Here is an alternative approach to defining tangent and cotangent spaces. It starts from the idea that a tangent vector of $M$ at the point $a$ should be a class of parametrized curves through $a$ that agree to first order.

(I am always a little unsure about how use phrases like ‘agreeing to order $k$ at a point’ . Here I will let it mean having the same value and the same derivatives, up to the derivatives of $k$th order. But that means that in one-variable calculus a function agrees to order 2 with the constant function 0 when it has a triple root, and so on.)

1 Jets

Given two smooth functions $\mathbb{R}^m \to \mathbb{R}^n$ taking 0 to 0, or better yet germs of such functions at 0, we say that they agree to order $k$ at 0 if all of the partial derivatives of order $\leq k$ coincide.

If $f$ and $g$ are smooth maps $(M,a) \to (N,b)$, i.e. smooth maps $M \to N$ taking $a \in M$ to $b \in N$ – or more generally if they are germs of such – then of course we say that they agree to order $k$ at $a$ if, when expressed in terms of coordinate charts, they agree to order $k$ in the sense above. That is, given charts $\Phi$ and $\Psi$ in $M$ and $N$ centered at $a$ and $b$, we look at whether the composed maps $\Psi \circ f \circ \Phi^{-1}$ and $\Psi \circ g \circ \Phi^{-1}$ agree to order $k$ at 0. This question is independent of the choice of charts, because according to the chain rule the $j$th partial derivatives of a composition depend only on the partial derivatives of order $\leq j$ of the maps that are being composed.

Call the equivalence class of $f$ under this relation the $k$-jet of $f$ at $a$, and denote the set of all such classes by $J^k_{a,b}(M,N)$. Right now we are mainly interested in the case $k = 1$. 

2 Tangent and cotangent vectors

Define the tangent space of $M$ at the point $a$ by $T_a M = J^1_{0,a}(\mathbb{R}, M)$. The equivalence class of the curve $\gamma$ is sometimes denoted by $\dot{\gamma}(0)$.

Define the cotangent space by $T^*_a M = J^1_{a,0}(M, \mathbb{R})$. Unlike the set $T_a M$ just defined, this one has an obvious vector space structure, inherited from the vector space structure of real-valued functions. In fact, it is obviously the same as the $\mathfrak{m}/\mathfrak{m}^2$ that I mentioned in class. (Recall that we have denoted the ring of germs of smooth real functions at $a$ by $\mathcal{O}_a \mathbb{R}$ and the maximal ideal corresponding to functions that vanish at $a$ by $\mathfrak{m}$, and that the smaller ideal consisting of functions that vanish to first order is $\mathfrak{m}^2$.) The cotangent vector represented by the function $f$ at $a$ is denoted by $d_a f$.

Extend this to cases where the function does not vanish at the point, writing $d_a f$ for the class of $f - f(a)$.

Composition

$$(\mathbb{R}, 0) \to (M, a) \to (\mathbb{R}, 0)$$

induces a map

$$J^1_{a,0}(M, \mathbb{R}) \times J^1_{0,a}(\mathbb{R}, M) \to J^1_{0,0}(\mathbb{R}, \mathbb{R})$$

If we identify $J^1_{0,0}(\mathbb{R}, \mathbb{R})$ with $\mathbb{R}$ by sending the 1-jet of $f$ at 0 to $f'(0)$, then this becomes a map

$$T^*_a M \times T_a M \to \mathbb{R}$$

Let us write it as

$$(\omega, v) \mapsto \langle \omega, v \rangle$$

Explicitly, $\langle d_a f, \dot{\gamma}(0) \rangle = (f \circ \gamma)'(0)$. For each $v \in T_a M$ the map $T^*_a M \to \mathbb{R}$ given by $\langle -, v \rangle$ is linear, because $f \circ g$ is a linear function of $f$. The resulting map $v \mapsto \langle -, v \rangle$ from $T_a M$ to the dual vector space of $T^*_a M$ is a bijection, by a computation using coordinates. In this way $T_a M$ gets a vector space structure, the unique such structure that makes $\langle \omega, v \rangle$ a linear function of $v$.

