Notes on a conjecture of Long and Reid

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1 Introduction

These notes explain some things I learned about the Long-Reid Conjecture when I unsuccessfully tried to solve it. I had thought for several weeks that I had solved the problem and I tried to write a paper about it. I quickly discovered a gap in the proof and abandoned the paper. These notes, which are a remnant of the abandoned paper, make no claims about solving the conjecture. However, they do have a clear statement of the problem and perhaps some interesting ideas. One of the nice ideas (in §4) is a very concrete way of describing the Serre tree and the action of $GL_2(\boldsymbol{Q})$ on it. Nic Brody has a similar understanding of these group actions, and probably he has also written about it.

A group Γ acts *properly* on a space X if for every compact $K \subset X$, the set

$$\{\gamma \in \Gamma |\ \gamma(K) \cap K \neq \emptyset\}$$

is finite. Here is the main question: *Can a surface group act properly on the product of infinite, finite valence, trees?* This is listed as *Question 1* in the paper [**FLSS**]. Constructions in Teichmuller theory give examples of a surface group acting properly on the product of two $\boldsymbol{R}$-trees, but so far nobody has found a way to replace $\boldsymbol{R}$-trees by finite valence trees.

Darren Long and Alan Reid have a candidate construction based on the action of $GL_2(\boldsymbol{Q})$ on Serre trees. Let T_p be the infinite regular tree of valence $p+1$. In Serre's famous book [**S**], called *Trees*, he describes how $GL_2(\boldsymbol{Q})$ naturally acts on T_p. The action is really defined for the larger group $GL_2(\boldsymbol{Q}_p)$

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of nonsingular 2×2 matrices with p-adic entries. We will formally recall how this works in §4, and also explain how we compute the action in practice.

Long and Reid consider a specific 2-generator group $\Gamma \subset GL_2(\boldsymbol{Q})$. Up to scaling and $GL_2(\boldsymbol{Q})$-conjugation, Long and Reid's group Γ is generated by a and b where

$$a = \begin{bmatrix} 9 & 2 \\ 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 & -16 \\ 4 & 83 \end{bmatrix}. \tag{1}$$

They conjecture that the Serre action of (a conjugate of) Γ on $X = T_2 \times T_3$ is proper. The advantage Γ has over the original group is that one of the commutators

$$c = a^{-1}bab^{-1} = \begin{bmatrix} -9 & -2 \\ 41 & 9 \end{bmatrix} \tag{2}$$

lies in $SL_2(\boldsymbol{Z})$ and therefore fixes the natural origin O in X, namely the point represented by the lattice $\boldsymbol{Z}_p^2$ of pairs of p-adic integers. This makes the orbit map $\gamma \to \gamma(O)$ especially attractive.

This is a great guess. As Long and Reid point out, Γ fits into a 2-parameter family of 2-generator groups which are orbifold fundamental groups of hyperbolic tori having one cone angle of π. The matrix entries are rational functions of the parameters. By cleverly choosing the parameters, Reid and Long controlled the prime factorization of the entries in a promising way. That is roughly how their guess arose.

Thanks to the nature of the hyperbolic quotient $\boldsymbol{H}^2/\Gamma$, the commutator c acts as an involution on the hyperbolic plane $\boldsymbol{H}^2$, and Γ preserves a tiling $\mathcal{T}$ by hyperbolic quadrilaterals having opposite sides of equal length. The Γ-orbit of the fixed point of c is the vertex set of $\mathcal{T}$. The stabilizer subgroup of each vertex point has order 2. See §3.

I will explore some of the geometry of the situation. In particular, I will construct a space Y_0, a parallelogram-tiled nonpositively curved translation surface, which is closely related to the group Γ and its actions. The parallelograms all have the same shape as the parallelogram $P \subset \boldsymbol{R}^2$ having vertices $\pm(3,1)$ and $\pm(3,3)$. In Y_0, there are 8 parallelograms fitting around a vertex, with the angles alternating acute and obtuse. All the cone angles are 4π. A group isomorphic to Γ acts isometrically on Y_0 with compact quotient. There is an equivariant map $\Phi : \boldsymbol{H}^2 \to Y_0$ which maps tiles to parallelograms. Using the orbit map discussed above (and identifying $\boldsymbol{H}^2$ with Y_0), we construct an equivariant and piecewise isometric mapping

$$\Psi : Y_0 \to X. \tag{3}$$

Ψ turns out to be an isometry in the complement of the vertices of Y_0. However, it is not a global isometric embedding, or even an embedding.

Conjecture 1.1 *If* $\Psi(a) = \Psi(b)$ *then* $\text{distance}(a, b) \leq 2$.

