Liouville-Arnold integrability of the pentagram map on closed polygons

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Abstract

The pentagram map is a discrete dynamical system defined on the moduli space of polygons in the projective plane. This map has recently attracted a considerable interest, mostly because its connection to a number of different domains, such as: classical projective geometry, algebraic combinatorics, moduli spaces, cluster algebras and integrable systems.

Integrability of the pentagram map was conjectured in [19] and proved in [15] for a larger space of twisted polygons. In this paper, we prove the initial conjecture that the pentagram map is completely integrable on the moduli space of closed polygons. In the case of convex polygons in the real projective plane, this result implies the existence of a toric foliation on the moduli space. The leaves of the foliation carry affine structure and the dynamics of the pentagram map is quasi-periodic. Our proof is based on an invariant Poisson structure on the space of twisted polygons. We prove that the Hamiltonian vector fields corresponding to the monodoromy invariants preserve the space of closed polygons and define an invariant affine structure on the level surfaces of the monodromy invariants.

Contents

1 Introduction

The *pentagram map* is a geometric construction which carries one polygon to another. Given an n-gon P, the vertices of the image $T(P)$ under the pentagram map are the intersection points of consecutive combinatorially shortest diagonals of P. The left side of Figure 1 shows the basic construction. The right hand side shows the second iterate of the pentagram map. The second iterate has the virtue that it acts in a canonical way on a labeled polygon, as indicated. The first iterate also acts on labeled polygons, but one must make a choice of labeling scheme; see Section 2.2. The simplest example of the pentagram map for pentagons was considered in [13]. In the case of arbitrary n, the map was introduced in [17] and further studied in [18, 19].

The pentagram map is defined on any polygon whose points are in general position, and also on some polygons whose points are not in general position. One sufficient condition for the pentagram map to be well defined is that every consecutive triple of points is not collinear. However, this last condition is not invariant under the pentagram map.

The pentagram map commutes with projective transformations and thereby induces a (generically defined) map

$$T : \mathcal{C}_n \to \mathcal{C}_n \tag{1.1}$$

where $\mathcal{C}_n$ is the moduli space of projective equivalence classes of n-gons in the projective plane. Mainly we are interested in the subspace $\mathcal{C}_n^0$ of projective classes of convex n-gons. The pentagram map is entirely defined on $\mathcal{C}_n^0$ and preserves this subspace.

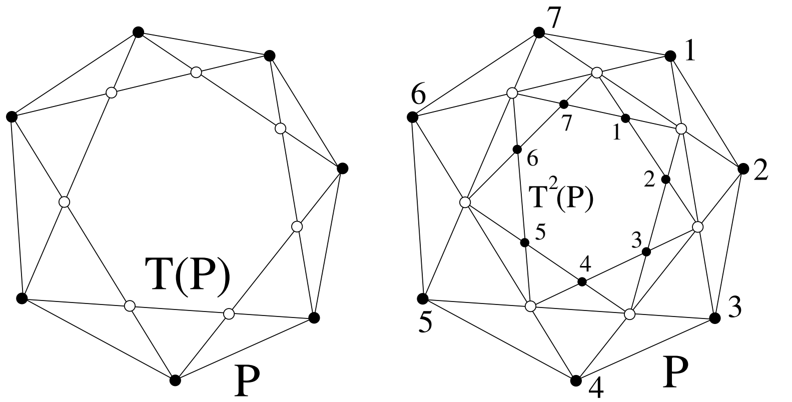

Figure 1: The pentagram map and its second iterate defined on a convex 7-gon

Note that the pentagram map can be defined over an arbitrary field. Usually, we restrict our considerations to the geometrically natural real case of convex n-gons in $\mathbb{RP}^2$. However, the complex case represents a special interest since the moduli space of n-gons in $\mathbb{CP}^2$ is a higher analog of the moduli space $\mathcal{M}_{0,n}$ (the moduli space of stable curves of genus zero with n distinct marked points). Unless specified, we will be using the general notation $\mathbb{P}^2$ for the projective plane and PGL_3 for the group of projective transformations.

1.1 Integrability problem and known results

The maps $T : \mathcal{C}_5 \to \mathcal{C}_5$ and $T : \mathcal{C}_6 \to \mathcal{C}_6$ are periodic. Indeed, there are maps $T' : \mathcal{C}_5 \to \mathcal{C}_5$ and $T' : \mathcal{C}_6 \to \mathcal{C}_6$, which differ from T only by composition with a cyclic relabelling, so that T' is the identity on $\mathcal{C}_5$ and an involution on $\mathcal{C}_6$. See [17]. (These alternate labeling schemes are only convenient for the cases $n = 5, 6$ so we do not use them below.)

The conjecture that the map (1.1) is completely integrable was formulated roughly in [17] and then more precisely in [19]. This conjecture was inspired by computer experiments in the case $n = 7$. Figure 2 presents (a two-dimensional projection of) an orbit of a convex heptagon in $\mathbb{RP}^2$. (In the example, the bilateral symmetry of the initial heptagon causes the orbit to have 2-dimensional closure, rather than a 3-dimensional closure, as one would expect from our main result below.)

The first results regarding the integrability of the pentagram map were proved for the pentagram map defined on a larger space, $\mathcal{P}_n$, of *twisted* n-gons. A series of T-invariant functions (or first integrals) called the *monodromy invariants*, was constructed in [19]. In [15] (see also [14] for a short version), the complete integrability of T on $\mathcal{P}_n$ was proved with the help of a T-invariant Poisson structure, such that the monodromy invariants Poisson-commute.

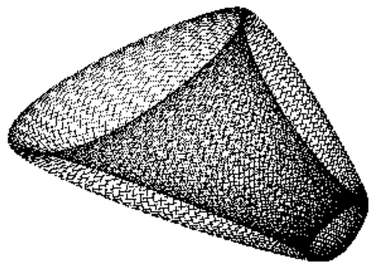

Figure 2: An orbit of the pentagram map on a heptagon

In [22], F. Soloviev found a Lax representation of the pentagram map and proved its algebraic-geometric integrability. The space of polygons (either $\mathcal{P}_n$ or $\mathcal{C}_n$) is parametrized in terms of a spectral curve with marked points and a divisor. The spectral curve is determined by the monodromy invariants, and the divisor corresponds to a point on a torus – the Jacobi variety of the spectral curve. These results allow one to construct explicit solutions formulas using Riemann theta functions (i.e., the variables that determine the polygon as explicit functions of time). Soloviev also deduces the invariant Poisson bracket of [15] from the Krichever-Phong universal formula.

Our result below has the same dynamical implications as that of Soloviev, in the case of real convex polygons. Soloviev's approach is by way of algebraic integrability, and it has the advantage that it identifies the invariant tori explicitly as certain Jacobi varieties. Our proof is in the framework of Liouville-Arnold integrability, and it is more direct and self-contained.

1.2 The main theorem

The main result of the present paper is to give a purely geometric proof of the following result.

Theorem 1. *Almost every point of $\mathcal{C}_n$ lies on a T-invariant algebraic submanifold of dimension*

$$d = \begin{cases} n-4, & n \text{ is odd} \\ n-5, & n \text{ is even.} \end{cases} \tag{1.2}$$

that has a T-invariant affine structure.

Recall that an affine structure on a d-dimensional manifold is defined by a locally free action of the d-dimensional Abelian Lie algebra, that is, by d commuting vector fields linearly independent at every point. (The vector fields, but not the canonical affine structure, depend on a choice of a basis for the Lie algebra.)

In the case of convex n-gons in the real projective plane, thanks to the compactness of the space established in [18], our result reads:

Corollary 1.1. *Almost every orbit in $\mathcal{C}_n^0$ lies on a finite union of smooth d-dimensional tori, where d is as in equation (1.2). This union of tori has a T-invariant affine structure.*

Hence, the orbit of almost every convex n-gon undergoes quasi-periodic motion under the pentagram map. The above statement is closely related to the integrability theorem in the Liouville–Arnold sense [1].

Let us also mention that the dimension of the invariant sets given by (1.2) is precisely a half of the dimension of $\mathcal{C}_n$, provided n is odd, which is a usual, generic, situation for an integrable system. If n is even, then $d = \frac{1}{2}\dim \mathcal{C}_n - 1$ so that one can talk of "hyper-integrability". Also, we remark that the cases $n = 5, 6$ are special in that the maps are periodic and hence the orbits are just finite sets of points.

Our approach is based on the results of [19] and [15]. We prove that the level sets of the monodromy invariants on the subspace $\mathcal{C}_n \subset \mathcal{P}_n$ are algebraic subvarieties of $\mathcal{C}_n$ of dimension (1.2). We then prove that the Hamiltonian vector fields corresponding to the invariant functions are tangent to $\mathcal{C}_n$ (and therefore to the level sets). Finally, we prove that the Hamiltonian vector fields define an affine structure on a generic level set. The main calculation, which establishes the needed independence of the monodromy invariants and their Hamiltonian vector fields, uses a trick that is similar in spirit to tropical algebra.

One point that is worth emphasizing is that our proof does not actually produce a symplectic (or Poisson) structure on the space $\mathcal{C}_n$. Rather, we use the Poisson structure on the ambient space $\mathcal{P}_n$, together with the invariants, to produce enough commuting flows on $\mathcal{C}_n$ in order to fill out the level sets.

1.3 Related topics

The pentagram map is a particular example of a discrete integrable system. The main motivation for studying this map is its relations to different subjects, such as: a) projective differential geometry; b) classical integrable systems and symplectic geometry; c) cluster algebras; d) algebraic combinatorics of Coxeter frieze patterns. All these relations may be beneficial not only for the study of the pentagram map, but also for the above mentioned subjects. Let us mention here some recent developments involving the pentagram map.

- The relation of T to the classical Boussinesq equation was essential for [15]. In particular, the Poisson bracket was obtained as a discretization of the (first) Adler-Gelfand-Dickey bracket related to the Boussinesq equation. We refer to [23, 24] and references therein for more information about different versions of the discrete Boussinesq equation.

- In [20], surprising results of elementary projective geometry are obtained in terms of the pentagram map, its iterations and generalizations.
- In [21], special relations amongst the monodromy invariants are established for polygons that are inscribed into a conic.
- In [2], the pentagram map is related to Lie-Poisson loop groups and to dimer models.
- The paper [11] concerns discretizations of Adler-Gelfand-Dickey flows as multi-dimensional generalizations of the pentagram map.
- A particularly interesting feature of the pentagram map is its relation to the theory of cluster algebras developed by Fomin and Zelevinsky, see [3]. This relation was noticed in [15] and developed in [7], where the pentagram map on the space of twisted n-gons is interpreted as a sequence of cluster algebra mutations, and an explicit formula for the iterations of T is calculated[1].
- Extending Glick's approach and developing the connection with cluster algebras, [6] introduces higher pentagram maps and proves their complete integrability using the machinery of weighted directed networks on surfaces.
- The structure of cluster manifold on the space $\mathcal{C}_n$ and the related notion of 2-frieze pattern are investigated in [12].
- The singularities of the pentagram map are studied in [8]. A typical singularity disappears after a finite number of iterations (a confinement phenomenon).
- A version of higher-dimensional pentagram map is introduced and studied in [9].

2 Integrability on the space of twisted n-gons

In this section, we explain the proof of the main result in our paper [15], the Liouville-Arnold integrability of the pentagram map on the space of twisted n-gons. While we omit some technical details, we take the opportunity to fill a gap in [15]: there we claimed that the monodromy invariants Poisson commute, but our proof there had a flaw. Here we present a correct proof of this fact.

2.1 The space $\mathcal{P}_n$

We recall the definition of the space of twisted n-gons.

[1] This can be understood as a version of integrability or "complete solvability".

A *twisted n-gon* is a map $\phi : \mathbb{Z} \to \mathbb{P}^2$ such that

$$v_{i+n} = M \circ v_i, \tag{2.1}$$

for all $i \in \mathbb{Z}$ and some fixed element $M \in \mathrm{PGL}_3$ called the *monodromy*. We denote by $\mathcal{P}_n$ the space of twisted n-gons modulo projective equivalence. The pentagram map extends to a generically defined map $T : \mathcal{P}_n \to \mathcal{P}_n$. The same geometric definition given for ordinary polygons works here (generically), and the construction commutes with projective transformations.

In the next section we will describe coordinates on $\mathcal{P}_n$. These coordinates identify $\mathcal{P}_n$ as an open dense subset of $\mathbb{R}^{2n}$. Sometimes we will simply identify $\mathcal{P}_n$ with $\mathbb{R}^{2n}$. The space $\mathcal{C}_n$ is much more complicated; it is an open dense subset of a codimension 8 subvariety of $\mathbb{R}^{2n}$.