Now consider maps $F : M \to N$ between manifolds. Composition

$$(\mathbb{R}, 0) \to (M, a) \to (N, b)$$

induces a map

$$J^1_{a,b}(M, N) \times T_a M \to T_b N$$

Let us write it as

$$(\{F\}, v) \mapsto F_* v$$

where $\{F\}$ is the 1-jet of $F$. Thus $F_* (\dot{\gamma}(0)) = (F \circ \gamma)(0)$.

Likewise composition

$$(M, a) \to (N, b) \to (\mathbb{R}, 0)$$

induces a map

$$T^*_b N \times J^1_{a,b}(M, N) \to T^*_a M$$

Let us write it as

$$(\omega, \{F\}) \mapsto F^* \omega$$
Thus \( F^*(d_af) = d_a(f \circ F) \).

The map \( F^* : T^*_aM \to T^*_bN \) is linear (again because \( f \circ F \) depends linearly on \( f \)). Denote by \( \mathcal{L}(V,W) \) the vector space of all linear maps \( V \to W \). A coordinate computation shows that our map sending \( \{ F \} \in J^1_{a,b}(M,N) \) to \( F^* \in \mathcal{L}(T^*_bN,T^*_aM) \) is a bijection.

Of course, since the tangent spaces are the duals of the cotangent spaces, \( \mathcal{L}(T^*_bN,T^*_aM) \) is canonically isomorphic to \( \mathcal{L}(T_aM,T_bN) \) by ‘adjoint’. We now verify that the map \( F^* : T_aM \to T_bN \) defined above is also linear, and moreover that this \( F^* \in \mathcal{L}(T_aM,T_bN) \) is the adjoint of \( F^* \in \mathcal{L}(T^*_bN,T^*_aM) \). The verification relies on the associativity of composition \((\mathbb{R},0) \to (M,a) \to (N,b) \to (\mathbb{R},0)\)

The 1-jet of \((f \circ F) \circ \gamma = f \circ (F \circ \gamma)\) is \(<F^*(d_af),\dot{\gamma}(0)> = <d_a(f),F^*(\dot{\gamma}(0))>,\) so \(<F^*\omega,v> = <\omega,F^*v>\)

For the composition \((\mathbb{R},0) \to (M,a) \to (N,b) \to (P,c)\)

the associative law \((G \circ F) \circ \gamma \) gives \((G \circ F)_*(\dot{\gamma}(0)) = G_*(F_*(\dot{\gamma}(0))),\) i.e. \((G \circ F)_* = G_* \circ F_*\). The equation \((G \circ F)^* = F^* \circ G^*\) can be obtained either by another such argument or by adjointness.

### 3 Higher jets

We have seen that \( J^1_{a,b}(M,N) \) is canonically a vector space; in fact it may be identified with the space of linear maps from \( T_aM \) to \( T_bN \). When coordinates are chosen, say \( y^i \) in \( N \) and \( x^j \) in \( M \), then this vector space gets identified with the space of \( n \times m \) matrices using the partial derivatives \( \frac{\partial y^i}{\partial x^j} \) at the origin as coordinates.

For \( k > 1 \) it is still true that coordinates in \( M \) and \( N \) give coordinates for \( J^k_{a,b}(M,N) \), namely the partial derivatives \( \frac{\partial^l y^i}{\partial x^j \cdots \partial x^j} \) of order \( 1 \leq l \leq k \) at the origin. Nevertheless, when \( k > 1 \) \( J^k_{a,b}(M,N) \) is not canonically a vector space. Of course, \( J^k_{a,0}(M,\mathbb{R}) \) is the vector space \( \mathfrak{m}/\mathfrak{m}^{k+1} \). But no linear structure on \( J^k_{a,b}(M,N) \) can be invariant under diffeomorphisms in \( N \). This is already the case for \( J^2_{0,b}(\mathbb{R},N) \) and \( N \) one-dimensional. When changing coordinates from \( y \) to \( z \) in \( N \) we obtain

\[
\frac{d^2 y}{dt^2} = \frac{dy}{dz} \frac{dz}{dt} + \frac{d^2 y}{dz^2} \left( \frac{dz}{dt} \right)^2
\]

The quadratic dependence on \( \frac{dz}{dt} \) is the problem.