Some Symplectic Geometry

1 The Goal

The purpose of these notes is to explain (to myself) the three basic facts about symplectic manifolds, Hamiltonian vector fields, and the Poisson bracket. I wrote these notes by filling in the proofs of the claims made on the Lie derivatives page of Wikipedia.

Let M be a smooth $(2n)$-dimensional manifold and let ω be a symplectic form on M. This means that ω is a closed nondegenerate 2-form. For any function $f : M \to \boldsymbol{R}$ we introduce the *Hamiltonian* H_f. It has the property that

$$\omega(H_f, W) = Wf = df(W); \tag{1}$$

for any vector field W. You need the nondegeneracy of ω to guarantee the existence of H_f. We also define the *Poisson bracket*

$$\{f, g\} = \omega(H_f, H_g) \tag{2}$$

Here are the three basic facts.

1. The flow generated by H_f preserves f. That is, H_f is tangent to the level sets of f. This fact is easy: $df(H_f) = \omega(H_f, H_f) = 0$. That's it.

2. The flow generatd by H_f preserves ω. That is, the flow is a symplectomorphism for each time value.

3. If $\{f, g\} = 0$ then H_f and H_g generate commuting flows.

These three basic facts are all you need to understand the miracle of completely integrable systems. A completely integrable system on M is a collection $f_1, ..., f_n$ of functions such that $\{f_i, f_j\} = 0$ for all i, j and such that the vector fields $\{H_1, ..., H_n\}$ are linearly independent.

The generic common level set L of $\{f_1, ..., f_n\}$ is an n-dimensional compact smooth manifold, and the vectors $H_1, ..., H_n$ generate pairwise commuting flows tangent to L. But then these flows give coordinate charts from L to $\boldsymbol{R}^n$ in which the overlap functions are translations. This forces L to be a torus, and each flow to be an isometric motion in the given coordinates.

The rest of the notes are devoted to proving Fact 2 and Fact 3.

2 The Lie Derivative

Let M be a smooth manifold and let V be a vector field on M. Suppose that M generates the flow $\phi_t : M \to M$. For a function f, we have

$$L_V f = \frac{d}{dt}(f \circ \phi_t) = Vf = df(V). \tag{3}$$

Here Vf is the directional derivative of f along V.

If W is another vector field, we define

$$L_V W = \frac{d}{dt}\Big((\phi_t^{-1})_*(W_{\phi_t})\Big) = [V, W]. \tag{4}$$

So, if we are interested at the derivative at the point p, we evaluate the vector field W at $\phi_t(p)$ and map the vector back to the tangent plane at p using the tangent map of ϕ_t^{-1}.

If ω is a differential form, we define

$$L_V \omega = \frac{d}{dt}\Big((\phi_t^{-1})^*(\omega_{\phi_t})\Big). \tag{5}$$

Suppose that ω is a 2-form and X, Y are vector fields. Then $\omega(X, Y)$ is a function. From the product rule

$$L_V(\omega(X, Y)) = (L_V \omega)(X, Y) + \omega([V, X], Y) + \omega(X, [V, Y]). \tag{6}$$

Equation 6 is one of the key equations we will use when establishing Fact 3 about symplectic geometry.

We introduce the contraction operator i_V, which maps $(n+1)$-forms to n-forms. Here is the formula

$$(i_V \beta)(X_1, ..., X_n) = \beta(V, X_1, ..., X_n). \tag{7}$$

We have Cartan's formula

$$L_V \beta = i_V(d\beta) + d(i_V \beta). \tag{8}$$

This holds for any differential form β. We wil prove Cartan's formula below, in the case we need. Cartan's formula is the key equation we need to establish Fact 2 about symplectic geometry.

3 Some Cases of Cartan's Formula

We need Cartan's formula for 1-forms and for closed 2-forms. Here we prove these 2 cases. For closed 2-forms, Cartan's formula reduces to

$$L_V\omega = d(i_V\omega). \tag{9}$$

Lemma 3.1 *If Cartan's formula holds for* 1*-forms, then Cartan's formula holds for closed* 2*-forms.*

Proof: Let ω be a closed 2-form. Cartan's formula is a local calculation, and so we may assume that $\omega = d\alpha$ where α is a closed 1-form. The pullback map commutes with the d-operator. Hence L and d commute. This gives us

$$L_V\omega = L_V(d\alpha) = d(L_V\alpha) = d(i_V d\alpha) + d(d(i_V\alpha)) = d(I_V\omega), \tag{10}$$

since $d^2 = 0$. ♠

Lemma 3.2 *Cartan's formula holds for* 1*-forms.*

Proof: Any 1-form can be expressed as a finite sum $\sum_i f_i dg_i$ for smooth functions f_i and g_i. So, it suffices to prove Cartan's formula for fdg. Using the fact that d and L commute, we have

$$L_V(fdg) = fL_V(dg) + (Vf)dg = fd(L_Vg) + (Vf)dg = fd(Vg) + (Vf)dg. \tag{11}$$

On the other hand

$$i_Vd(fdg) = i_V(df \wedge dg) = i_V(df \otimes dg - dg \otimes df) = (Vf)dg - (Vg)df, \tag{12}$$

and

$$d(i_V(fdg)) = d(fVg) = fd(Vg) + (Vg)df. \tag{13}$$

Adding the last two equations, we get that

$$i_Vd(fdg) + d(I_V(fdg)) = fd(Vg) + (Vf)dg = L_V(fdg), \tag{14}$$

so it works. ♠

4 Proof of the Facts

Fact 2: We first prove Fact 2. This amounts to showing that $L_V\omega = 0$ when $V = H_f$. Using the special case of Cartan's formula, we have

$$L_{H_f}\omega = d(i_{H_f}(\omega)) = d(df) = 0.$$

The point here is that $i_{H_f}(\omega)(X) = \omega(H_f, X) = df(X)$, by definition. That's the proof.

Fact 3: We will show that $H_{\{f,g\}} = [H_f, H_g]$, the Lie bracket of H_f and H_g. When $\{f, g\} = 0$ it means that $[H_f, H_g] = 0$, and this means that H_f and H_g generate commuting flows.

Let $V = H_f$ and $W = H_g$. Below we will derive the identity.

$$i_{[V,W]}\omega = d(i_V i_W \omega). \tag{15}$$

Assuming this identity, we get the following for any vector field X:

$$\omega([H_f, H_g], X) = \omega([V, W], X) = i_{[V,W]}\omega(X) =$$

$$d(i_V i_W \omega)(X) = X\omega(V, W) = X\{f, g\} = \omega(H_{\{f,g\}}, X). \tag{16}$$

This proves what we want. It only remains to prove Equation 15. Choose X to be a vector field which commutes with V. We have the identity

$$L_V(\omega(W, X)) = (L_V\omega)(W, X) + \omega(L_V W, X) + \omega(W, L_V X) = \omega([V, W], X). \tag{17}$$

Here we have used the fact that $L_V\omega = 0$ and $L_V W = [V, W]$ and $L_V X = 0$. Since Equation 17 is true for any choice of commuting X, and we can arrange for such a vector field to be arbitrary at a point of interest to us, we get

$$L_V(i_W(\omega)) = i_{[V,W]}\omega. \tag{18}$$

Let $\alpha = i_W(\omega)$. Note that $\alpha = dg$. Hence $d\alpha = 0$. Applying Cartan's formula to α, we have

$$L_V(i_W(\omega)) = L_V\alpha = d(i_V\alpha) = d(i_V i_W \omega). \tag{19}$$

Equation 15 comes from putting together Equations 18 and 19.