Remark 2.1. If $n \neq 3m$, then it seems useful to impose the simple condition that v_i, v_{i+1}, v_{i+2} are in general position for all i. With this condition, $\mathcal{P}_n$ is isomorphic to the space of difference equations of the form

$$V_i = a_i\, V_{i-1} - b_i\, V_{i-2} + V_{i-3}, \tag{2.2}$$

where $a_i, b_i \in \mathbb{C}$ or $\mathbb{R}$ are n-periodic: $a_{i+n} = a_i$ and $b_{i+n} = b_i$, for all i. Therefore, $\mathcal{P}_n$ is just a $2n$-dimensional vector space, provided $n \neq 3m$. Let us also mention that the spectral theory of difference operators of type (2.2) is a classical domain (see [10] and references therein).

2.2 The corner coordinates

Following [19], we define local coordinates $(x_1, \ldots, x_{2n})$ on the space $\mathcal{P}_n$ and give the explicit formula for the pentagram map.

Recall that the (inverse) *cross ratio* of 4 collinear points in $\mathbb{P}^2$ is given by

$$[t_1, t_2, t_3, t_4] = \frac{(t_1 - t_2)\,(t_3 - t_4)}{(t_1 - t_3)\,(t_2 - t_4)}, \tag{2.3}$$

In this formulation, the line containing the points is identified with $\mathbb{R} \cup \infty$ by a projective transformation so that $t_i \in \mathbb{R}$ for all i. Any identification yields the same final answer.

We define

$$\begin{aligned} x_{2i-1} &= [v_{i-2},\, v_{i-1},\, ((v_{i-2}, v_{i-1}) \cap (v_i, v_{i+1}))\,,\, ((v_{i-2}, v_{i-1}) \cap (v_{i+1}, v_{i+2}))] \\ x_{2i} &= [v_{i+2},\, v_{i+1},\, ((v_{i+2}, v_{i+1}) \cap (v_i, v_{i-1}))\,,\, ((v_{i+2}, v_{i+1}) \cap (v_{i-1}, v_{i-2}))] \end{aligned} \tag{2.4}$$

where (v, w) stands for the line through $v, w \in \mathbb{P}^2$, see Figure 3. The functions $(x_1, \ldots, x_{2n})$ are cyclically ordered: $x_{i+2n} = x_i$. They provide a system of local coordinates on the space $\mathcal{P}_n$ called the *corner invariants*, cf. [19].

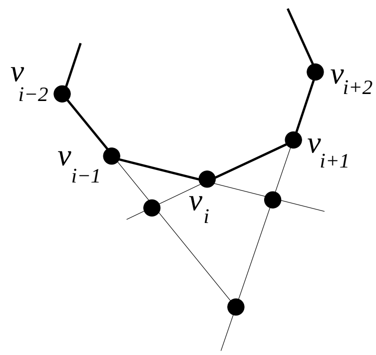

Figure 3: Definition of the corner invariants

Remark 2.2. a) The right hand side of the second equation is obtained from the right hand side of the first equation just by swapping the roles played by $(+)$ and $(-)$. In light of this fact, it might seem more natural to label the variables so that the second equation defines x_{2i+1} rather than x_{2i+0}. The corner invariants would then be indexed by odd integers. In Section 5 we will present an alternate labelling scheme which makes the indices work out better.

b) Continuing in the same vein, we remark that there are two useful ways to label the corner invariants. In [19] one uses the variables $x_1, x_2, x_3, x_4, \ldots$ whereas in [15, 21] one uses the variables $x_1, y_1, x_2, y_2, \ldots$. The explicit correspondence between the two labeling schemes is $x_{2i-1} \to x_i$, $x_{2i} \to y_i$. We call the former convention the *flag convention* whereas we call the latter convention the *vertex convention*. The reason for the names is that the variables x_1, x_2, x_3, x_4 naturally correspond to the flags of a polygon, as we will see in Section 5. The variables x_i, y_i correspond to the two flags incident to the ith vertex.

Let us give an explicit formula for the pentagram map in the corner coordinates. Following [15], we will choose the *right* labelling[2] of the vertices of $T(P)$, see Figure 4. One then has (see [19]):

$$T^* x_{2i-1} = x_{2i-1} \, \frac{1 - x_{2i-3}\, x_{2i-2}}{1 - x_{2i+1}\, x_{2i+2}}, \qquad T^* x_{2i} = x_{2i+2} \, \frac{1 - x_{2i+3}\, x_{2i+4}}{1 - x_{2i-1}\, x_{2i}}, \tag{2.5}$$

where $T^* x_i$ stands for the pull-back of the coordinate functions.

2.3 Rescaling and the spectral parameter

Equation (2.5) has an immediate consequence: a *scaling symmetry* of the pentagram map.

[2]To avoid this choice between the left or right labelling one can consider the square T^2 of the pentagram map.
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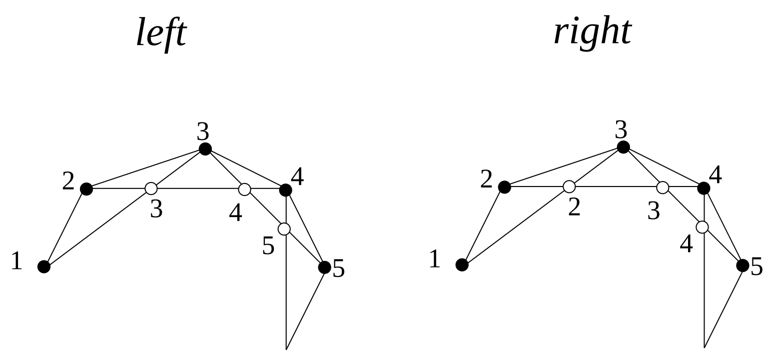

Figure 4: Left and right labelling

Consider a one-parameter group $\mathbb{R}^*$ (or $\mathbb{C}^*$ in the complex case) acting on the space $\mathcal{P}_n$ multiplying the coordinates by s or s^{-1} according to parity:

$$R_t : (x_1, x_2, x_3 \ldots, x_{2n}) \to (s\,x_1,\ s^{-1}\,x_2,\ s\,x_3,\ \ldots,\ s^{-1}x_{2n}). \tag{2.6}$$

It follows from (2.5), that the pentagram map commutes with the rescaling operation.

We will call the parameter s of the rescaling symmetry the *spectral parameter* since it defines a one-parameter deformation of the monodromy, M_s. Note that the notion of spectral parameter is extremely useful in the theory of integrable systems.

2.4 The Poisson bracket

Recall that a *Poisson bracket* on a manifold is a Lie bracket $\{.,.\}$ on the space of functions satisfying the Leibniz rule:

$$\{F, GH\} = \{F, G\}H + G\{F, H\},$$

for all functions F, G and H. The Poisson bracket is an essential ingredient of the Liouville-Arnold integrability [1].

Define the following Poisson structure on $\mathcal{P}_n$. For the coordinate functions we set

$$\{x_i,\ x_{i+2}\} = (-1)^i\,x_i\,x_{i+2}, \tag{2.7}$$

and all other brackets vanish. In other words, the Poisson bracket $\{x_i, x_j\}$ of two coordinate functions is different from zero if and only if $|i - j| = 2$. The Leibniz rule then allows one to extend the Poisson bracket to all polynomial (and rational) functions. Note that the Jacobi identity obviously holds. Indeed, the bracket (2.7) has constant coefficients when considered in the logarithmic coordinates $\log x_i$.

Action on Monomials: Here is a more explicit description of how the bracket acts on monomials. Given monomials A and B, we form a bipartite graph, where the top vertices and the bottom vertices are both indexed by the set $\{1, ..., 2n\}$. We join the top vertices a_i to the bottom vertices $b_{i\pm 2}$ iff x_i appears in A and $x_{i\pm 2}$ appears in B. Indices are taken cyclically, as usual. We label the edge joining a_i to $b_{i\pm 2}$ with $(\pm)$ if i is even and with $(\mp)$ if i is odd. Then $\{A, B\}/AB$ is the number of $(+)$ signs minus the number of $(-)$ signs. One derives this description by induction and Leibniz's rule. Figure 5 illustrates this for $n = 6$ and $A = x_3x_4x_5x_9$ and $B = x_1x_5x_6x_7$. In this case $\{A, B\}/AB = 1$. The thick lines are labelled with $(-)$ and thin ones with $(+)$.

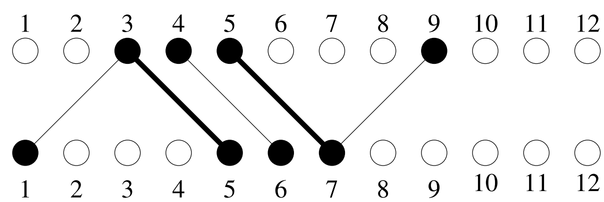

Figure 5: Graphical calculation of $\{x_3x_4x_5x_9, x_1x_5x_6x_7\}$

Alternatively, one may orient the graph according to the sign of the bracket between the respective variables: the thin edges are oriented downward and the thick one upward. The coefficient $\{A, B\}/AB$ is the intersection number of this oriented graph with the horizontal line separating the top and bottom parts.

Note that the above definition is valid in the case of monomials with powers of variables. One then obtains multiple edges with multiplicity given by the product of the corresponding powers.

Proposition 2.3. *The pentagram map preserves the Poisson bracket (2.7).*

Proof. This is an easy consequence of formula (2.5), see [15] (Lemma 2.9), for the details. □

Recall that a Poisson structure is a way to associate a vector field to a function. Given a function f on $\mathcal{P}_n$, the corresponding vector field X_f is called the *Hamiltonian vector field* defined by $X_f(g) = \{f, g\}$ for every function g. In the case of the bracket (2.7), the explicit formula is as follows:

$$X_f = \sum_{i-j=2} (-1)^{\frac{i+j}{2}} \; x_i \, x_j \left(\frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i} \right). \tag{2.8}$$

Note that the definitions of the Poisson structure in terms of the bracket of coordinate functions (2.7) and in terms of the Hamiltonian vector fields (2.8) are equivalent.

Geometrically speaking, Hamiltonian vector fields are defined as the image of the map

$$X : T_x^* \mathcal{P}_n \to T_x \mathcal{P}_n \tag{2.9}$$

at arbitrary point $x \in \mathcal{P}_n$. The kernel of X at a generic point is spanned by the differentials of the Casimir functions, that is, the functions that Poisson commute with all functions.

Remark 2.4. The cluster algebra approach of [7] also provides a Poisson bracket, invariant with respect to the pentagram map (see the book [5]). It can be checked that this cluster Poisson bracket is induced by the bracket (2.7).

2.5 The rank of the Poisson bracket and the Casimir functions

The corank of a Poisson structure is the codimension of the generic symplectic leaves. In other words, this is the dimension of the kernel of the map X in (2.9), that is, the dimension of the space generated by the differentials of the Casimir functions. In our situation, in which everything in sight is algebraic, the corank is generically constant.

Proposition 2.5. *The Poisson bracket (2.7) has corank* 2 *if n is odd and corank* 4 *if n is even; the functions*

$$O_n = x_1 x_3 \cdots x_{2n-1}, \qquad E_n = x_2 x_4 \cdots x_{2n} \tag{2.10}$$

for arbitrary n and the functions

$$O_{\frac{n}{2}} = \prod_{1 \le i \le \frac{n}{2}} x_{4i-1} + \prod_{1 \le i \le \frac{n}{2}} x_{4i+1}, \qquad E_{\frac{n}{2}} = \prod_{1 \le i \le \frac{n}{2}} x_{4i} + \prod_{1 \le i \le \frac{n}{2}} x_{4i+2}, \tag{2.11}$$

for even n, are the Casimirs of the Poisson bracket (2.7).

Proof. First, one checks that the functions (2.10) and (2.11) are indeed Casimir functions (for arbitrary n and for even n, respectively). To this end, it suffices to consider the brackets of (2.10) and (2.11), if n is even, with the coordinate functions x_i.

Second, one checks that the corank of the Poisson bracket is equal to 2, for odd n and 4, for even n. The corank is easily calculated in the coordinates $\log x_i$, see [15], Section 2.6 for the details. $\Box$

It follows that the Casimir functions are of the form $F\left(O_n, E_n\right)$, if n is odd, and of the form $F(O_{n/2}, E_{n/2}, O_n, E_n)$, if n is even. In both cases the generic symplectic leaves of the Poisson structure have dimension $4[(n-1)/2]$.

Remark 2.6. If n is even, then the Casimir functions can be written in a more simple manner:

$$\left\{ \prod_{1 \le i \le \frac{n}{2}} x_{4i-1}, \quad \prod_{1 \le i \le \frac{n}{2}} x_{4i+1}, \quad \prod_{1 \le i \le \frac{n}{2}} x_{4i}, \quad \prod_{1 \le i \le \frac{n}{2}} x_{4i+2} \right\}$$

instead of (2.10) and (2.11).

2.6 Two constructions of the monodromy invariants

The second main ingredient of the Liouville-Arnold theory is a set of Poisson-commuting invariant functions. In this section, we recall the construction [19] of a set of first integrals of the pentagram map

$$O_1, \dots, O_{[\frac{n}{2}]}, O_n, \; E_1, \dots, E_{[\frac{n}{2}]}, E_n$$

called the *monodromy invariants*. In other words, we will define $n+1$ invariant function on $\mathcal{P}_n$, if n is odd, and $n+2$ invariant function on $\mathcal{P}_n$, if n is even. The monodromy invariants are polynomial in the coordinates (2.4). Algebraic independence of these polynomials was proved in [19]. Note that O_n and E_n are the Casimir functions (2.10) and, for even n, the functions $O_{\frac{n}{2}}$ and $E_{\frac{n}{2}}$ are as in (2.11).

