Smooth manifolds are global geometric objects whose local structure is all about smooth mappings. The word smooth can be used in different ways, but in this course it will mean $C^\infty$. For a more detailed account than the following, see pages 581-586 in Lee’s book.

If $f : U \to \mathbb{R}^m$ is a mapping whose domain $U$ is an open subset of $\mathbb{R}^n$ then we say that $f$ is differentiable at a point $a \in U$ if there is a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that the limit of $|f(x) - f(a) - L(x-a)|/|x-a|$ as $x \to a$ is zero. Differentiability at $a$ implies continuity at $a$. The linear map $L$ is unique if it exists, because for every vector $v \in \mathbb{R}^n$ the vector $L(v)$ can be described as a directional derivative, the limit of $(f(a + tv) - f(a))/t$ as $t \to 0$. $L$ is called the derivative of $f$ at $a$ and denoted by $f'(a)$.

In particular, if $f$ is differentiable at $a$ then the (first-order) partial derivatives of $f$ exist, and the matrix expression of $f'(a)$ with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ is the usual Jacobian matrix.

The chain rule holds in the sense that when $g$ and $f$ are composable and both $f'(a)$ and $g'(f(a))$ exist then $(g \circ f)'(a)$ exists and equals the composition $g'(f(a)) \circ f'(a)$. The proof is not hard.

If $f'(x)$ exists for every $x \in U$ then we say that $f$ is differentiable. The function $f' : U \to Hom(\mathbb{R}^n, \mathbb{R}^m) \cong \mathbb{R}^{mn}$ may or not be differentiable. If it is, then its derivative at a point $a$ is a linear map $\mathbb{R}^n \to Hom(\mathbb{R}^n, \mathbb{R}^m)$. This second derivative of $f$ at $a$ can then be interpreted as a bilinear map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$, and so on. We say that $f$ is a $C^r$ mapping if the $k$th order derivative at $a$ exists and is continuous for all $1 \leq k \leq r$, and a $C^\infty$ mapping if the $k$th order derivative at $a$ exists for all $k \geq 1$.

The existence and continuity of first-order partial derivatives implies $C^1$. The existence and continuity of partial derivatives of order $\leq r$ implies $C^r$. 

1
$C^2$ implies the equality of mixed second-order derivatives.