Conjecture 1.1 is really just a reformulation of the Long-Reid Conjecture. If Γ does not act properly on X then we can find an infinite sequence of words in Γ which move the origin in X a bounded distance. This gives an infinite number of vertices in Y_0 that Ψ maps to a bounded neighborhood of the origin. By the pidgeonhole principle, Ψ must map two distant vertices to the same point, contradicting Conjecture 1.1.

What is attractive about Conjecture 1.1 is that it involves a concrete map of an infinite translation surface into an infinite square complex. Everything in sight is a piecewise flat object. I have no idea how prove Conjecture 1.1, however.

These notes are organized as follows. In §2 I will construct the translation surface and discuss how it relates to Γ. In §3 I will discuss how Γ acts on $\boldsymbol{H}^2$, the hyperbolic plane. In §4 I will explain how to compute the action of Γ on $T_2 \times T_3$. In §5 I will prove two technical lemmas left over from §2.

2 Some Geometric Structure

2.1 The Hyperbolic Tiling

Let $\boldsymbol{H}^2$ denote the hyperbolic plane. Let $\Gamma = \langle a, b\rangle$ be the group given in Equation 1. Let $c = [A, b] = AbaB$ be the commutator of A and B. Here $A = a^{-1}$ and $B = b^{-1}$. Let $v_0 \in \boldsymbol{H}^2$ be the fixed point of the involution c.

Γ preserves a tiling $\mathcal{T}$ of $\boldsymbol{H}^2$ by hyperbolic quadrilaterals having opposite sides of equal length and adjacent sides of unequal lengths. We call these quadrilaterals *tiles*. The vertex set of $\mathcal{T}$ is the Γ-orbit of v_0. Figure 2 shows a hand-drawn but accurate drawing of one flower of $\mathcal{T}$ in $\boldsymbol{H}^2$. The numbers indicate vertices $v_j = w_j(v_0)$, where $w_1, ..., w_{16}$ are certain words we list in §3.2.

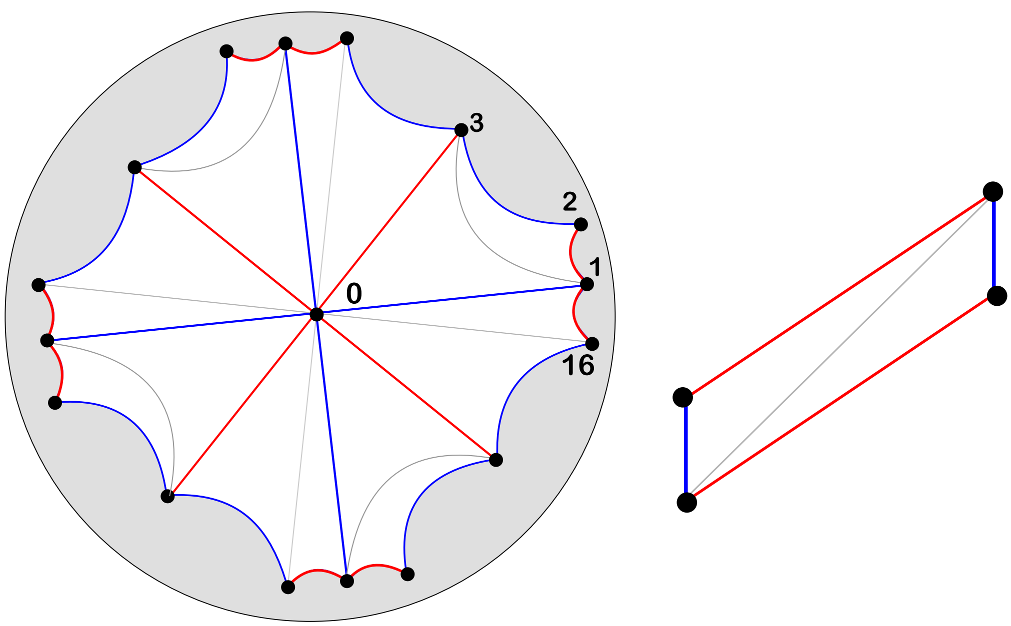

Figure 1. The hyperbolic tiling and a Euclidean parallelogram

We color the edges of the tiling red and blue as shown in Figure 1. We also distinguish one diagonal in each tile. The pattern is that the distinguished diagonals in adjacent tiles do not share any vertices. When distinguished or undistinguished diagonals meet at a vertex, they meet at right angles.

2.2 The Infinite Translation Surface

Let P be the Euclidean parallelogram having vertices $\pm(3,1)$ and $\pm(3,3)$. The right side of Figure 1 shows a Euclidean parallelogram having the same shape as P. We color the short sides of P blue and the long sides red. We also distinguish the long diagonal of P. In other words, we decorate P as in Figure 1. Let Y_0 be the space discussed in the introduction. Again, we build Y_0 by fitting 8 copies of P around each vertex, so that the angles alternate acute and obtuse.