The indexing of the function O_i, E_j corresponds to their *weight*. More precisely, we define the weight of the coordinate functions by

$$|x_{2i+1}| = 1, \qquad |x_{2i}| = -1. \tag{2.12}$$

Then, $|O_k| = k$ and $|E_k| = -k$. We give two definitions of the monodromy invariants. In [19] it is proved that the two definitions are equivalent.

A. *The geometric definition.* Given a twisted n-gon (2.1), the corresponding monodromy has a unique lift to SL_3. By slightly abusing notation, we again denote this matrix by M. The two traces, $\mathrm{tr}(M)$ and $\mathrm{tr}(M^{-1})$, are preserved by the pentagram map (this is a consequence of the projective invariance of T). These traces are rational functions in the corner invariants. Consider the following two functions:

$$\widetilde{\Omega}_1 = \mathrm{tr}(M)\, O_n^{\frac{2}{3}}\, E_n^{\frac{1}{3}}, \qquad \widetilde{\Omega}_2 = \mathrm{tr}(M^{-1})\, O_n^{\frac{1}{3}}\, E_n^{\frac{2}{3}}.$$

It turns out that $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ are polynomials in the corner invariants (see [19]). Since the pentagram map preserves the monodromy, and O_n and E_n are invariants, the two functions $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ are also invariants. We then have:

$$\widetilde{\Omega}_1 = \sum_{k=0}^{[n/2]} O_k, \qquad \widetilde{\Omega}_2 = \sum_{k=0}^{[n/2]} E_k, \tag{2.13}$$

where O_k has weight k and E_k has weight $-k$ and where we set

$$O_0 = E_0 = 1,$$

for the sake of convenience. The pentagram map preserves each homogeneous component individually because it commutes with the rescaling (2.6).

Notice also that, if n is even, then $O_{\frac{n}{2}}$ and $E_{\frac{n}{2}}$ are precisely the Casimir functions (2.11). However, the invariants O_n and E_n do not enter the formula (2.13).

B. *The combinatorial definition.* Together with the coordinate functions x_i, we consider the following "elementary monomials"

$$X_i := x_{i-1}\, x_i\, x_{i+1}, \qquad i = 1, \ldots, 2n. \tag{2.14}$$

Let $O(X, x)$ be a monomial of the form

$$O = X_{i_1} \cdots X_{i_s}\, x_{j_1} \cdots x_{j_t},$$

where $i_1, \ldots, i_s$ are even and $j_1, \ldots, j_t$ are odd. Such a monomial is called *admissible* if the following conditions hold for all relevant indices a, b:

- $|i_a - i_b| > 4$.
- $|j_a - j_b| > 2$.
- $|i_a - j_b| > 3$.

It turns out that this is equivalent to the statement that there are no repeated factors (of the form X_i^k or x_j^k with $k > 1$), and furthermore that the Poisson brackets $\{X_{i_r}, X_{i_u}\}$ and $\{X_{i_r}, x_{j_u}\}$ and $\{x_{j_r}, x_{j_u}\}$ of all the elementary monomials entering O vanish.

The weight of the above monomial is

$$|O| = s + t,$$

see (2.12). For every admissible monomial, we also define the *sign* of O via

$$\mathrm{sign}(O) := (-1)^t.$$

The invariant O_k is defined as the alternated sum of all the admissible monomials of weight k:

$$O_k = \sum_{|O|=k} \mathrm{sign}(O)\, O, \qquad k \in \left\{1, 2, \ldots, \left[\frac{n}{2}\right]\right\}. \tag{2.15}$$

It is proved in [19] that this definition of O_k coincides with (2.13).

Example 2.7. The first two invariants are:

$$O_1 = \sum_{i=1}^{n} \left(X_{2i} - x_{2i+1}\right), \qquad O_2 = \sum_{|i-j|\geq 2} \left(x_{2i+1}\, x_{2j+1} - X_{2i}\, x_{2j+1} + X_{2i}\, X_{2j+2}\right),$$

for $n = 5$ the above formulas simplify, see [15].

The definition of the functions E_k is exactly the same, except that the roles of *even* and *odd* are swapped.

Remark 2.8. There is an elegant way to define the monodromy invariants in terms of determinants. See [21].

2.7 The monodromy invariants Poisson commute

The goal is this section is to prove the following result.

Theorem 2. $\{O_i, O_j\} = \{O_i, E_j\} = \{E_i, E_j\} = 0$ *for all relevant indices i and j. Hence, the corresponding Hamiltonian vector fields commute.*

We prove first that $\{O_i, O_j\} = 0$. The only monomials which can appear in $\{O_i, O_j\}$ have exponents (of X_i and x_j) in the set $\{1, 2\}$. In our proof, we sometimes view the monomial μ as a *mapping* $\mu : \{1, ..., 2n\} \to \{0, 1, 2\}$. Here $\mu(i)$ is the exponent of x_i in μ. The *support* of μ (as a map) is exactly the set of indices of variables which appear in μ (as a monomial).

For a polynomial P and a monomial μ, we define $P|_\mu$ as the monomial μ times its coefficient in P. We call μ *good* if $\{O_i, O_j\}|_\mu = 0$ for all indices i, j. We will prove that all monomials are good.

We say that μ *decomposes* into μ_1 and μ_2 if (as monomials) $\mu = \mu_1\mu_2$, and (as maps) the supports of μ_1 and μ_2 are separated by at least 2 empty spaces, in the cyclic sense. It follows that the Poisson bracket of a variable with the index in support of μ_1 and a variable with the index in support of μ_2 vanishes. If we cannot factor μ this way, we call μ *indecomposable*.

Lemma 2.9. *If μ decomposes into μ_1 and μ_2, and both μ_1 and μ_2 are good, then μ is good.*

Proof. For any monomial F whose support is contained in the support of μ, we have a unique decomposition $F = F_1F_2$ where F_1 (respectively F_2) denotes the monomial obtained from F by setting to 1 all the variables having indices in the support of μ_2 (respectively μ_1). For example, $\mu = x_1x_5x_7$ decomposes into $\mu_1 = x_1$ and $\mu_2 = x_5x_7$. If $F = x_1x_5$ then $F_1 = x_1$ and $F_2 = x_5$.

Assume that the support of F is contained in the support of μ. Note that F is an admissible monomial in O_i if and only if F_1 and F_2 are admissible monomials in O_{i_1} and O_{i_2}, respectively, where $i_1 + i_2 = i$. Furthermore, $\text{sign}(F) = \text{sign}(F_1)\text{sign}(F_2)$.

Let $\bar{O}_i$ be the sum of the terms in O_i whose support is contained in the support of μ, and likewise for $\bar{O}_j$:

$$\bar{O}_i = \sum \text{sign}(A)A, \quad \bar{O}_j = \sum \text{sign}(B)B.$$

Then

$$\{O_i, O_j\}|_\mu = \{\bar{O}_i, \bar{O}_j\}|_\mu = \sum \text{sign}(A)\text{sign}(B)\{A, B\}|_\mu =^*$$

$$\sum \text{sign}(A_1)\text{sign}(B_1)\text{sign}(A_2)\text{sign}(B_2) \left(\{A_1, B_1\}|_{\mu_1}(A_2B_2)|_{\mu_2} + (A_1B_1)|_{\mu_1}\{A_2, B_2\}|_{\mu_2}\right) =$$

$$\sum_{i_1+i_2=i,\ j_1+j_2=j} \left(\{\bar{O}_{i_1}, \bar{O}_{j_1}\}|_{\mu_1}(\bar{O}_{i_2}\bar{O}_{j_2})|_{\mu_2} + (\bar{O}_{i_1}\bar{O}_{j_1})|_{\mu_1}\{\bar{O}_{i_2}\bar{O}_{j_2}\}|_{\mu_2}\right) =$$

$$\sum_{i_1+i_2=i,\ j_1+j_2=j} \left(\{O_{i_1}, O_{j_1}\}|_{\mu_1}(O_{i_2}O_{j_2})|_{\mu_2} + (O_{i_1}O_{j_1})|_{\mu_1}\{O_{i_2}O_{j_2}\}|_{\mu_2}\right),$$

where the starred equality is due to the large separation between the supports of A and B and Leibniz's rule. The last term vanishes because μ_1 and μ_2 are assumed to be good. We conclude that μ is good as well. $\square$

Let us record how the Poisson bracket interacts with the elementary monomials. We have

$$\{X_i, X_{i+2}\} = (-1)^{i+1}\, X_i X_{i+2}, \qquad \{X_i, X_{i+4}\} = (-1)^{i+1}\, X_i X_{i+4}, \tag{2.16}$$

and

$$\{x_i, X_j\} = \begin{cases} (-1)^i\, x_i X_j, & j = i+1,\, i+2,\, i+3, \\ (-1)^{i+1}\, x_i X_j, & j = i-3,\, i-2,\, i-1, \end{cases} \tag{2.17}$$

All other brackets $\{X_i, X_j\}$, as well as $\{X_i, x_j\}$, vanish.

The Poisson bracket $\{O_i, O_j\}$ is a sum of monomials of the form

$$\mu = X_{i_1} \cdots X_{i_s} x_{j_1} \cdots x_{j_t}, \tag{2.18}$$

where $i_1, \dots, i_s$ are even and $j_1, \dots, j_t$ are odd. Using Lemma 2.9, we assume that μ is indecomposable, and we want to prove that μ is good.

If μ contains x_a^2 then either $\mu = x_a^2$ or μ is decomposable. (This follows from the admissibility condition.) Likewise, if μ contains X_a^2 then either $\mu = X_a^2$ or μ is decomposable. These two "singleton" cases are trivial, so henceforth we will assume that μ does not contain the square of an elementary monomial.

Define an oriented graph Γ_μ whose vertex set is $\{X_{i_1}, \dots, X_{i_s}, x_{j_1}, \dots, x_{j_t}\}$, the elementary monomials which appear in μ. Join vertex v_1 with v_2 by an oriented edge from v_1 to v_2 if $\{v_1, v_2\} = v_1 v_2$. Recall that all the non-zero brackets of elementary monomials are listed in (2.16) and (2.17), and they all have coefficients ± 1.

Note that if Γ_μ is disconnected then μ is decomposable. Hence Γ_μ is a connected graph.

Lemma 2.10. *Γ_μ has no 3-cycles (i.e., triangles), no cycles of odd length, and no vertices having in-degree, or out-degree, greater than* 1.

Proof. (i) Assume there is a triangle (a, b, c) of elementary monomials. Then two of the three involved elementary monomials belong to the decomposition of the same polynomial, either O_i or O_j; assume that $a, b \in O_i$. The monomials a and b are joined by an arrow, thus their Poisson bracket does not vanish. This leads to a contradiction since all the monomials in O_i are admissible (see Section 2.6, definition B).

(ii) The argument for odd cycles is the same as in (i): the monomials in the cycle should alternate between O_i or O_j, and since the cycle is odd, two adjacent monomials will land on the same O_i.

(iii) We will show that no vertex of Γ_μ has out-degree greater than 1. The in-degree case is similar. Suppose that some vertex X_i has at least 2 outgoing arrows. Since the indices i

are even and the indices j are odd, X_i can be joined by an *outgoing* arrow to the following vertices (provided they belong to the graph): X_{i-2}, X_{i-4}, x_{i-1}, x_{i-3}. Indeed, the coefficient of the bracket in all these cases is $+1$. However, any two vertices from this list, would also be joined together by an edge. This would give us a triangle, which we have already ruled out. A similar (but simpler) consideration applies to a vertex x_i: it can be joined by outgoing arrows to X_{i-1} and X_{i-1} which are connected by an edge. The argument for two incoming arrows is similar. $\Box$

It follows that Γ_μ is of the form:

$$a_1 \longrightarrow a_2 \longrightarrow a_3 \longrightarrow \cdots \longrightarrow a_\ell, \tag{2.19}$$

or Γ_μ is a cycle of even length; here a_i are elementary monomials appearing in (2.18).

Suppose ℓ is odd and assume that $i > j$. Let A and B be admissible monomials in O_i and O_j, respectively, such that AB is a scalar multiple of μ (otherwise, trivially, $\{A,B\}|_\mu = 0$). By the admissibility condition, $A = a_1a_3\dots a_\ell$ and $B = a_2a_4\dots a_{\ell-1}$. By the Leibniz rule,

$$\{A,B\} = \left(\frac{\{a_1,a_2\}}{a_1a_2} + \frac{\{a_3,a_2\}}{a_2a_3} + \frac{\{a_3,a_4\}}{a_3a_4} + \cdots\right) AB = (1 - 1 + 1 \mp \cdots)\, AB = 0 \tag{2.20}$$

since the number of terms is even. Alternatively, one notices that the intersection number of Γ_μ with the separating line of the bibartite graph is zero; see Section 2.4.

