**] P. Pansu, *Metriques de Carnot Caratheodory et Quasi-Isometries des Espaces Symetriques de Rang Un*, Annals of Math, 1989.

[**RS**] E. Rips and Z. Sela, *Structure and Rigidity in Hyperbolic Groups*, Geometric and Functional Analysis, vol 4, 1994.

[**S1**] R. Schwartz, *The Quasi-Isometry Classification of Rank One Lattices*, publ. IHES, vol. 82, 1995

[**S2**] R. Schwartz, *Quasi Isometric Rigidity and Diophantine Approximation*, preprint, 1995.

[**Su**] D. Sullivan, *On the Ergodic Theory at Infinity of an Arbitrary Discrete Group of Hyperbolic Motions*, proceedings of the 1978 Stony Brook conference on Riemann surfaces, Princeton University Press.

[**T1**] W. Thurston, *Three Dimensional Manifolds, Kleinian Groups and Hyperbolic Geometry*, Proceedings of Symposia in Pure Mathematics, Vol 39

(1983), Part 1.

T2] W. Thurston, *The Geometry and Topoogy of Three Manifolds*, Princeton University Lecture Notes, 1978.

[**T3**] W. Thurston, *A Norm on Homology*, Memoirs of the AMS, 1986.

[**Tu**] P. Tukia, *On Quasiconformal Groups*, J. D' Analyse, 1986.