Lemma 2.1 *Up to an order* 2 *rotation, there is a unique homeomorphism from a hyperbolic tile* τ *to* P *which maps geodesic segments in* τ *to line segments in* P *and respects the decorations.*

Proof: This proof only uses the fact that τ is a hyperbolic quadrilateral whose opposite sides have the same length. We work in the Klein model, so that the geodesics are straight line segments and the isometry group acts by projective automorphisms of the unit disk. This means that under a suitable isometry, τ is a Euclidean rectangle. In this case, there are two decoration respecting affine maps from τ to P. These maps differ by an order 2 rotation, and they are the unique maps which do the job. ♠

Here is the quasi-isometric map $\Phi : \boldsymbol{H}^2 \to Y_0$. We first choose the homeomorphism from Lemma 2.1 and then we extend from tile to tile so as to make the whole map a homeomorphism. What makes Φ fit together across tile edges is that the maps from Lemma 2.1 send the hyperbolic midpoints of the tiles to the midpoints of edges of the parallelograms. The map Φ is a kind of combinatorial isomorphism in which we are just switching the geometry of the tiles.

2.3 The Orbit Map

We identify the space Y_0 with $\boldsymbol{H}^2$ in the way determined by the map Φ just discussed. Really, we are just changing the local geometry but keeping everything else – the combinatorics, decorations, and group action – the same. One could think of Y_0 as $\boldsymbol{H}^2$ with a different Γ-invariant metric. Figure 1 will be our guide sometimes.

We first establish some notation. Recall that $X = T_2 \times T_3$ is the product two Serre trees, the 2-adic tree and the 3-adic tree. Let $\pi_p : X \to T_p$

be projection for $p = 2, 3$. As mentioned in the introduction, each matrix $M \in \Gamma$ acts on T_2 and T_3 by the Serre action.

Given a pair of points $\psi_1, \psi_2 \in X$ we associate a pair of integers (d_2, d_3), where d_p is the distance from $\pi_p(\psi_1)$ to $\pi_p(\psi_2)$ in T_p. We call (d_2, d_3) the *pitch* of the pair (ψ_1, ψ_2). Likewise we define the pitch of a geodesic segment in X to be the pitch of its pair of endpoints. The usual metric on X computes the distance between ψ_1 and ψ_2 as $\sqrt{d_2^2 + d_3^2}$.

Let V denote the vertex set of Y_0. Let $v_0 \in V$ be the vertex fixed by c. The element $c \in \Gamma$ also fixes the origin $O \in X$, as we mentioned in the introduction. We define the orbit map $\Psi : V \to Y$:

$$\Psi(v) = \gamma(O), \qquad v = \gamma(v_0). \tag{4}$$

The stabilizer of v_0 in Γ is just the order 2 group generated by c. Thus Ψ is a well-defined map.

Now we wish to extend Ψ to the rest of Y_0. We first present some motivating calculations. Here, and below, the vertex v_j corresponds to the number j in Figure 1. We have $v_j = w_j(O)$ where w_j is the jth word listed in §3.2. Let $\psi_j = \Psi(v_j)$. We compute the following for the vertices v_0, v_1, v_2, v_3 of a tile.

1. Blue edge: The pitch of (ψ_0, ψ_1) is $(0, 2)$.
2. Blue edge: The pitch of (ψ_2, ψ_3) is $(0, 2)$.
3. Red edge: The pitch of (ψ_1, ψ_2) is $(6, 4)$.
4. Red edge: The pitch of (ψ_3, ψ_0) is $(6, 4)$.
5. Distinguished diagonal: The pitch of (ψ_1, ψ_3) is $(6, 6)$.
6. Undistinguished diagonal: The pitch of (ψ_2, ψ_0) is $(6, 2)$.

In short, the pitches precisely calculate the displacement vectors of the corresponding edges of the parallelograms comprising Y_0. This, of course, is why we defined Y_0 this way. These calculations suggest the following result, which we also establish by direct calculation in §4.

Lemma 2.2 *The points $\psi_0, \psi_1, \psi_2, \psi_3$ are the vertices of a geodesically embedded Euclidean parallelogram which is isometric to a tile of Y_0.*

Thus, we can extend our map Ψ to all of Y_0 by making its restriction to each tile the kind of isometry which we just discussed. Ψ sends the distinguished diagonals in each tile to a segment of pitch $(6,6)$. The map sends the undistinguished diagonals to segments of pitch $(6,2)$.

By construction, Ψ is an isometric embedding when restricted to the interior of each tile. We also establish the following result, which actually subsumes Lemma 2.2.