Suppose now that ℓ is even. Again let A be an admissible monomial in O_i and B that in O_j such that AB is a scalar multiple of μ. Then either $A = a_1a_3\dots a_{\ell-1}$ and $B = a_2a_4\dots a_\ell$, or the other way around. In both cases, $i = j = \ell/2$, contradicting the assumption that $i > j$. The same argument works if Γ_μ is a cycle of even length: the vertices must alternate between A and B, hence the weights of A and B are equal and thus $i = j$, a contradiction.

The same proof (or symmetry) shows that $\{E_i, E_j\} = 0$ as well.

To prove that $\{O_i, E_j\} = 0$, we will start out with the same set-up as above. Lemma 2.9 works again, and gets us to the indecomposable case. This time we will analyze the structure of an indecomposable μ without the aid of the bipartite graph. As above, we sometimes think of μ as a map from $\{1,...,2n\}$ to $\{0,1,2\}$.

Lemma 2.11. *No terms contribute to $\{O_i, E_j\}|_\mu$ unless μ has the form*

$$x_i^\alpha (X_{i+3}X_{i+6}X_{i+9}\cdots X_{i+3\ell}) x_{i+3\ell+2}^\beta, \tag{2.21}$$

where $\alpha, \beta \in \{0,1\}$. (This way of expressing μ is not necessarily unique.)

Proof. Let (A,B) be a supposedly contributing pair. Suppose μ contains the variables x_{a-1} and x_{a+1} but not x_a. If a is even (respectively odd), then B (respectively A), has an odd-indexed (respectively even-indexed) variable but not both the adjacent even-indexed (respectively odd-indexed) variables. This is a contradiction. This shows that the support of μ is a consecutive string of indices.

Now suppose that μ has the square of some variable. If the support of μ is contained in 3 indices, the result is trivial, so we suppose otherwise. Suppose that μ contains the square of an odd-indexed variable. Say μ contains x_5^2. Then B contains $x_4x_5x_6$ but not x_k when $k \in \{1,2,3,7,8,9\}$. Moreover, A contains x_5, but not both x_4 and x_6. If neither x_4 nor x_6 appears in A, then x_k does not appear in A for $k \in \{2,3,4,6,7,8\}$. But then μ does not contain x_k for $k = 2,3,7,8$. Hence μ is decomposable. If x_4 appears in A, then $x_3x_4x_5$ appears in A and μ does not contain x_k for $k \in \{0,1,2,6,7,8\}$. But then x_k does not appear in μ for $k = 1,2,7,8$ and μ is decomposable. The proof is the same when A contains x_6. A similar argument works when, μ contains the square of an even-indexed variable.

Now we know that μ is a consecutive string of indices and μ takes on the value 1 on any index in its support. The admissibility of A and B now forces the structure claimed in the lemma. $\square$

Just as in the odd-odd case, the way the elementary monomials are assigned to A and B alternates – this follows from parity considerations. Using Leibniz's rule, we see that $\{A,B\} = 0$ unless $\alpha + \beta = 1$. In this case, there are exactly two ways to express μ as in Equation 2.21 and correspondingly there are exactly two terms contributing to $\{O_i, O_j\}|_\mu$ and they cancel. To illustrate this principle, we will consider the example where $i = 2$ and $j = 1$ and $\ell = 2$. After suitably shifting the indices, we have

$$\mu = x_1x_2x_3x_4x_5x_6x_7 = x_1X_3X_6 = X_2X_5x_7,$$

the two terms $\{x_1X_6, X_3\}$ and $\{X_2x_7, X_5\}$ cancel.

This completes the proof that $\{O_i, E_j\} = 0$.

In [19] it is proved that the monodromy invariants are algebraically independent. The argument is rather complicated, but it is very similar in spirit to the related independence proof we give in Section 4. The algebraic independence result combines with Theorem 2 to establish the integrability of the pentagram map on the space $\mathcal{P}_n$. Indeed, the Poisson bracket (2.7) defines a symplectic foliation on $\mathcal{P}_n$, the symplectic leaves being locally described as levels of the Casimir functions, see Proposition 2.5. The number of the remaining invariants is exactly half of the dimension of the symplectic leaves. The classical Liouville-Arnold theorem [1] is then applied.

3 Integrability on $\mathcal{C}_n$ modulo a calculation

The general plan of the proof of Theorem 1 is as follows.

1. We show that the Hamiltonian vector fields on $\mathcal{P}_n$ corresponding to the monodromy invariants are *tangent* to the subspace $\mathcal{C}_n$,

2. We restrict the monodromy invariants to $\mathcal{C}_n$ and show that the dimension of a generic level set is $n-4$ if n is odd and $n-5$ if n is even.

3. We show that there are exactly the same number of independent Hamiltonian vector fields.

In this section, we prove the first statement and also show that the dimension of the level sets is *at most* $n-4$ if n is odd and $n-5$ if n is even, and similarly for the number of independent Hamiltonian vector fields. The final step of the proof that this upper bound is actually the lower one will be done in the next two sections. This final step is a nontrivial calculation that comprises the bulk of the paper.

3.1 The Hamiltonian vector fields are tangent to $\mathcal{C}_n$

The space $\mathcal{C}_n$ is a subvariety of $\mathcal{P}_n$ having codimension 8. It turns out that one can give explicit equations for this variety. See Lemma 5.3. (These equations do not play a role in our proof, but they are useful to have.)

The following statement is essentially a consequence of Theorem 2. This is an important step of the proof of Theorem 1.

Proposition 3.1. *The Hamiltonian vector field on $\mathcal{P}_n$ corresponding to a monodromy invariant is tangent to $\mathcal{C}_n$.*

Proof. The space $\mathcal{P}_n$ is foliated by isomonodromic submanifolds that are generically of codimension 2 and are defined by the condition that the monodromy has fixed eigenvalues. Hence the isomonodromic submanifolds can be defined as the level surfaces of two functions, $\mathrm{tr}(M)$ and $\mathrm{tr}(M^{-1})$. This foliation is singular, and $\mathcal{C}_n$ is a singular leaf of codimension 8. We note that the versal deformation of $\mathcal{C}_n$ is locally isomorphic to SL(3) partitioned into the conjugacy equivalence classes.

Consider a monodromy invariant, $F\,(= O_i$ or $E_i)$, and its Hamiltonian vector field, X_F. We know that the Poisson bracket $\{F, \mathrm{tr}(M)\} = 0$, since all monodromy invariants Poisson commute and $\mathrm{tr}(M)$ is a sum of monodromy invariants. Hence X_F is tangent to the generic leaves of the isomonodromic foliation on $\mathcal{P}_n$. Let us show that X_F is tangent to $\mathcal{C}_n$ as well.

In a nutshell, this follows from the observation that the tangent space to $\mathcal{C}_n$ at a smooth point x_0 is the intersection of the limiting positions of the tangent spaces to the isomonodromic leaves at points x as x tends to x_0. Assume then that X_F is transverse to $\mathcal{C}_n$ at point $x_0 \in \mathcal{C}_n$. Then X_F will be also transverse to an isomonodromic leaf at some point x close to x_0, yielding a contradiction.

More precisely, we can apply a projective transformation so that the vertices V_1, V_2, V_3, V_4 of a twisted n-gon $V_1, V_2, \ldots$ become the vertices of a standard square. This gives a local identification of $\mathcal{P}_n$ with the set of tuples $(V_5, \ldots, V_n; M)$ where M is the monodromy, the projective transformation that takes the quadruple (V_1, V_2, V_3, V_4) to $(V_{n+1}, V_{n+2}, V_{n+3}, V_{n+4})$. The space of closed n-gons is characterized by the condition that M is the identity. Thus we have locally identified $\mathcal{P}_n$ with $\mathcal{C}_n \times \mathrm{SL}(3)$. In particular, we have a projection $\mathcal{P}_n \to \mathrm{SL}(3)$, and the preimage of the identity is $\mathcal{C}_n$. The isomonodromic leaves project to the conjugacy equivalent classes in SL(3).

Thus our proof reduces to the following fact about the group SL(3) (which holds for SL(n) as well).

Lemma 3.2. *Consider the singular foliation of* SL(3) *by the conjugacy equivalence classes, and let T_X be the tangent space to this foliation at $X \in \mathrm{SL}(3)$. Then the intersection, over all X, of the limiting positions of the spaces T_X, as $X \to 1\!\!1$, is trivial (here $1\!\!1 \in \mathrm{SL}(3)$ is the identity).*

Proof. Let $B \in \mathrm{SL}(3)$, and let $B + \varepsilon C$ be an infinitesimal deformation within the conjugacy equivalence class. Then

$$\mathrm{tr}\,(B + \varepsilon C) = \mathrm{tr}(B), \qquad \mathrm{tr}\left((B + \varepsilon C)^2\right) = \mathrm{tr}\left(B^2\right),$$

hence $\mathrm{tr}(C) = 0$ and $\mathrm{tr}(BC) = 0$, and also $\mathrm{tr}(B^{-1}C) = 0$ since $\det(B + \varepsilon C) = 1$. Thus the tangent space to a conjugacy equivalent class of B is given by

$$\mathrm{tr}(C) = \mathrm{tr}(BC) = \mathrm{tr}(B^{-1}C) = 0.$$

Now let $B = 1\!\!1 + \varepsilon A$, a point in an infinitesimal neighborhood of the identity $1\!\!1$; we have $\mathrm{tr}(A) = 0$. Then our conditions on C implies $\mathrm{tr}(C) = \mathrm{tr}(AC) = 0$. Since $\mathrm{tr}(AC)$ is a non-degenerate quadratic form, an element $C \in \mathrm{sl}(3)$ satisfying $\mathrm{tr}(AC) = 0$ for all $A \in \mathrm{sl}(3)$ has to be zero. $\Box$

In view of what we said above, this implies the proposition. $\Box$

3.2 Identities between the monodromy invariants

In this section, we consider the restriction of the monodromy invariants from the space of all twisted n-gons to the space $\mathcal{C}_n$ of closed n-gons. We show that these restrictions satisfy 5 non-trivial relations, whereas their differentials, considered as covectors in $\mathcal{P}_n$ whose foot-points belong to $\mathcal{C}_n$, satisfy 3 non-trivial relations. These relations are also mentioned in [15] and [22]. In Sections 4 and 5, we will prove that there are no other relations between the monodromy invariants on $\mathcal{C}_n$ and their differentials along $\mathcal{C}_n$.

We remark that, strictly speaking, the identities established in this section are not needed for the proof of our main result. For the main result, all we need to know is that there are enough commuting flows to fill out what could be (*a priori*, without the results in this section) a union of level sets of the monodromy invariants. Thus, the reader interested only in the main result can skip this section.

Theorem 3. *(i) The restrictions of the monodromy integrals to $\mathcal{C}_n$ satisfy the following five identities:*

$$\begin{array}{c}
\displaystyle\sum_{j=0}^{[n/2]} O_j = 3\, E_n^{\frac{1}{3}}\, O_n^{\frac{2}{3}}, \qquad \sum_{j=0}^{[n/2]} E_j = 3\, E_n^{\frac{2}{3}}\, O_n^{\frac{1}{3}}, \\
\displaystyle\sum_{j=1}^{[n/2]} j\, O_j = n\, E_n^{\frac{1}{3}}\, O_n^{\frac{2}{3}}, \qquad \sum_{j=1}^{[n/2]} j\, E_j = n\, E_n^{\frac{2}{3}}\, O_n^{\frac{1}{3}}, \\
\displaystyle E_n^{\frac{1}{3}} \sum_{j=1}^{[n/2]} j^2\, O_j \;=\; O_n^{\frac{1}{3}} \sum_{j=1}^{[n/2]} j^2\, E_j.
\end{array} \tag{3.1}$$

(ii) The differentials of the monodromy integrals along $\mathcal{C}_n$ satisfy the three identities:

$$\begin{aligned}
\sum_{j=1}^{[n/2]} dO_j &= 2\, E_n^{\frac{1}{3}}\, O_n^{-\frac{1}{3}}\, dO_n + E_n^{-\frac{2}{3}}\, O_n^{\frac{2}{3}} dE_n, \\
\sum_{j=1}^{[n/2]} dE_j &= 2\, E_n^{-\frac{1}{3}}\, O_n^{\frac{1}{3}} dE_n + E_n^{\frac{2}{3}}\, O_n^{-\frac{2}{3}}\, dO_n, \\
O_n^{\frac{1}{3}} \left(\sum_{j=1}^{[n/2]} j\ dE_j \right) + E_n^{\frac{1}{3}} \left(\sum_{j=1}^{[n/2]} j\ dO_j \right) &= n\, E_n^{\frac{2}{3}} O_n^{\frac{2}{3}} \left(E_n^{-1} dE_n + O_n^{-1} dO_n \right).
\end{aligned} \tag{3.2}$$

Proof. Recall that the monodromy invariants O_j are the homogeneous components of the polynomial $O_n^{2/3} E_n^{1/3}\, \mathrm{tr}(M)$ with respect to the rescaling (2.6), where $s = e^t$ for convenience. Likewise, the monodromy invariants E_j are homogeneous components of $O_n^{1/3} E_n^{2/3}\, \mathrm{tr}(M^{-1})$. Recall also that $O_0 = E_0 = 1$.

