Here is a discussion of the concept of $m$-dimensional smooth submanifold of $\mathbb{R}^n$. Later it will be superseded by the general concept of submanifold of an abstract manifold, but right now I want to get some ideas across by looking at this concrete case.

1 Curves in the plane

Let us start with the case $m = 1$, $n = 2$. I will give several definitions of smooth curve in the plane and show that they are all logically equivalent.

Let $C \subset \mathbb{R}^2$ be a subset and let $P = (a, b) \in C$ be a point. The following four conditions are equivalent, and if they hold for every $P \in C$ we say that $C$ is a smooth curve in $\mathbb{R}^2$.

1. For some open set $U \subset \mathbb{R}^2$ containing $P$ there exists a diffeomorphism $\Phi : U \to \Phi(U)$ from $U$ to some open subset of $\mathbb{R}^2$ such that $\Phi(C \cap U) = (\mathbb{R} \times 0) \cap \Phi(U)$ and $\Phi(P) = (0, 0)$.

2. (Locally $C$ can be given a regular parametrization.) For some open subset $V$ of $\mathbb{R}$ there is a smooth map $\phi : V \to \mathbb{R}^2$ such that $\phi(V)$ is a neighborhood of $P$ in $C$, $\phi(0) = P$, and the derivative $\phi'(0)$ is not zero.

3. (Locally $C$ is a graph.) For some open set $U \subset \mathbb{R}^2$ containing $P$, the set $U \cap C$ can be described either as the set of all pairs $(x, y)$ with $y = f(x)$ and $x \in J$ for some smooth $f$ defined in an open interval $J \in \mathbb{R}$, or as the set of all pairs $(x, y)$ with $x = g(y)$ and $y \in J$ for some open interval $J \in \mathbb{R}$.

4. (Locally $C$ is a regular level set.) For some open set $U \subset \mathbb{R}^2$ containing $P$ there exists a smooth map $\psi : U \to \mathbb{R}$ such that $\psi^{-1}(0) = U \cap C$ and $\psi'(P)$ is not zero.
We outline the proof:

(1) implies (2) trivially. Define \( \phi \) by \( \phi(u) = \Phi^{-1}(u,0) \) (the domain \( V \) being the set of all \( u \) such that \( (u,0) \in \Phi(U) \)).

(2) implies (3) using the Inverse Function Theorem in one dimension. Write \( \phi(u) = (\phi_1(u), \phi_2(u)) \). Either \( \phi'_1(0) \) or \( \phi'_2(0) \) is nonzero, say the former. Restricting to a smaller interval if necessary, we can assume that \( \phi_1 \) has an inverse. Let \( f = \phi_2 \circ \phi_1^{-1} \). If \( \phi'_2(0) \neq 0 \) then let \( g = \phi_1 \circ \phi_2^{-1} \).

(3) easily implies (4). If \( C \) is described locally by \( y = f(x) \) then let \( \psi(x,y) = y - f(x) \); if it is described by \( x = g(y) \) then let \( \psi(x,y) = x - g(y) \).

(4) implies (1) using the Inverse Function Theorem in two dimensions. One of the partial derivatives of \( \psi \) at \( P \) is nonzero, say \( \partial_2 \psi \). Define \( \Phi : U \to \mathbb{R}^2 \) by \( \Phi(x,y) = (x-a, \psi(x,y)) \). The derivative of \( \Phi \) at \( P \) is an invertible two by two matrix, so after restricting to a smaller open neighborhood of \( P \) the map \( \Phi \) becomes a diffeomorphism to its image.

2 The general case

Now let \( 0 \leq m \leq n \). Let \( M \subset \mathbb{R}^n \) be a subset and let \( P \in M \) be a point. The following four conditions are equivalent, and if they hold for every \( P \in M \) we say that \( M \) is a smooth \( m \)-dimensional manifold in \( \mathbb{R}^n \).

1. For some open set \( U \subset \mathbb{R}^n \) containing \( P \) there exists a diffeomorphism \( \Phi : U \to \Phi(U) \) from \( U \) to some open subset of \( \mathbb{R}^n \) such that \( \Phi(M \cap U) = (\mathbb{R}^m \times 0) \cap \Phi(U) \) and \( \Phi(P) = 0 \).

2. (Regular parametrization) For some open subset \( V \) of \( \mathbb{R}^m \) there is a smooth map \( \phi : V \to \mathbb{R}^n \) such that \( \phi(V) \) is a neighborhood of \( P \) in \( M \), \( \phi(0) = P \), and the \( n \times m \) derivative matrix \( \phi'(0) \) has rank \( m \) (the maximum possible).

3. (Graph) For some open set \( U \subset \mathbb{R}^n \) containing \( P \), the set \( U \cap M \) is related by some permutation of the \( n \) standard coordinates in \( \mathbb{R}^n \) to the set of all pairs \( (x,y) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \) with \( y = f(x) \) for some smooth \( f \) defined for \( x \) in some open set \( W \subset \mathbb{R}^m \).

4. (Regular level set) For some open set \( U \subset \mathbb{R}^n \) containing \( P \) there exists a smooth map \( \psi : U \to \mathbb{R}^{n-m} \) such that \( \psi^{-1}(0) = U \cap M \) and the \( (n-m) \times n \) derivative matrix \( \psi'(P) \) has rank \( n-m \) (the maximum possible).

The arguments are essentially the same as in the case \( m = 1, n = 2 \). Here are details for the two most interesting steps.