Lemma 2.3 *Let τ_1 and τ_2 be two adjacent tiles of Y_0. The restriction of Ψ to the interior of $\tau_1 \cup \tau_2$ is an isometric embedding.*

Lemma 2.3 shows that Ψ is a local isometric embedding of $Y_0 - V$ into X. In other words, the only place where Ψ may not be a local isometry is at the vertices.

Here we explain why $\Psi : Y_0 \to X$ cannot be an embedding onto its image $Y = \Psi(Y_0)$. If Ψ was an embedding, then the fact that Ψ is a local isometry away from vertices would force the link $K_{3,4}$ of a vertex in X to contain an embedded 8-cycle. Since $K_{3,4}$ only has 7 vertices, this is impossible. Hence Ψ cannot be an embedding, let alone an isometric embedding.

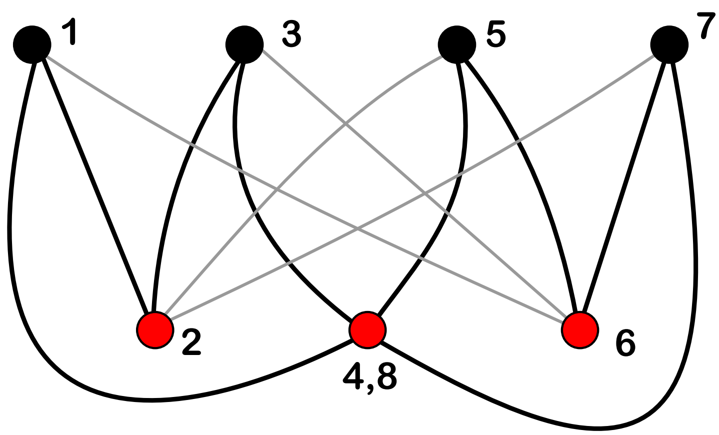

Figure 2. An immersed 8-cycle in $K_{3,4}$.

At the same time, $K_{3,4}$ does have an immersed 8-cycle which is homeomorphic to a figure 8 and invariant under an involution which sends (i) to $(i+4)$ mod 8. Figure 2 shows an example. The 1-neighborhood of Y at each vertex $v \in Y$ is isometric to a suitably metrized cone on such an 8-cycle. The action of the non-identity stabilizer of v in Γ implements the involution just mentioned.

3 The Hyperbolic Picture

3.1 The Group

Recall that Γ is generated by a and b from Equation 1 and $c = AbaB$. Again,

$$a = \begin{bmatrix} 9 & 2 \\ 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 & -16 \\ 4 & 83 \end{bmatrix}, \qquad c = \begin{bmatrix} -9 & -2 \\ 41 & 9 \end{bmatrix} \tag{5}$$

We let Γ act on the upper half plane by the action:

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}(z) = \frac{\alpha z + \gamma}{\beta z + \delta}. \tag{6}$$

Note that this action is non-standard: It is the transpose of the usual action. The fixed point of c is

$$z_0 = \frac{9}{2} + \frac{1}{2}i.$$

For Figure 2, we map z_0 to the origin of the unit disk using the map

$$T(w) = \frac{w' - i}{w' + i}, \qquad w' = 2w - 9.$$

3.2 The Tiling

For convenience, we repeat Figure 2. Here are the words we mentioned in §2.1.

1. $w_1 = A$.
2. $w_2 = ba$.
3. $w_3 = baa$.
4. $w_4 = baaa$.
5. $w_5 = a$.
6. $w_6 = ABc$.
7. $w_7 = AABc$.
8. $w_8 = AABcAc$.

The other words are determined by the rule that $w_{8+j} = w_j c$. In terms of group actions, our notation is such that $d_1 ... d_n$ means "First apply d_1, then d_2, etc."