Denote for simplicity $O_n^{1/3} E_n^{2/3} = U$, $O_n^{2/3} E_n^{1/3} = V$. Notice that the monodromy matrix M has the unit determinant. Let $e^{\lambda_1},\ e^{\lambda_2},\ e^{\lambda_2}$ be the eigenvalues of M. One has

$$\lambda_1 + \lambda_2 + \lambda_3 \equiv 0. \tag{3.3}$$

We consider a one-parameter family of n-gons depending on the rescaling parameter t, such that for $t = 0$, the n-gon belongs to $\mathcal{C}_n$. The monodromy $M = M_t$ also depends on t so that we think of λ_i as functions of the corner coordinates $(x_1, \ldots, x_{2n})$ and of t. For $t = 0$, one has: $\lambda_i = 0$, $i = 1, 2, 3$ since $M_0 = \mathrm{Id}$.

The eigenvalues of M^{-1} are $e^{-\lambda_1},\ e^{-\lambda_2},\ e^{-\lambda_2}$. Since the weights of O_i and E_j are j and $-j$ respectively, the definition of the integrals writes as follows:

$$e^{\frac{nt}{3}}\,V\left(e^{\lambda_1}+e^{\lambda_2}+e^{\lambda_2}\right)=\sum_{j=0}^{[n/2]}e^{tj}\,O_j,\qquad e^{-\frac{nt}{3}}\,U\left(e^{-\lambda_1}+e^{-\lambda_2}+e^{-\lambda_2}\right)=\sum_{j=0}^{[n/2]}e^{-tj}\,E_j.$$

which we rewrite as

$$V\left(e^{\lambda_1}+e^{\lambda_2}+e^{\lambda_2}\right)=\sum_{j=0}^{[n/2]}e^{t(j-\frac{n}{3})}\,O_j,\qquad U\left(e^{-\lambda_1}+e^{-\lambda_2}+e^{-\lambda_2}\right)=\sum_{j=0}^{[n/2]}e^{-t(j-\frac{n}{3})}\,E_j. \tag{3.4}$$

Setting $t=0$ in these formulas yields the first two identities in (3.1). Next, differentiate these equations in t :

$$V\sum_{i=1}^{3}\lambda_i'\,e^{\lambda_i}=\sum_{j=0}^{[n/2]}\left(j-\frac{n}{3}\right)e^{tj}\,O_j,$$

where $\lambda_i'=d\lambda_i/dt$, and similarly for E_j. Set $t=0$, then the left-hand-side vanishes because $\sum\lambda_i'=0$ due to (3.3). Hence

$$\sum_{j=0}^{[n/2]}j\,O_j=\frac{n}{3}\sum_{j=0}^{[n/2]}O_j=n\,V$$

due to the first identity in (3.1) and similarly for E_j. One thus obtains the third and the fourth identity in (3.1).

To obtain the fifth equation in (3.1), differentiate the equations (3.4) with respect to t twice to get

$$\begin{aligned}V\left(\sum_{i=1}^{3}(\lambda_i''+\lambda_i'^2)\,e^{\lambda_i}\right)&=\sum_{j=0}^{[n/2]}\left(j-\frac{n}{3}\right)^2e^{tj}\,O_j,\\ U\left(\sum_{i=1}^{3}\left(-\lambda_i''+\lambda_i'^2\right)\,e^{\lambda_i}\right)&=\sum_{j=0}^{[n/2]}\left(j-\frac{n}{3}\right)^2e^{-tj}\,E_j.\end{aligned}$$

Divide the first equality by V, the second by U, subtract one from another, and set $t=0$:

$$2\sum_{i=1}^{3}\lambda_i''=V^{-1}\sum_{j=0}^{[n/2]}\left(j-\frac{n}{3}\right)^2O_j-U^{-1}\sum_{j=0}^{[n/2]}\left(j-\frac{n}{3}\right)^2E_j.$$

The left hand side vanishes, due to (3.3), so

$$V^{-1}\sum_{j=0}^{[n/2]}\left(j-\frac{n}{3}\right)^2O_j=U^{-1}\sum_{j=0}^{[n/2]}\left(j-\frac{n}{3}\right)^2E_j. \tag{3.5}$$

Therefore
$$V^{-1}\sum_{j=0}^{[n/2]} j^2\,O_j-\frac{2n}{3}\,V^{-1}\sum_{j=0}^{[n/2]} j\,O_j+V^{-1}\,\frac{n^2}{9}\sum_{j=0}^{[n/2]} O_j=$$
$$U^{-1}\sum_{j=0}^{[n/2]} j^2\,E_j-\frac{2n}{3}\,U^{-1}\sum_{j=0}^{[n/2]} j\,E_j+U^{-1}\,\frac{n^2}{9}\sum_{j=0}^{[n/2]} E_j.$$
The second and the third terms on the left and the right hand sides are pairwise equal, due to the first four identities in (3.1). This implies the fifth identity (3.1).

To prove (3.2), take differentials of (3.4):

$$V\sum_{i=1}^{3} e^{\lambda_i}\,d\lambda_i+\Big(\sum_{i=1}^{3} e^{\lambda_i}\Big)\,dV=$$
$$\Big(\sum_{j=0}^{[n/2]}\Big(j-\frac{n}{3}\Big)\,e^{t(j-\frac{n}{3})}\,O_j\Big)dt+\sum_{j=0}^{[n/2]} e^{t(j-\frac{n}{3})}\,dO_j,$$

and

$$-U\sum_{i=1}^{3} e^{-\lambda_i}\,d\lambda_i+\Big(\sum_{i=1}^{3} e^{-\lambda_i}\Big)\,dU=$$
$$-\Big(\sum_{j=0}^{[n/2]}\Big(j-\frac{n}{3}\Big)e^{-t(j-\frac{n}{3})}E_j\Big)\,dt+\sum_{j=0}^{[n/2]} e^{-t(j-\frac{n}{3})}\,dE_j.$$

Set $t=0$: the first terms on the right hand sides vanish due to (3.3), and the first parentheses on the right hand sides vanish due to (3.1). We get

$$\sum_{j=0}^{[n/2]} dO_j=3\,dV,\qquad \sum_{j=0}^{[n/2]} dE_j=3\,dU,$$

the first two identities in (3.2).

Finally, differentiate the above equations with respect to t and set $t=0$ to obtain:

$$\begin{aligned}
V\sum_{i=1}^{3}\lambda_i'\,d\lambda_i+V\sum_{i=1}^{3} d(\lambda_i')+\Big(\sum_{i=1}^{3}\lambda_i'\Big)\,dV &= \Big(\sum_{j=0}^{[n/2]}\Big(j-\frac{n}{3}\Big)^2 O_j\Big)\,dt+\sum_{j=0}^{[n/2]}\Big(j-\frac{n}{3}\Big)\,dO_j,\\
U\sum_{i=1}^{3}\lambda_i'\,d\lambda_i-U\sum_{i=1}^{3} d(\lambda_i')+\Big(\sum_{i=1}^{3}\lambda_i'\Big)\,dU &= \Big(\sum_{j=0}^{[n/2]}\Big(j-\frac{n}{3}\Big)^2 E_j\Big)\,dt-\sum_{j=0}^{[n/2]}\Big(j-\frac{n}{3}\Big)\,dE_j.
\end{aligned}$$

Once again, the second and the third sums on the left hand sides vanish, due to (3.3). Divide the first equation by V, the second by U, and subtract one from another, using (3.5):

$$V^{-1}\sum_{j=0}^{[n/2]}\left(j-\frac{n}{3}\right)dO_j+U^{-1}\sum_{j=0}^{[n/2]}\left(j-\frac{n}{3}\right)dE_j=0.$$

Hence

$$V^{-1}\sum_{j=0}^{[n/2]}j\,dO_j+U^{-1}\sum_{j=0}^{[n/2]}j\,dE_j=\frac{n}{3}\Big(V^{-1}\sum_{j=0}^{[n/2]}dO_j+U^{-1}\sum_{j=0}^{[n/2]}dE_j\Big).$$

Due to the first two identities in (3.2), the right-hand-side equals $n\,(O_n^{-1}\,dO_n+E_n^{-1}\,dE_n)$. This yields the third identity in (3.2). Theorem 3 is proved. $\Box$

Remark 3.3. a) Let $\mathcal{E}$ be the Euler vector field that generates the scaling. Then

$$\mathcal{E}(O_j)=j\,O_j,\qquad \mathcal{E}(E_j)=-j\,E_j.$$

If one evaluates the differentials in the identities (3.2) on $\mathcal{E}$, one obtains the last three identities in (3.1). This is a check that (3.1) and (3.2) are consistent with each other.

b) Equivalently, (3.2) can be rewritten as

$$\begin{aligned}
3\,dO_n &= 2\,E_n^{-\frac{1}{3}}\,O_n^{\frac{1}{3}}\Big(\textstyle\sum_{j=1}^{[n/2]}dO_j\Big)-E_n^{-\frac{2}{3}}\,O_n^{\frac{2}{3}}\Big(\textstyle\sum_{j=1}^{[n/2]}dE_j\Big),\\
3\,dE_n &= 2\,E_n^{\frac{1}{3}}\,O_n^{-\frac{1}{3}}\Big(\textstyle\sum_{j=1}^{[n/2]}dE_j\Big)-E_n^{\frac{2}{3}}\,O_n^{-\frac{2}{3}}\Big(\textstyle\sum_{j=1}^{[n/2]}dO_j\Big),\\
0 &= O_n^{\frac{1}{3}}\Big(3\sum_{j=1}^{[n/2]}j\;dE_j-n\sum_{j=1}^{[n/2]}dE_j\Big)+E_n^{\frac{1}{3}}\Big(3\sum_{j=1}^{[n/2]}j\,dO_j-n\sum_{j=1}^{[n/2]}dO_j\Big).
\end{aligned}$$

c) The identities (3.1) and (3.2) are satisfied in a larger subspace than $\mathcal{C}_n$, consisting of twisted polygons whose monodromy has *equal eigenvalues.* This subspace has codimension 2 in $\mathcal{P}_n$.

d) In both cases, n odd and n even, the kernel of the Poisson map X (2.9) (spanned by the differentials of the Casimir functions) has zero intersection with the subspace of $T^*\mathcal{P}_n$ spanned by the relations 3.2.

3.3 Reducing the proof to a one-point computation

For ease of exposition, we will give our proof only in the odd case, and we set $n\geq 7$ odd. Modulo changing some of the indices, the even case is similar. We will explain everything in terms of the odd case and, at the end of this section, briefly explain what happens in the even case.

Let $\mathcal{M}$ denote the algebra generated by the monodromy invariants. In the Section 4 we make the following calculations.

1. There exist elements $F_1, ..., F_{n-2} \in \mathcal{M}$ and a point $p \in \mathcal{C}_n$ such that the differentials $dF_1, ..., dF_{n-2}$ are linearly independent at p. Therefore, $dF_1, ..., dF_{n-2}$ are linearly independent at almost all $q \in \mathcal{C}_n$.

2. There exist elements $G_1, ..., G_{n-4} \in \mathcal{M}$ and a point $p \in \mathcal{C}_n$ such that the differentials $dG_1|_{T_p\mathcal{C}_n}, ..., dG_{n-4}|_{T_p\mathcal{C}_n}$ are linearly independent. Therefore, $dG_1|_{T_q\mathcal{C}_n}, ..., dG_{n-4}|_{T_q\mathcal{C}_n}$ are linearly independent at almost all $q \in \mathcal{C}_n$.

In Calculation 1, we are computing the differentials on the ambient space $\mathcal{P}_n$ but evaluating them at a point of $\mathcal{C}_n$. In Calculation 2, we are computing the differentials on the ambient space, evaluating them at a point of $\mathcal{C}_n$, *and* restricting the resulting linear functionals to the tangent space of $\mathcal{C}_n$. In both calculations, we are actually evaluating at points in $\mathbb{C}_n^0$. In each case, what allows us to make a conclusion about generic points is that the monodromy invariants are algebraic.

Calculation 2 combines with Theorem 3 to show that there are exactly $n-4$ algebraically independent monodromy invariants, when restricted to $\mathcal{C}_n$. Hence, the generic common level set of the monodromy invariants O_i, E_i, restricted to $\mathcal{C}_n$, has dimension $n-4$.

Next, we wish to prove that these level sets have locally free action of the abelian group $\mathbb{R}^d$ (or $\mathbb{C}^d$ in the complex case). For $F \in \mathcal{M}$, the Hamiltonian vector field X_F is tangent to $\mathcal{C}_n$, by Proposition 3.1, and also tangent to the common level set of functions in $\mathcal{M}$. Finally, by Theorem 2, the Hamiltonian vector fields all commute with each other (i.e., define an action of the Abelian Lie algebra). The following lemma finishes our proof.