(2) implies (3) using the Inverse Function Theorem in \( m \) dimensions. After some permutation of coordinates we can assume that the first \( m \) rows of \( \phi'(0) \) constitute an invertible \( m \times m \) matrix. Write \( \phi(u) = (\phi_1(u), \phi_2(u)) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \), so that \( \phi'_1(0) \) is invertible. Restricting to a smaller
domain if necessary, we can assume that $\phi_1$ has an inverse. Let $f$ be $\phi_2 \circ \phi_1^{-1}$.

(4) implies (1) using the Inverse Function Theorem in $n$ dimensions. After composing with a permutation we can assume that the last $n - m$ columns of $\psi'(P)$ constitute an invertible $(n - m) \times (n - m)$ matrix. Define $\Phi : U \rightarrow \mathbb{R}^m \times \mathbb{R}^{n-m}$ by $\Phi(x, y) = (x - a, \psi(x, y))$ where $P = (a, b)$. The derivative of $\Phi$ at $P$ is an invertible $n \times n$ matrix, so after restricting to a smaller open neighborhood $\Phi$ becomes a diffeomorphism to its image.

The fact that (4) implies (3) is a version of the Implicit Function Theorem.

3 Tangent Spaces

To a smooth $m$-manifold $M \subset \mathbb{R}^n$ and a point $P \in M$ is associated an $m$-dimensional vector subspace of $\mathbb{R}^n$, the tangent space $T_P M$. We can describe it in four ways corresponding to (1) through (4) above.

If $\Phi$ is a diffeomorphism as in (1) then we let $T_P M$ be $\Phi'(P)^{-1}(\mathbb{R}^m \times 0)$. To see that this is well-defined, first note that it does not change if $\Phi$ is replaced by its restriction to a smaller open neighborhood of $P$. Then suppose that $\Phi_1$ and $\Phi_2$ are two diffeomorphisms as in (1) both having the same domain. Writing $\Phi_1 = \Psi \circ \Phi_2$, we have $\Phi_1'(P) = \Psi'(0) \circ \Phi_2'(P)$. Since the diffeomorphism $\Psi : \Phi_2(U) \rightarrow \Phi_1(U)$ preserves (a neighborhood of 0 in) $\mathbb{R}^m \times 0$, the linear isomorphism $\Psi'(0) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ preserves $\mathbb{R}^m \times 0$ and therefore $\Phi_1'(P)^{-1}(\mathbb{R}^m \times 0) = \Phi_2'(P)^{-1}(\Psi'(0)^{-1}(0)(\mathbb{R}^m \times 0))$.

Given a parametrization $\phi$ of a neighborhood of $P$ in $M$ as in (2), $T_P M = \phi'(0)(\mathbb{R}^m)$.

If $M$ is locally the graph of $f$ [altered by a permutation of the coordinates in $\mathbb{R}^n$] and $P = (a, b) = (a, f(a))$ then $T_P M$ is the graph of the linear map $f'(a) : \mathbb{R}^m \rightarrow \mathbb{R}^{n-m}$ [similarly altered].

Given a map $\psi$ as in (4) such that $M$ is locally $\psi^{-1}(0)$, the space $T_P M$ is the kernel of $\psi'(P) : \mathbb{R}^n \rightarrow \mathbb{R}^{n-m}$.

4 Transverse Intersections

Let $M_1$ and $M_2$ be smooth submanifolds of $\mathbb{R}^n$ with dimensions $m_1$ and $m_2$. We say that $M_1$ and $M_2$ are transverse (or intersect transversely) at $P \in M_1 \cap M_2$ if $T_P M_1 + T_P M_2 = \mathbb{R}^n$, or equivalently if the intersection $T_P M_1 \cap T_P M_2$ has vector space dimension $m_1 + m_2 - n$. Notice that if $m_1 + m_2 \geq n$ then this is the lowest possible dimension for the intersection of vector spaces of dimensions $m_1$ and $m_2$ in an $n$-dimensional vector space. We say simply that $M_1$ and $M_2$ are transverse if they are transverse at every point of intersection. If $m_1 + m_2 < n$ then $M_1$ and $M_2$ can never be transverse at a point, so the only way they can be transverse is by having empty intersection.
We show that if $M_1$ and $M_2$ are transverse and $m_1 + m_2 \geq n$ then $M_1 \cap M_2$ is a smooth submanifold of $\mathbb{R}^n$ with dimension $m_1 + m_2 - n$, and that $T_P(M_1 \cap M_2) = T_P M_1 \cap T_P M_2$ for every $P$. For this we use the regular level set point of view: For a suitable open neighborhood $U$ of $P \in M_1 \cap M_2$ choose $\psi_1 : U \to \mathbb{R}^{n-m_1}$ such that $\psi_1^{-1}(0) = U \cap M_1$ and $\psi_2 : U \to \mathbb{R}^{n-m_2}$ such that $\psi_2^{-1}(0) = U \cap M_2$, both with derivatives of maximal rank, and observe that because of the transversality the combined map

$$
\psi = (\psi_1, \psi_2) : U \to \mathbb{R}^{n-m_1} \times \mathbb{R}^{n-m_2} = \mathbb{R}^{n-(m_1+m_2-n)}
$$

also has derivative of maximal rank: the kernel of

$$
\psi'(P) : \mathbb{R}^n \to \mathbb{R}^{n-(m_1+m_2-n)}
$$

is the intersection of $\ker(\psi_1'(P)) = T_PM_1$ and $\ker(\psi_2'(P)) = T_PM_2$. 