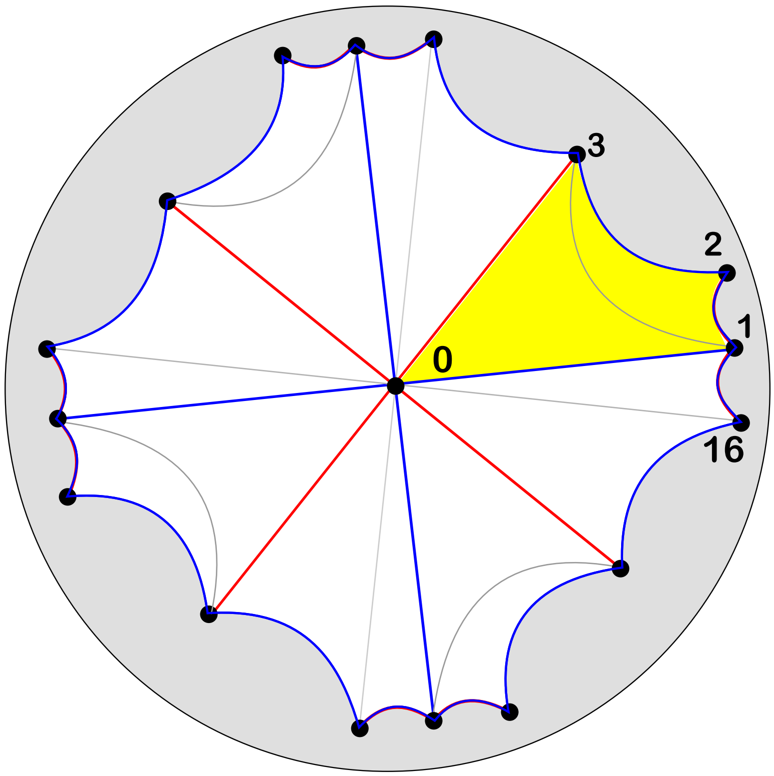

Figure 3. The hyperbolic tiling

3.3 The Geometry of the Group

The vertex v_0 in the Poincare model corresponding to the word $w_1 = A$ is given by

$$T \circ A(z_0) = \frac{9}{10} + \frac{i}{10}.$$

This looks about right in the picture. In the upper half plane, the corresponding point is

$$z_1 = A(z_0) = \frac{-1}{2} + \frac{9}{2}i.$$

We also compute

$$z_2 = (b.a)(z_0) = \frac{684}{425} + \frac{288}{425}i,$$

$$z_3 = (b.a.a)(z_0) = \frac{29484}{8345} + \frac{2592}{8345} i.$$

The points z_0, z_1, z_2, z_3 are the vertices of one of the yellow quadrilateral τ in Figure 3.

We can use these calculations to get the information we have used about the hyperbolic action of Γ.

Lemma 3.1 *The opposite sides of τ have the same length. These sides are paired by the elements A and baa.*

Proof: We compute that $A(z_3) = z_2$. Hence A pairs the red sides of τ and they have the same length. We likewise compute that $baa(z_1) = z_2$. Hence baa pairs the blue sides of τ and they have the same length. ♠

Lemma 3.2 Γ *acts properly on $\boldsymbol{H}^2$ and τ is a fundamental domain for the action. Moreover, the stabilizer of z_0 is the order 2 subgroup consisting of c and the identity.*

Proof: (sketch) Note that A and baa pair the opposite sides of τ. We compute that $[A, baa] = c$. This fact implies that 8 translates of τ fit around each vertex of τ. Finally A and baa generate Γ. The Poincare Polyhedron Theorem now gives the desired results. ♠

Lemma 3.3 *The red and blue edges in the tiling fit together to form a union of geodesics.*

Proof: This follows from the Poincare Theorem as well, but we can also do a direct calculation. We compute that $v_5 = iv_1$ and $v_7 = iv_3$. Therefore, around a vertex, consecutive red or blue edges form a right angle. This suffices. ♠

4 Calculating the Serre Action

4.1 Formal Definition of the Serre Action

Let $\boldsymbol{Q}_p$ denote the p-adic field and let $\boldsymbol{Z}_p$ denote the ring of p-adic integers. A *rank two* $\boldsymbol{Z}_p$ *module* is a a subset of $(\boldsymbol{Q}_p)^2$ which is invariant under addition and $\boldsymbol{Z}_p$ scaling. Two such modules Λ_1 and Λ_2 are declared *equivalent* if there is some $r \in \boldsymbol{Q}_p$ such that $r\Lambda_1 = \Lambda_2$, and *adjacent* if there is some $r \in \boldsymbol{Q}_p$ such that $r\Lambda_1 \subset \Lambda_2$ and $\Lambda_2/(r\Lambda_1)$ is isomorphic to $\boldsymbol{Z}/p$.

Serre forms a graph whose vertices are equivalence classes of rank 2 $\boldsymbol{Z}_p$ modules and joins them by an edge if two equivalence classes have adjacent representatives. Serre then proves that the resulting graph is the infinite regular tree of valence T_p! The group $GL_2(\boldsymbol{Q}_p)$ acts on these rank two modules and preserves both the relations of equivalence and adjacency. This gives the Serre action on T_p.

4.2 An Integer Model for Serre Trees

I found these amazing results very hard to understand until I started thinking about it in terms of something much easier to understand – the connection between $\boldsymbol{Z}_p$ and a rooted $p+1$-valent tree. This tree has one vertex of degree p – the root – and the remaining vertices have degree $p+1$.