Lemma 3.4. *The Hamiltonian vector fields of the monodromy invariants generically span the monodromy level sets on $\mathcal{C}_n$.*

Proof. Let $\wedge^1\mathcal{P}_n$ denote the space of 1-forms on $\mathcal{P}_n$. Let $\mathcal{X}$ denote the space of vector fields on $\mathcal{C}_n$. Let $d\mathcal{M} \subset \wedge^1\mathcal{P}_n$ denote the image of $\mathcal{M}$ under the d-operator. Calculation 1 shows that the vector space $d\mathcal{M}$ generically has dimension $n-2$ when evaluated at points of $\mathcal{C}^n$. At the same time, we have the Poisson map $X : d\mathcal{M} \to \mathcal{X}$, given by

$$X(dF) = X_F,$$

see (2.9). In the odd case, the map X has 2 dimensional kernel, see Remark 3.3 d). Hence, X has $n-4$ dimensional image, as desired. $\Box$

Now we explain explicitly how the results above give us the quasi-periodic motion in the case of closed convex polygons. We know from the work in [17] that the monodromy level sets on $\mathcal{C}_n^0$ are compact. By Sard's Theorem, and by the calculations above, almost every level set is a smooth compact manifold of dimension $m = n - 4$. By Sard's Theorem again, and by the dimension count above, almost every level set L possesses a framing by Hamiltonian vector fields. That is, there are m Hamiltonian vector fields on L which are linearly independent at each point and which define commuting flows. These vector fields define local coordinate charts from L into $\mathbb{R}^m$, such that the overlap functions are translations. Therefore L is a finite union of affine m-dimensional tori. The whole structure is invariant under the pentagram map, and so the pentagram map is a translation of L relative to the affine structure on L. This is the quasi-periodic motion. Even more explicitly, some finite power of the pentagram map preserves each connected component of L and is a constant shift on each connected component.

The Even Case: In the even case, we have the following calculations:

1. There exist elements $F_1, ..., F_{n-1} \in \mathcal{M}$ and a point $p \in \mathcal{C}_n$ such that the differentials $dF, ..., dF_{n-1}$ are linearly independent at p. Therefore, $dF, ..., dF_{n-1}$ are linearly independent at almost all $q \in \mathcal{C}_n$.

2. There exists elements $G_1, ..., G_{n-3} \in \mathcal{M}$ and a point $p \in \mathcal{C}_n$ such that the differentials $dG_1|_{T_p\mathcal{C}_n}, ..., dG_{n-3}|_{T_p\mathcal{C}_n}$ are linearly independent. Therefore, $dG_1|_{T_p\mathcal{C}_n}, ..., dG_{n-3}|_{T_p\mathcal{C}_n}$ are linearly independent at almost all $q \in \mathcal{C}_n$.

In this case, the common level sets generically have dimension $n-5$ and, again, the Hamiltonian vector fields generically span these level sets. The situation is summarized in the following table.

	Invariants	Casimirs	Level sets / Hamiltonian fields
n odd	$n+1$	2	$d = n - 4$
n even	$n+2$	4	$d = n - 5$

4 The linear independence calculation

4.1 Overview

For any given (smallish) value of n, one can make the calculations directly, at a random point, and see that it works. The difficulty is that we need to make one calculation for each n. One might say that the idea behind our calculations is tropicalization. The monodromy invariants and their gradients are polynomials with an enormous number of terms. We only need to make our calculation at one point, but we will consider a 1-parameter family of points, depending on a parameter u. As $u \to 0$, the different variables tend to 0 at different rates. This sets up a kind of hierarchy (or filtration) on the the monomials comprising the polynomials of interest to us, and only the "heftiest" monomials in this hierarchy matter. This reduces the whole problem to a combinatorial exercise.

We take $n \geq 7$ odd. Let $m = (n-1)/2$. Recall that $\mathcal{M}$ is spanned by

$$O_1, ..., O_m, O_n, E_1, ..., E_m, E_n.$$

We define

$$A_{k,\pm} = O_k \pm E_k. \tag{4.1}$$

For the first calculation, we use the monodromy invariants

$$A_{3,+}, ..., A_{m,+}, A_{n,+}, A_{2,-}, ..., A_{m,-}, A_{n,-}. \tag{4.2}$$

For the second calculation, we use the monodromy invariants

$$A_{3,-}, ..., A_{m,-}, A_{3,+}, ..., A_{m,+}, A_{n,+}. \tag{4.3}$$

The point we use is of the form $p = P^u$, where P^u is an n-gon having corner invariants

$$a, b, c, d, u^1, u^2, u^3, u^4, ..., u^4, u^3, u^2, u^1, d, c, b, a, \tag{4.4}$$

Here

- $a = O(u^{(n-4)(n-3)/2})$.
- $b = 1 + O(u)$
- $c = 1 + O(u)$.
- $d = 1 + O(u)$.

We will show that the results hold when u is sufficiently small. Here we are using the big O notation, so that $O(u)$ represents an expression that is at most Cu in size, for a constant C that does not depend on u.

We will construct P^u in the next section. Our first calculation requires only the information presented above. The second calculation, which is almost exactly the same as the first calculation, requires some auxilliary justification. In order to justify the calculation we make, we need to make some estimates on the tangent space T_{P^u} to $\mathcal{C}_n$ at P^u. We will also do this in the next section.

In Section 4.2 and Section 4.3 we will explain our two calculations in general terms. In Section 4.4 we will define the concept of the *heft* of a monomial, and we will use this concept to put a kind of ordering on the monomials that appear in the monodromy invariants of interest to us. Following the analysis of the heft, we complete the details of our calculations.

4.2 The first calculation in broad terms

Let ∇ denote the gradient on $\mathbb{R}^{2n}$. Let $\widetilde{\nabla}$ denote the *normalized gradient*:

$$\widetilde{\nabla} F = \lambda^{-1} \nabla F; \qquad \lambda = \|\nabla F\|_\infty. \tag{4.5}$$

In practice, we never end up dividing by zero. So, the largest entry in $\widetilde{\nabla} F$ is ± 1.

If F is a monodromy invariant, the coordinates of $\widetilde{\nabla} F(P^u)$ have a power series in u. We define ΨF to be the result of setting all terms except the constant term to 0. We call ΨF the *asymptotic gradient*. Thus, if

$$\widetilde{\nabla} F(P^u) = (1 - u^3 \cdots, -1 + u \cdots, u^2 \cdots, ...)$$

then $\Psi F = (1, -1, 0, ...)$.

Lemma 4.1. *Suppose that $\Psi F_1, ..., \Psi F_k$ are linearly independent. Then likewise $\nabla F_1, ..., \nabla F_k$ are linearly independent at P^u for u sufficiently small. Equivalently, the same goes for $dF_1, ..., dF_k$.*

Proof. Since $\Psi F_1, ..., \Psi F_k$ are independent there is some $\epsilon > 0$ such that a sum of the form

$$\left| \sum b_j \Psi F_j \right| < \epsilon; \qquad \max |b_j| = 1$$

is impossible.

Suppose for the sake of contradiction that the gradients are linearly dependent at P^u for all sufficiently small u. Then the normalized gradients are also linearly dependent at P^u for all sufficiently small u. We may write

$$\sum b_j \widetilde{\nabla} F_j \cdot e_i = 0; \qquad \max |b_j| = 1. \tag{4.6}$$

for the standard basis vectors $e_1, ..., e_{2n}$. The coefficients b_j possibly depend on u, but this doesn't bother us.

We have the bound

$$\left| b_j \widetilde{\nabla} F_j - b_j \Psi F_j \right| = O(u). \tag{4.7}$$

Hence

$$\sum_j b_j \Psi F_j \cdot e_i = O(u) \tag{4.8}$$

for all basis vectors e_i. Therefore, we can take u small enough so that

$$\left| \sum b_j \Psi F_j \right| < \epsilon; \qquad \max |b_j| = 1,$$

in contradiction to what we said at the beginning of the proof. $\square$

Remark 4.2. The idea of the proof of the previous lemma is simple: given a matrix, algebraically dependent on a parameter u, the rank of the matrix is greatest in a Zariski open subset of the parameter space and can only drop for special values of the parameter (zero, in our case).

We form a matrix M_+ whose rows are ΨF, where F is each of the A_+ invariants. We similarly form the matrix M_-.

Lemma 4.3. *Each row of M_+ is orthogonal to each row of M_-.*

Proof. Consider the map $T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ which simply reverses the coordinates. We have $E_k \circ T = O_k$ for all k and moreover $T(P^u) = P^u$. Letting dT be the differential of T, we have

$$dT(\nabla A_{k,\pm}) = \pm \nabla A_{k,\pm}. \tag{4.9}$$

Our lemma follows immediately from this equation, and from the fact that T is an isometric involution. $\square$

In view of Lemmas 4.1 and Lemma 4.3, our first calculation follows from the statements that M_+ and M_- have full rank.

For the matrix M_+, we consider the minor m_+ consisting of columns

$$1, 6, 7, 10, 11, 14, 15, 18, 19, ...$$

until we have a square matrix. We will prove below that m_+ has the following form (shown in the case $n = 13$.)

$$\begin{bmatrix} 0 & \pm 1 & \pm 1 & \pm 1 & \pm 1 \\ 0 & 0 & \pm 1 & \pm 1 & \pm 1 \\ 0 & 0 & 0 & \pm 1 & \pm 1 \\ 0 & 0 & 0 & 0 & \pm 1 \\ \pm 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.10}$$

This matrix always has full rank. Hence M_+ has full rank.

For the matrix M_- we consider the minor m_- consisting of columns

$$1, \mathbf{3}, 6, 7, 10, 11, 14, 15, 18, 19, ...$$

The only difference here is that column 3 is inserted. The resulting matrix has exactly the same structure as just described. Hence M_- has full rank.

4.3 The second calculation in broad terms

Let $T = T_{P^u}(\mathcal{C}_n)$ denote the tangent space to $\mathcal{C}_n$ at P^u. Let $\{e_k\}$ denote the standard basis for $\mathbb{R}^{2n}$. Let $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n-8}$ denote the map which strips off the first and last 4 coordinates.

Define

$$\nabla_8 = \pi \circ \nabla. \tag{4.11}$$

We define the normalized version $\widetilde{\nabla}_8$ exactly as we defined $\widetilde{\nabla}$. Likewise we define $\Psi_8 G$ for any monodromy function G.

For a collection of vectors $v_5, \ldots, v_{2n-4}$ to be specified in the next lemma, we form the vector

$$\Upsilon_8 G = (D_{v_5} G, ..., D_{v_{2n-4}} G) \tag{4.12}$$

made from the directional derivatives of G along these vectors. Note, by way of analogy, that

$$\nabla_8 G = (D_{e_5} G, ..., D_{e_{2n-4}} G). \tag{4.13}$$

We define the normalized version $\widetilde{\Upsilon}_8$ exactly as we defined $\widetilde{\nabla}_8$.

In the next section, we will establish the following result.

Lemma 4.4 (Justification)**.** *There is a basic $v_5, ..., v_{2n-4}$ for $T_{P^u}(\mathcal{C}_n)$ such that $\pi(v_k) = e_k$ for all k and*

$$\widetilde{\Upsilon}_8 G - \widetilde{\nabla}_8 G = O(u).$$

Corollary 4.5. *Suppose that $\Psi_8 G_1, ..., \Psi_8 G_k$ are linearly independent. Then the restrictions of $dG_1, ..., dG_k$ to $T_{P^u}(\mathcal{C}_n)$ are linearly independent for u sufficiently small.*

Proof. Given our basis, Ψ_8 represents the constant term approximation of both $\widetilde{\Upsilon}_8$ and $\widetilde{\nabla}_8$. So, the same proof as in Lemma 4.1 shows that the vectors $\widetilde{\Upsilon}_8 G_j$ are linearly independent. This is equivalent to the conclusion of our corollary. □

Using the invariants listed in (4.3), we form the matrices M_+ and M_- just as above, using Ψ_8 in place of Ψ. Lemma 4.3 again shows that each row of M_+ is orthogonal to each row of M_-. Hence, we can finish the second calculation by showing that both M_+ and M_- have full rank.

For M_- we create a square minor m_- using the columns

$$2, 3, 6, 7, 10, 11, 14, 15, ...$$

Again, we continue until we have a square. It turns out that m_- has the form

$$\begin{bmatrix} \pm 1 & \pm 1 & \pm 1 & \pm 1 \\ 0 & \pm 1 & \pm 1 & \pm 1 \\ 0 & 0 & \pm 1 & \pm 1 \\ 0 & 0 & 0 & \pm 1 \end{bmatrix} \tag{4.14}$$

Hence M_- has full rank.

For M_+ we create a square minor m_+ using the same columns, but extending out one further (on account of the larger matrix size.) It turns out that m_+ has the same form as m_-. Hence M_+ has full rank.

