Moebius Maps Preserve Circles

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Moebius transformations are maps of the form

$$z \to \frac{az+b}{cz+d}$$

where $a, b, c, d \in \boldsymbol{C}$ are complex numbers such that $ad - cb \neq 0$. These form a group G and act on the Riemann sphere $\boldsymbol{C} \cup \infty$ as homeomorphisms. A *circle* in $\boldsymbol{C} \cup \infty$ is either a circle of $\boldsymbol{C}$ or else the union of a line with ∞. We call the latter *extended lines*. In particular, $C_0 = \boldsymbol{R} \cup \infty$ is an extended line. In these notes I'll give a strange proof that Moebius transformations map circles to circles. The proof is based on 4 properties.

1. If γ is a loop in $\boldsymbol{C} \cup \infty$ which is not an extended line, then there is some circle D such that $\gamma \cap D$ contains at least 3 points.

2. For any circle C, there is some $T_C \in G$ such that $T(C_0) = C$.

3. If (a_1, a_2, a_3) and (b_1, b_2, b_3) are two triples of distinct points on C_0, then $\exists R \in G$ such that $R(C_0) = C_0$ and $R(a_i) = b_i$ for $i = 1, 2, 3$.

4. $R \in G$ is determined by where it takes 3 distinct points of C_0.

Main Argument: Let $M \in G$ and let C be a circle. Let $\gamma = M(C)$. If γ is an extended line, we are done. Otherwise let D be the circle from Property 1. Let $L = T_D \circ M \circ T_C^{-1}$. By construction, $L(C_0) \cap C_0$ contains 3 points b_1, b_2, b_3. Let $a_i = L^{-1}(b_i)$ for $i = 1, 2, 3$. Let $R \in G$ be given by Property 2. Then R and L agree on a_1, a_2, a_3. But then, by Property 3, $R = L$, which forces $L(C_0) = C_0$. but then $\gamma = D$ and γ is a circle.

Property 1: γ has 3 non-collinear points. Every 3 non-collinear points $a, b, c \in \gamma$ lie in the circle D of radius $|x - a|$ centered at x, where x is the intersection of the perpendicular bisectors of the segments $\overline{ab}$ and $\overline{bc}$.

Property 2: Using similarities, we reduce to the case when C is the unit circle. The Moebius transformation $T(z) = (z+i)/(z-i)$ evidently maps C_0 into C, and the upper halfplane outside the unit disk, and the lower halfplane inside the unit disk. Since T is a homeomorphism, we must have $T(C_0) = C$.

Property 3: By the group property, it suffices to consider the case when $(b_1, b_2, b_3) = (0, 1, \infty)$. The map

$$T(z) = \frac{-(a_2 - a_3)(a_1 - z)}{(a_1 - a_2)(a_3 - z)}$$

is a Moebius transformation and has all the properties.

Property 4: Using Property 3, and the group property, it suffices to show that a Moebius transformation is the identity provided that it fixes $(0, 1, \infty)$. Starting with $T(z) = (az + b)/(cz + d)$, and plugging $T(0) = 0$ gives $b = 0$. Plugging in $T(1) = 1$ gives $a = c + d$. Plugging in $T(\infty) = \infty$ gives $c = 0$. We're left with $T(z) = z$.