Here is the description. $\boldsymbol{Z}_p$ is the inverse limit of the chain of homomorphisms

$$\cdots \to \boldsymbol{Z}/p^3 \to \boldsymbol{Z}/p^2 \to \boldsymbol{Z}/p^1 \to \boldsymbol{Z}/p^0.$$

Here $\boldsymbol{Z}/p^0$ is just defined to be $\{0\}$. We form a tree by joining each element in $\boldsymbol{Z}/p^k$ to the element in $\boldsymbol{Z}/p^{k-1}$ to which it reduces. This gives us our rooted $p+1$ valent regular tree. The ring $\boldsymbol{Z}_p$ can be identified with the set of infinite paths in the tree. It turns out that we can make the Serre tree T_p look very much like this much simpler object. In particular, we can take all the representatives to have an integer basis. We will first deal with the cases $p = 2$ and then consider the case $p = 3$ somewhat more briefly.

The case $p = 2$: We can take representatives for these modules which are certain lower triangular integer matrices, in which the diagonal entries are powers of 2. The module is the $\boldsymbol{Z}_p$ span of the rows. We build the tree in layers, with the nth layer consisting of certain such matrices of determinant 2^n. The left two-thirds of the tree has precisely the structure coming from our rooted tree. The right one-third of the tree is a kind of inversion of the left two-thirds. Thus, we really just have to think about the rooted tree construction and remember how the inversion works. Beautifully, each lattice manifestly has index 2 in the one it joins to on the layer above. So, one can see by inspection that this really is a correct model for T_2. Figure 4 shows the first 4 layers of the Serre Tree T_2.

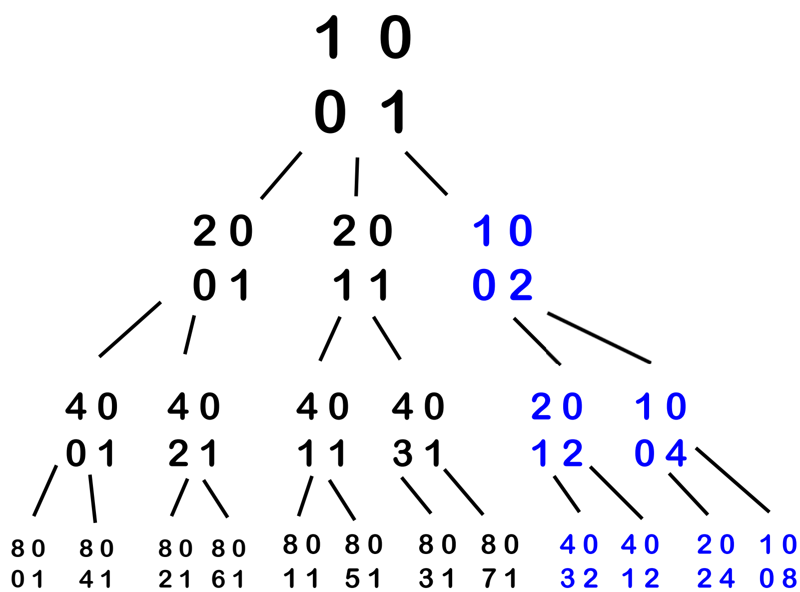

Figure 4. The beginning of T_2.

T_3 works the same way, but with powers of 3 replacing powers of 2. Figure 5 shows the first 3 layers of T_3.

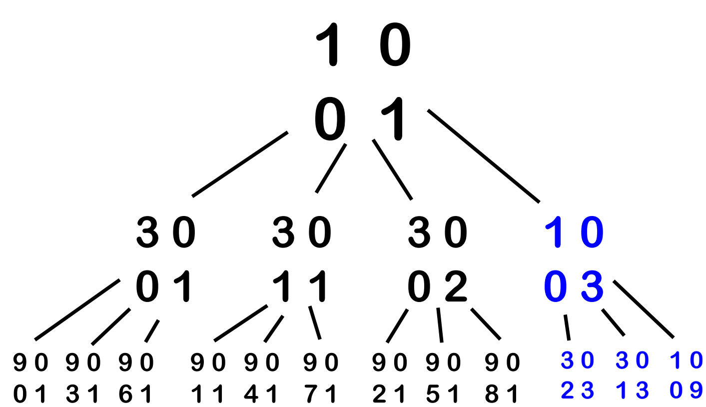

Figure 5. The beginning of T_3.

4.3 Computing the Serre Action in Principle

In principle, we compute the Serre action on our model just by taking the relevant matrices, multiplying them out, and then matching the answer to the representative in our model. Even though we don't really compute things this way, we include an example to illustrate our convention for multiplying matrices. Let

$$M = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

be our group element and let

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

be a lattice in our model. We compute

$$M(\Lambda) = \Lambda \times M = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$$

This is not in our model, but note that the module in question is the span of $U = (2,1)$ and $V = (0,4)$. This is also the span of U and $4U - V = (8,0)$. Thus, in terms of the Serre action,

$$M(\Lambda) = \begin{bmatrix} 8 & 0 \\ 2 & 1 \end{bmatrix}.$$

This one is in our model.