4.4 The heft

Any monomial in the variables $x_1, ..., x_{2n}$, when evaluated at P^u, has a power series expansion in u. We define the *heft* of the monomial to be the smallest exponent that appears in this series. For instance, the heft of $u^2 + u^3$ is 2. We define the heft of a polynomial to be the minimum heft of the monomials that comprise it. Given a polynomial F, we define heft of ∇F to be the minimum heft, taken over all partial derivatives $\partial F/\partial x_j$.

We call a monomial term of $\partial F/\partial x_k$ *hefty* if its heft realizes the heft of ∇F. We define $H_k F$ to be the sum of the hefty monomials in $\partial F/\partial x_k$. Each monomial occurs with sign ± 1. We define $|H_k F| \in \mathbb{Z}$ to be the sum of the coefficients of the hefty terms in $H_k F$. We say that F is *good* if $|H_k F| \neq 0$ for at least one index k. If F is good then

$$\Psi F = C(|H_1 F|, ..., |H_{2n} F|), \tag{4.15}$$

for some nonzero constant C that depends on F. It turns out that $C = \pm 1$ in all cases.

We say that F is *great* if F is *good* and $|H_k F| \neq 0$ for at least one index k which is not amongst the first or last 4 indices. When F is great, not only does equation (4.15) hold, but we also have

$$\Psi_8 F = C(|H_5 F|, ..., |H_{2n-4} F|), \tag{4.16}$$

Lemma 4.6. *Let $k = 2, 3$. Then $A_{k,\pm}$ is great and $\nabla A_{k,\pm}$ has heft* 0.

Proof. Let $F = A_{k,\pm}$. Consider the case $k = 2$. The argument turns out to be the same in the $(+)$ and $(-)$ cases. We say that an *outer variable* is one of the first or last 4 variables in $\mathbb{R}^{2n}$, and we call the remaining variables *inner*. Since $x_2 x_6$ and $x_6 x_{2n-2}$ are both terms of F, we see that

$$H_6 F = x_2 + x_{2n-2} + ...$$

In particular, ∇F has heft 0. Any term in $H_6 F$ involves only the outer 8 variables, and a short case-by-case analysis shows that there are no other possibilities besides the two terms listed above. Hence $|H_6 F| = 2$. This shows that F is great.

Now consider the case $k = 3$. The argument turns out to be the same in the $(+)$ and $(-)$ cases. Since $x_2 x_6 x_{2n-2}$ is a term of F we see that

$$H_6 F = x_2 x_{2n-2} + ...$$

The rest of the proof is as in the previous case, with the only difference being that $|H_6 F| = 1$ in this case. $\square$

From now on, we fix some $F = A_{k,\pm}$ with $3 < k \leq m$. Let $\alpha_1, \alpha_2, ...$ be the terms of the following sequence

$$0, 0, 0, 2, 3, 6, 7, 10, 11, 14, 15, ... \tag{4.17}$$

Lemma 4.7. *∇F has heft at most $\alpha_1 + ... + \alpha_k$.*

Proof. We describe a specific term in ∇F having heft $\alpha_1 + ... + \alpha_k$. We make a monomial using the indices

$$2, 2n-2, 6, 2n-6, 10, 2n-10, ... \tag{4.18}$$

stopping when we have used $k-1$ numbers. The monomial corresponding to these indices has heft

$$0 + 0 + 0 + 2 + 3 + 6 + 7 + 10 + 11 ... = \alpha_1 + ... + \alpha_k.$$

Thinking of our indices cyclically, we see that our integers lie in an interval of length $4k-7$. So, between the largest index in (4.18) that is less than n and the smallest index greater than n there is an unoccupied stretch of at least 9 integers. The point here is that

$$9 + (4k-7) \leq 9 + 4m - 7 = 9 + 2(n-1) - 7 = 2n.$$

Given that the unoccupied stretch has at least 9 consecutive integers, there is at least 1 (and in fact at least 2) even indices j such that the monomial

$$m = \pm x_j x_2 x_{2n-2} x_6 x_{2n-6} x_{10} ...$$

is a term of F. But then $\partial m / \partial x_j$ has heft $\alpha_1 + ... + \alpha_k$. $\square$

We mention that (4.18) is one of two obvious ways to make a term of heft $\alpha_1 + ... + \alpha_k$. The other way is to take the *mirror image*, namely

$$2n-1, 3, 2n-5, 7, 2n-9, 11, ... \tag{4.19}$$

Lemma 4.8. *If $\partial F / \partial x_j$ has a hefty term, then j is an inner variable.*

Proof. For ease of exposition, we will consider the case when j is one of the first 4 variables. Let $(i_1, ..., i_d)$ be the sequence of indices which appear in a term m' of $\partial F / \partial x_j$. The corresponding term m in F has index sequence $(j, i_1, ..., i_d)$, where these numbers are not necessarily written in order. We know that at least one of the indices, say a, is an inner variable. By construction $\partial m / \partial x_a$ has smaller heft than m'. Hence $\partial F / \partial x_j$ has no hefty terms. Hence j is an inner variable. $\square$

Lemma 4.9. *Suppose the monomial $\pm x_{i_1} ... x_{i_a}$ is a hefty term of $\partial F / \partial x_j$. Then $a = k-1$ and $i_1, ..., i_{k-1}$ are either as in (4.18) or as in equation (4.19).*

Proof. We have to play the following game: We have a grid of $2n$ dots. The first and last dot are labelled $(n-3)(n-4)/2$. The remaining 6 outer dots are labelled 0. The inner dots are labelled $1, 2, 3, ..., 3, 2, 1$. Say that a *block* is a collection of d dots in a row for $d = 1, 2, 3$. We must pick out either k or $k-1$ blocks in such a way that the total sum of the corresponding dots is as small as possible, and the (cyclically reckoned) spacing between consecutive blocks is at least 4. That is, at least 3 "unoccupied dots" must appear between every two blocks.

It is easy to see that one should use $k-1$ blocks, all having size 1. Moreover, half (or half minus one) of the blocks should crowd as much as possible to the left and half minus one (or half) of the blocks should crowd as much as possible to the right. A short case by case analysis of the placement of the first and last blocks shows that one must have precisely the choices made in (4.18) and (4.19). □

Corollary 4.10. *Let $F = A_{k,\pm}$, with $k \geq 2$. Then F is good. If $k \leq m$ then F is great, and the heft of ∇F is $\alpha_1 + ... + \alpha_k$.*

Proof. In light of the results above, the only nontrivial result is that F is great when $3 < k \leq m$. The construction in connection with (4.18) produces a hefty term of $\partial F/\partial x_j$ for some inner index j. The key observation is that, for parity considerations, the mirror term corresponding to (4.19) is not a term of $\partial F/\partial x_j$. In one case j must be odd and in the other case j must be even. Hence, there is only 1 hefty term in $\partial F/\partial x_j$. □

As regards the heft, we have done everything but analyze the Casimirs. Recall that

$$O_n = x_1 x_3 ... x_{2n-1}; \qquad E_n = x_2 x_4 ... x_{2n}. \tag{4.20}$$

Lemma 4.11. *$A_{n,\pm}$ is good and $\nabla A_{n,+}$ has heft*

$$\frac{(n-3)(n-4)}{2}.$$

Moreover,

$$\Psi A_{n,\pm} = (1, 0, ..., 0, \pm 1).$$

Proof. Let F be either of these functions. Clearly the hefty terms of ∇F are the ones which omit the first and last variables. From here, this lemma is an exercise in arithmetic. □

A similar argument proves

Lemma 4.12. *$A_{n,\pm}$ is good and $\nabla_8 A_{n,+}$ has heft $(n-4)^2$. Moreover,*

$$\Psi A_{n,\pm} = (0, ..., 0, \pm 1, 1, 0, ..., 0),$$

with the 2 middle indices being nonzero.

4.5 Completion of the first calculation

To complete the first calculation, we need to analyze the matrix made from the asymptotic gradients $\Psi F_1, \Psi F_2, \ldots$. We deal with the first two in a calculational way.

Lemma 4.13. $\Psi A_{2,\pm} = (0,0,\pm 1,0,0,1,\pm 1,...,1,\pm 1,0,0,1,0,0)$.

Proof. Let $F = A_{2,\pm}$. We know that F has heft 0, so the hefty terms in ∇F are monomials which only involve the outer indices. Hence, when $8 \leq j \leq 2n-8$ the result only depends on the parity of j and neither the value of j nor the value of n. For the remaining indices, the result is also independent of n. Thus, a calculation in the case (say) $n = 13$ is general enough to rigorously establish the whole pattern. This is what we did. $\square$

Lemma 4.14. $\Psi A_{3,\pm} = (0,0,0,0,0,1,\pm 1,...,1,\pm 1,0,0,1,0,0)$.

Proof. Same method as the previous result. $\square$

Now we are ready to analyze the minors m_+ and m_- described in connection with the first calculation. When we say that a certain part of one of these matrices has the form given by (4.10), we understand that (4.10) gives a smallish member of an infinite family of matrices, all having the same general type. So, we mean to take the corresponding member of this family which has the correct size.

We say that a given row or column of one of our matrices *checks* if it matches the form given by (4.10). We will give the argument for m_+. The case for m_- is essentially the same.

Lemma 4.15. *The first column of m_+ checks.*

Proof. By Lemma 4.11, the first coordinate of $\Psi A_{n,+}$ is ± 1. By Lemmas 4.8, 4.13, and 4.14., we have $\Psi A_{k,+}$ is zero for $k < n$. This is equivalent to the lemma. $\square$

Lemma 4.16. *The first row of m_+ checks and the last row of m_+ checks.*

Proof. The first statement follows immediately from Lemma 4.14. The second statement follows immediately from Lemma 4.11. $\square$

Now we finish the proof. Consider the ith row of m_+. Let $k = i + 2$. In light of the trivial cases taken care of above, we can assume that $3 < k \leq m$. Let $F = A_{k,+}$. As we discussed in the proof of Corollary 4.10, each polynomial $\partial A/\partial F_j$ has either 0 or 1 hefty terms.

Assume that j is even. Let $J \subset \{1, ..., 2n\}$ be the unoccupied stretch from Lemma 4.7. Let $J' \subset J$ denote the smaller set obtained by removing the first and last 3 members from J. It follows from the construction in Lemma 4.7 that $\partial F/\partial j$ has a hefty term if and only if $j \in J'$. Thus the jth entry of the kth row is ± 1 if and only if $j \in J'$. Similar considerations hold when j is odd. It is an exercise to show that the conditions we have given translate precisely into the form given in (4.10). Hence m_+ checks.

Remark 4.17. One can approach the proof differently. When we move from row k to row $k+2$ the corresponding interval $J' = (a, b)$ changes to the new interval $J' = (a+4, b-4)$. From this fact, and from our choice of minors, it follows easily that row k checks if and only if row $k+2$ checks. At the same time, when n is replaced by $n+2$, the interval $J' = (a, b)$ changes to $J' = (a, b+4)$. This translates into the statement that row k checks for n if and only if row k checks for $n+2$. All this reduces the whole problem to a computer calculation of the first few cases. We did the calculation up to the case $n = 13$ and this suffices.

4.6 Completion of the second calculation

We make all the same definitions and conventions for the second calculation, using the matrix (family) in (4.14) in place of the matrix (family) in (4.10). The argument for the second calculation is really just the same as the argument for the first calculation. Essentially, we just ignore the outer 8 coordinates and see what we get. What makes this work is that all the functions except $A_{n,\pm}$ are great – the inner indices determine the heft. To handle the last row of m_+, which involves the Casimir $A_{n,+}$, we use Lemma 4.12 in place of Lemma 4.11.

It remains to establish the Justification Lemma 4.4. It is convenient to define

$$\delta = \frac{(n-4)(n-5)}{2}. \tag{4.21}$$

We also mention several other pieces of notation and terminology. When we line up the indices $5, ..., 2n-4$, there are 2 *middle indices*. When $n = 7$ the middle indices of $5, 6, 7, 8, 9, 10$ are 7 and 8. Let $\pi^\perp$ denote the projection from $\mathbb{R}^{2n}$ onto $\mathbb{R}^8$ obtained by stringing out the first and last 4 coordinates.

Lemma 4.18 (Tangent Estimate)**.** *The following properties of $\pi^\perp(v_j)$ hold:*

- *All coordinates are $O(1)$.*
- *Coordinates 3 and 6 are $O(u)$.*
- *Except when j is one of the middle two indices, coordinates 1 and 8 are $O(u^{\delta+1})$.*
- *When j is the first middle index, coordinate 1 is $u^\delta + O(u^{\delta+1})$ and coordinate 8 is $O(u^{\delta+1})$.*
- *When j is the second middle index, coordinate 8 is $u^\delta + O(u^{\delta+1})$ and coordinate 1 is $O(u^{\delta+1})$.*

Proof. We prove this in the next section. $\square$

Lemma 4.19. *The Justification Lemma holds for $F = A_{n,+}$.*

Proof. A direct calculation shows that, up to $O(u^{\delta+1})$,

$$\widetilde{\nabla} F = (1, 0, ..., 0, u^{\delta}, u^{\delta}, 0, ..., 0, 1) \tag{4.22}$$

Hence

$$\widetilde{\nabla}_8 F = (0, ..., 0, 1, 1, 0, ...0) + O(u). \tag{4.23}$$

Let Z be the first coordinate of ∇F. If j is not a middle index, we have

$$D_{v_j} F = \nabla F \cdot v_j = Z \times O(u^{\delta+1}). \tag{4.24}$$

This estimate comes from the Tangent Estimate Lemma 4.18.

