The other day I made a few slips at the end of my discussion of the tangent bundle of $S^2$. Let me correct them and say a few more words about vector bundles on spheres in general.

1 Tangent bundles of $S^n$

I showed that a smooth vector bundle on $\mathbb{R}^n$ is always trivial. Now consider a rank $k$ smooth vector bundle on $S^n$. Covering $S^n$ by open sets $U_+$ and $U_-$, the complements of the south and north poles, we have a trivialization of $E$ on $U_+$, an isomorphism

$$\Phi_+: \pi^{-1}(U_+) \to U_+ \times \mathbb{R}^k$$

between the restricted bundle and the trivial bundle. Likewise for $U_-$:

$$\Phi_-: \pi^{-1}(U_-) \to U_- \times \mathbb{R}^k$$

Together these give an isomorphism between the trivial bundle $(U_+ \cap U_-) \times \mathbb{R}^k$ and itself, which may be expressed by a smooth map $\tau: U_+ \cap U_- \to GL_k(\mathbb{R}^k)$,

$$\Phi_+(\Phi_-^{-1}(x, v)) = (x, \tau(x)v)$$

If the bundle $E$ has a nowhere-vanishing global section $\sigma: S^n \to E$, then we may describe $\sigma$ in $U_+$ by a function $f_+: U_+ \to \mathbb{R}^k - 0$:

$$\Phi_+(\sigma(x)) = (x, f_+(x))$$

and in $U_-$ by a function $f_-: U_- \to \mathbb{R}^k - 0$:

$$\Phi_-(\sigma(x)) = (x, f_-(x)).$$
For \( x \in U_+ \cap U_- \) we then have \( f_+(x) = \tau(x)f_-(x) \).

Restricting further to \( x \in S^{n-1} \subset U_+ \cap U_- \), we have that the functions \( f_+, f_- : S^{n-1} \to \mathbb{R}^k - \{0\} \) are homotopic to constant maps, because they extend to the contractible spaces \( U_+, U_- \) respectively. It follows that for a constant \( u \in \mathbb{R}^k - \{0\} \) the function \( x \mapsto \tau(x)u \) from \( S^{n-1} \) to \( \mathbb{R}^{k-1} \) is homotopic to a constant.

This leads to a contradiction in some cases. For the tangent bundle of \( S^n \), if the trivializations on \( U_+ \) and \( U_- \) are made using stereographic projection then the resulting \( \tau \) is given by a very nice formula: for \( x \in S^{n-1} \) the element \( \tau(x) \in GL_n(\mathbb{R}^n) \) is reflection along the vector \( x \). (I said this in class in the case \( n = 2 \), but said something wrong at the end.) The mapping \( x \mapsto \tau(x)u \) in this case is generically two to one. It takes both \( u \) and \(-u\) to \(-u\). It takes all unit vectors orthogonal to \( U \) to \( u \). Its degree is 0 if \( n \) is odd, but if \( n \) is even then the degree is 2 and we have a contradiction to the existence of a nowhere-vanishing tangent vector field on \( S^n \).

2 Classifying vector bundles on \( S^n \)

Conversely any such map determines a rank \( k \) vector bundle on \( S^n \) by gluing together \( U_+ \times \mathbb{R}^k \) and \( U_- \times \mathbb{R}^k \). \((x, v) \in U_- \times \mathbb{R}^k \) is identified with \((x, \tau(x)v) \in U_+ \times \mathbb{R}^k \).

Isomorphism classes of vector bundles on \( S^n \) correspond one to one with classes of maps \( \tau : U_+ \cap U_- \to GL_k(\mathbb{R}^k) \). As argued in class, two maps \( \tau \) and \( \tau' \) are equivalent for this purpose if and only if there exist maps \( \rho_+, \rho_- : U_+ \cap U_- \to GL_k(\mathbb{R}^k) \), the one extending to \( U_+ \) and the other to \( U_- \), such that for \( x \in U_+ \cap U_- \) we have \( \tau(x) = \rho_+(x)\tau(x)\rho_-(x) \).

This can be analyzed further. A convenient form of the answer is that such bundles correspond to maps \( \tau : S^{n-1} \to GL_k(\mathbb{R}^k) \) up to the relation of multiplying on right and left by maps that extend to \( D^n \). This the same as multiplying on right and left by maps that are homotopic to constants. This can also be expressed as homotopy classes of maps \( S^{n-1} \to GL_k(\mathbb{R}^k) \) up to multiplication on right and left by constants; or based homotopy classes up to conjugation by constants.

For example, rank 2 bundles on \( S^2 \) correspond to elements of \( \pi_1(GL_2(\mathbb{R})) \cong \pi_1(SO(2)) \cong \mathbb{Z} \) up to sign change. For \( k > 2 \) we have \( \pi_1(GL_k(\mathbb{R}^k)) \cong \mathbb{Z}/2\mathbb{Z} \), so that there is only one nontrivial bundle of that kind.

For another example, rank \( k \) vector bundles on \( S^1 \) correspond to path components (based homotopy classes from \( S^0 \)) of \( GL_k(\mathbb{R}) \) up to conjugation, a two element set if \( k > 0 \).