Remarks:
(i) To make the method here more in line with the hyperbolic geometry action, we could instead take the left transpose action

$$M(\Lambda) = (M^t \times \Lambda^t)^t.$$

I somewhat regret not using more standard conventions but I consider it a miracle that my program actually works.
(ii) When we are trying to figure out the equivalence, we never need to deal with $\boldsymbol{Z}_p$. Instead we can work with the ring R_p consisting of fractions of the form a/b where b is relatively prime to p. In other words, in our calculation above, we are free to take integer combinations of the form $uU + vV$ with $u, v \in R_p$. We can also scale by elements of R_p.

4.4 Computing the Serre Action in Practice

I gave the example above mainly to explain how the action works in principle. For large words, it seems really quite difficult to find the equivalence. Sometimes, though not for the example above, we really need to work with the ring R_p. Instead of doing this, I found two tricks for doing and certifying the calculations. One trick is to simply write the generators of Γ in terms of shorter generators and compute with these instead. The second trick is somewhat akin to Hensel's lemma. I will explain these tricks in turn. My code implements these tricks.

Simpler Generators: We introduce the simpler matrices

$$\mu_0 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mu_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mu_4 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \qquad \mu_6 = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}. \quad (7)$$

and set $\mu_{j+1} = \mu_j^{-1}$ for $j = 0, 2, 4, 6$. These matrices do not belong to Γ but we can use them to generate the generators a and b of Γ. From trial and error, I found that

$$a = (35300242), \qquad b = (3536633334444444433242). \quad (8)$$

This notation means that $a = \mu_3\mu_5\mu_3\mu_0...$ and likewise for b.

Guided Lattice Finding: Even computing with the simpler matrices is tough. Here is our trick. I think of this trick as being similar to Hensel's Lemma, at least in spirit. We explain the trick for T_2. It works essentially the same way for T_3. We define the *full reduction* of a lattice Λ in our model to be the sequence $\Lambda_0, ..., \Lambda_n = \Lambda$ comprising the path connecting the level 0 lattice to Λ. Let θ be one of our 8 matrices above. We first precompute $\theta(\Lambda_0)$. This is easy.

Next, assuming that we have computed $\theta(\Lambda_k)$, we know that $A = \theta(\Lambda_{k+1})$ must be one of the 3 lattices B_1, B_2, B_3 adjacent to $\theta(\Lambda_k)$. So, we only need to find one of three lattices and certify equivalence. To do this, we apply an auxiliary matching routine which finds a sufficient condition for A and some B_j to match. There is no theoretical reason to think that my matching routine always works, but it did work for about a million randomly chosen words in Γ having length about 25. The matching routine breaks when we try it with some other smallish matrices, but we don't care.

Simple Matching: First we explain a method which sometimes certifies that lattices E and F match. Let M^* denote the integer matrix such that $MM^* = \det(M)$.

- We divide out by the gcd of the entries of E to get a new matrix E'.
- We divide out by the gcd of the entries of F to get a new matrix F'.
- We let $G = (E')^* F'$.
- We check whether $G' \in \pm SL_2(\boldsymbol{Z})$. If so, then E and F represent the same point in the Serre tree.

This method is not very powerful, so we augment it with something better.

Enhanced Matching: Our enhanced matching method is pretty *ad hoc*. The approach is to construct several lattices equivalent to E and then try to match one of these with F using the simple matching algorithm. First of all, given a vector $U \in \boldsymbol{Z}^2$ we let $U^* \in \boldsymbol{Z}^2$ denote the vector U/u where u is largest common odd divisor of the components of U. For instance, if $U = (12, 18)$ then $u = 3$ and $U^* = (4, 6)$. Let E be the span of U and V.

- E_1 is the span of U^* and V^*. Call these vectors U_1 and U_2.
- E_2 is the span of U_1 and $(U_1 + V_1)^*$.
- E_3 is the span of U_1 and $(U_1 - V_1)^*$.

The matrices E_0, E_1, E_2 all represent the same point in the Serre tree. We try to match F with each of these. The method succeeds if we get at least one match. Like the simple method, this method is not guaranteed to work. However, it does all our calculations for us just fine.

5 The Technical Lemmas

5.1 Embedding a Single Tile

Here we prove Lemma 2.2. We denote geodesic segments in T_2 and T_3 by their endpoints. Such segments will be denoted by square brackets. Thus $[E,F]_2$ will represent a geodesic segment in T_2. The words corresponding to the 4 vertices of the quadrilateral τ discussed above are

- $w_0 = \emptyset$
- $w_1 = A$.
- $w_2 = ba$.
- $w_3 = baa$.

Here w_0 is the empty word. This word acts as the identity on the space X. Let Λ_0 denote the level 0 lattice in the Serre tree.

$$w_0(\Lambda_0) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right). \tag{9}$$

$$w_1(\Lambda_0) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 4 & 1 \end{bmatrix} \right). \tag{10}$$

$$w_2(\Lambda_0) = \left(\begin{bmatrix} 64 & 0 \\ 44 & 1 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \right). \tag{11}$$

$$w_3(\Lambda_0) = \left(\begin{bmatrix} 64 & 0 \\ 44 & 1 \end{bmatrix}, \begin{bmatrix} 81 & 0 \\ 0 & 1 \end{bmatrix} \right). \tag{12}$$

These calculations give us the pitches advertised in §2.3. In X, all 4 vertices lie in the product

$$\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 64 & 0 \\ 44 & 1 \end{bmatrix} \right]_2 \times \left[\begin{bmatrix} 9 & 0 \\ 4 & 1 \end{bmatrix}, \begin{bmatrix} 81 & 0 \\ 0 & 1 \end{bmatrix} \right]_3. \tag{13}$$

This is a 6×6 flat Euclidean rectangle in X. This proves Lemma 2.2 for the quadrilateral τ. Given that τ is a fundamental domain for the action of Γ on $\boldsymbol{H}^2$, the action of Γ on X transitively permutes the sets of interest to us. Hence, by symmetry, Lemma 2.2 holds for all of the parallelogram tiles. To be sure, we also calculated the pitches for all 8 quadrilaterals in Figure 5.

5.2 Embedding Adjacent Tiles

Here we prove Lemma 2.3. Let τ_2 be the tile adjecent to $\tau_1 = \tau$ across the red diagonal edge. The vertices of τ_2 are v_0, v_3, v_4, v_5. We compute

$$w_4(\Lambda_0) = \left(\begin{bmatrix} 64 & 0 \\ 44 & 1 \end{bmatrix}, \begin{bmatrix} 729 & 0 \\ 0 & 1 \end{bmatrix} \right). \qquad w_5(\Lambda_0) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \right). \tag{14}$$

This time all the vertices of $\Psi(\tau_1 \cup \tau_2)$ are contained in the product

$$\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 64 & 0 \\ 44 & 1 \end{bmatrix} \right]_2 \times \left[\begin{bmatrix} 9 & 0 \\ 4 & 1 \end{bmatrix}, \begin{bmatrix} 729 & 0 \\ 0 & 1 \end{bmatrix} \right]_3. \tag{15}$$

This is a flat 6×8 rectangle. The two flat rectangles $\Psi(\tau_1)$ and $\Psi(\tau_2)$ share an edge and have the same shape. They either coincide or lie on opposite sides of the same edge. Since they don't coincide, they lie on opposite sides of the same edge. Hence $\Psi(\tau_1 \cup \tau_2)$ is another flat parallelogram. By symmetry, Lemma 2.3 holds for all tiles which are adjacent across a red edge.

Let τ_3 be the tile in Figure 5 that is adjacent to τ_1 across a blue edge. The vertices of τ_3 are v_0, v_{15}, v_{16}, v_1. We have

$$w_1(\Lambda_0) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 4 & 1 \end{bmatrix} \right). \tag{16}$$

$$w_2(\Lambda_0) = \left(\begin{bmatrix} 64 & 0 \\ 44 & 1 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \right). \tag{17}$$

$$w_3(\Lambda_0) = \left(\begin{bmatrix} 64 & 0 \\ 44 & 1 \end{bmatrix}, \begin{bmatrix} 81 & 0 \\ 0 & 1 \end{bmatrix} \right). \tag{18}$$

$$w_{15}(\Lambda_0) = \left(\begin{bmatrix} 4 & 0 \\ 3 & 16 \end{bmatrix}, \begin{bmatrix} 81 & 0 \\ 76 & 1 \end{bmatrix} \right). \tag{19}$$

$$w_{16}(\Lambda_0) = \left(\begin{bmatrix} 4 & 0 \\ 3 & 16 \end{bmatrix}, \begin{bmatrix} 729 & 0 \\ 643 & 1 \end{bmatrix} \right). \tag{20}$$

All these vertices lie in the product

$$\left[\begin{bmatrix} 64 & 0 \\ 44 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 3 & 16 \end{bmatrix} \right]_2 \times \left[\begin{bmatrix} 729 & 0 \\ 643 & 1 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 4 & 1 \end{bmatrix} \right]_3. \tag{21}$$

This works because $643 \equiv 76 \bmod 81$ and $76 \equiv 4 \bmod 9$. The same argument as in the red case finishes the proof. This completes the proof of Lemma 2.3.

6 References

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