If j is the first middle index, then

$$D_{v_j} F = \nabla F \cdot v_j = Z \times 2O(\delta). \tag{4.25}$$

The first contribution comes from coordinate 1, and is justified by the Tangent Estimate Lemma, and the second contribution comes from coordinate j.

The above calculations show that

$$\widetilde{\Upsilon}_8 F = (0, ..., 0, 1, 1, 0, ...0) + O(u). \tag{4.26}$$

Hence $\widetilde{\nabla}_8 F = \widetilde{\Upsilon}_8 F + O(u)$. $\square$

Now suppose that F is one of the relevant monodromy invariants, but not the Casimir.
Our analysis establishes

Lemma 4.20. *Both $\pi^{\perp}(\widetilde{\nabla} F)$ and $\pi^{\perp}(\nabla F)$ have the following properties.*

1. *All coordinates are at most $1 + O(u)$ in size.*
2. *All coordinates except coordinates 3 and 6 are $O(u)$.*

Proof. This is immediate from our analysis of the heft of ∇F. $\square$

Lemma 4.21. *One has*

$$\widetilde{\nabla}_8 F \cdot e_j = \widetilde{\nabla} F \cdot v_j + O(u).$$

Proof. Combining the Tangent Estimate Lemma with Lemma 4.20, we see that

$$\pi^{\perp}(\widetilde{\nabla} F)\cdot\pi^{\perp}(v_j)=O(u).$$

Hence

$$\widetilde{\nabla} F\cdot v_j=\pi\circ\widetilde{\nabla} F\cdot e_j+O(u). \tag{4.27}$$

From Property 1 above, we see that

$$\|\nabla_8 F\|_\infty=\|\nabla F\|_\infty+O(u).$$

Therefore

$$\widetilde{\nabla}_8 F=\pi\circ\widetilde{\nabla} F+O(u). \tag{4.28}$$

Combining equations (4.27) and (4.28), we get the result of the lemma. $\square$

Lemma 4.22. *Setting* $\lambda=\|\nabla F\|_\infty$, *we have*

$$\lambda^{-1}(\Upsilon_8 F)_j=(\widetilde{\Upsilon}_8 F)_j+O(u).$$

Here $(X)_j$ *is the* j*th coordinate of* X.

Proof. Combining the Tangent Estimate Lemma 4.18 with Lemma 4.20, we have

$$\pi^{\perp}\circ\nabla F\cdot\pi^{\perp}(v_j)=O(u).$$

Therefore

$$\|\Upsilon_8 F\|_\infty=\|\nabla_8 F\|_\infty+O(u).$$

Combining this with equation (4.28), we have

$$\|\Upsilon_8 F\|_\infty=\|\nabla F\|_\infty+O(u).$$

Our lemma follows immediately. $\square$

By definition, we have

$$\widetilde{\nabla} F\cdot v_j=\lambda^{-1}\nabla F\cdot v_j=\lambda^{-1}(\Upsilon_8 F)_j; \qquad \lambda=\|\nabla F\|_\infty. \tag{4.29}$$

Combining this last equation with our two lemmas, we have

$$(\widetilde{\nabla}_8 F)_j=\widetilde{\nabla}_8 F\cdot e_j=(\widetilde{\Upsilon}_8 F)_j+O(u). \tag{4.30}$$

This holds for all j. This completes the proof of the Justification Lemma.

5 The polygon and its tangent space

The goal of this section is to construct the polygon P^u and prove the Tangent Lemma, which estimates the tangent space $T_{P^u}(C)$. We will begin by repackaging some of the material worked out in [19]. The results here are self-contained, though our main formula relies on the work done in [19]. In order to remain consistent with the formulas in [19], we will use a slightly different labelling convention for polygons.

5.1 Polygonal rays

We say that a *polygonal ray* is an infinite list of points $P_{-7}, P_{-3}, P_1, P_5, ...$ in the projective plane. We normalize so that (in homogeneous coordinates)

$$P_{-7} = (0,0,1), \qquad P_{-3} = (1,0,1), \qquad P_1 = (1,1,1), \qquad P_5 = (0,1,1). \tag{5.1}$$

The first 4 points are normalized to be the vertices of the positive unit square, starting at the origin, and going counterclockwise. Here we are interpreting these points in the usual affine patch $z = 1$. This polygonal ray defines lines:

$$L_{-5+k} = P_{-7+k}P_{-3+k}; \qquad k = 0,4,8,... \tag{5.2}$$

We denote by LL' the intersection $L \cap L'$. Similarly, PP' is the line containing P and P'.

The pairs of points and lines determine flags, as follows:

$$F_{-6+k} = (P_{-7+k}, L_{-5_k}), \qquad F_{-4+k} = (P_{-3+k}, L_{-5+k}), \qquad k = 0,4,8,12... \tag{5.3}$$

The corner invariants were defined in Section 2.2. In this section we relate the definition there to our labelling convention here. We define

$$\chi(F_{0+k}) = [P_{-7+k}, P_{-3+k}, L_{-5+k}L_{3+k}, L_{-5+k}L_{7+k}], \qquad k = 0,4,8,... \tag{5.4}$$

$$\chi(F_{2+k}) = [P_{9+k}, P_{5+k}, L_{7+k}L_{-1+k}, L_{7+k}L_{-5+k}], \qquad k = 0,4,8,... \tag{5.5}$$

Here we are using the inverse cross ratio, as in equation 2.3. Referring to the corner invariants, we have

$$x_k = \chi(F_{2k}); \qquad x_{k+1} = \chi(F_{2k+2}); \qquad k = 0,2,4,... \tag{5.6}$$

Remark 5.1. Notice that it is impossible to define $\chi(F_{-2})$ because we would need to know about a point P_{-11}, which we have not suppled. Likewise, it is impossible to define $\chi(F_{-4})$ because we would need to know about L_{-9}, which we have not supplied. Thus, the invariants $x_0, x_1, x_2, ...$ are well defined for our polygonal ray.

Cross product in vector form: Since we are going to be computing a lot of these cross ratios, we mention a formula that works quite well. We represent both points and lines in homogeneous coordinates, so that (a, b, c) represents the line corresponding to the equation $ax + bx + cz = 0$. We define $V * W$ to be the coordinate-wise product of V and W. Of course, $V * W$ is also a vector. Let $(\times)$ denote the cross product. We have

$$(\chi, \chi, \chi) = \frac{(A \times B) * (C \times D)}{(A \times C) * (B \times D)}. \tag{5.7}$$

Here χ is the inverse cross ratio of the points or lines represented by these vectors. It may happen that some coordinates in the denominator vanish. In this case, one needs to interpret this equation as a kind of limit of nearby perturbations. This formula works whenever A, B, C, D represent either collinear points or concurrent lines in the projective plane.

5.2 The reconstruction formulas

Referring to the definition of the monodromy invariants, we define O_a^b to be the sum over all odd admissible monomials in the variables $x_0, x_1, x_2, ...$ which do not involve any variables with indices $i \leq a$ or $i \geq b$. For instance

$$O_1^1 = 1, \qquad O_1^3 = 1, \qquad O_1^5 = 1 - x_3, \qquad O_1^7 = 1 - x_3 + x_3 x_4 x_5.$$

We also note that, when $a < 0$, the polynomial O_a^b is independent of the value of a. For this reason, when $a < 0$ we simply write O^b in place of O_a^b. The corresponding set S^b consists of admissible sequences, all of terms are less than b.

Given a list $(x_0, x_1, x_2, ...)$ we seek a polygonal ray which has this list as its corner invariants. Here is the formula.

$$P_{9+2k} = (O^{3+k} - O_1^{3+k} + x_0 x_1 O_3^{3+k}, O^{3+k}, O^{3+k} + x_0 x_1 O_3^{3+k}), \qquad k = 0, 2, 4, ... \tag{5.8}$$

We would also like a formula for reconstructing the lines of a polygonal ray. We start with the obvious:

$$L_{-5} = (0, 1, 0); \qquad L_{-1} = (-1, 0, 1); \qquad L_3 = (0, -1, 1). \tag{5.9}$$

For the remaining points, we define polynomials E_a^b exactly as we defined O_a^b except we interchange the uses of *even* and *odd*. Thus, for instance $E_2^6 = 1 - x^4$. Here is the formula.

$$L_{7+2k} = (E^{2+k} - E_0^{2+k}, E_0^{2+k} - x_0 E_2^{2+k}, -E^{2+k}), \qquad k = 0, 2, 4... \tag{5.10}$$

Remark 5.2. These formulas are equivalent to equations 19 and 20 in [19], but the normalization of the first 4 points is different, and the roles of points and lines have been switched. We got the above formulas by applying a suitable projective duality to the polygonal ray in [19].

We mention one important connection between our various reconstruction formulas. The following is an immediate consequence of Lemma 3.2 in [19]:

$$P_{5+k} \times P_{9+k} = -(x_1 x_3 x_5, ..., x_{k/2+1}) L_{7+k}, \qquad k = 0, 4, 8... \tag{5.11}$$

We close this section with a characterization of the moduli space of closed polygons within X. We do not need this result for our proofs, but it is nice to know.[3]

Lemma 5.3. *The invariant $x_1, ..., x_{2n}$ define a closed polygon if and only if O^{2n-5} and all its cyclic shifts vanish.*

Proof. We can think of a closed polygon as an n-periodic infinite ray. The periodicity implies that $P_{4n-7} = P_{-7} = (0, 0, 1)$. Since $4n - 7 = 2k + 9$ for $k = 2n - 8$, equation (5.8) tells us that $O^{2n-5} = 0$. Considering equation 5.10, we see that $E^{2n-6} = E_0^{2n-6} = 0$. But E_0^{2n-6} is a cyclic shift of O^{2n-5}. Hence, if P is closed then O^{2n-5} and all its cyclic shifts vanish.

Conversely, if O^{2n-5} and all its shifts vanish then $P_{4n-7} \in L_{-5}$ and $P_{-3} \in L_{4n-5}$. Likewise $P_{4n-3} \in L_{-1}$ and $P_1 \in L_{4n-1}$, and so on. This situation forces $P_{4n-3} = P_{-3}$. Shifting the indices, we see that $P_{4n+1} = P_1$, and so on. $\square$

Remark 5.4. Observe that O^{2n-5} involves exactly $2n - 7$ consecutive corner invariants. If the first $2n - 8$ are specified, then the next variable can be found by solving $O^{2n-5} = 0$. Thus, Lemma 5.3 gives an algorithmic way to find a closed n-gon whose first $2n - 8$ corner invariants are specified.

5.3 The polygon

We start with an infinite periodic list of variables which starts out

$$(u, u^2, ..., u^{n-4}, u^{n-4}, ..., u^2, u^1, ...) \tag{5.12}$$

and has period $2n - 8$. We let X_u denote the polygonal ray associated to this infinite list. Once u is sufficiently small, the first n points of X_u are well defined. We define P^u to be the n-gon made from the first n-points of X_u, and we take u small enough so that this definition makes sense.

The first $2n - 8$ corner invariants of P^u, which we now identify with $x_0, ..., x_{2n-9}$, are the ones listed in equation (5.12). However, when it comes time to compute $x_{2n-8}, ..., x_{2n-1}$, we do not use the relevant points of X_u but rather substitute in the corresponding point of P^u. Thus, the remaining 8 corner invariants change. We write the corner invariants of P^u as

$$a, b, c, d, u, u^2, u^3, ..., u^3, u^2, u, d', c', b', a'. \tag{5.13}$$

[3] One could give an alternative proof of Proposition 3.1 computing the Poisson bracket of the polynomials of Lemma 5.3 with the monodromy invariants.

It follows from symmetry that $e = e'$ for each $e \in \{a, b, c, d\}$. This symmetry here is that the first $2n - 8$ invariants determine P, and their palindromic nature forces P to be self-dual: the projective duality carries P to the dual polygon made from the lines extending the sides of P.

Lemma 5.5. *$e = 1 + O(u)$ for each $e \in \{b, c, d\}$.*

Proof. We set $P_{-11} = (X, Y, Z)$ and $L_{-13} = (U, V, W)$. We have

$$L_{-9} = (1, 0, 0) \times (X, Y, Z) = (-Y, X, Z). \tag{5.14}$$

Equations 5.8 and 5.10 tell us

$$(X, Y, Z) = (1, 0, 0) + O(u); \qquad (U, V, W) = (0, 1, -1) + O(u). \tag{5.15}$$

We compute

$$b = \chi(F_{-6}) = \chi(P_1, P_{-3}, L_{-1}L_{-9}, L_{-1}L_{-13}) = \frac{UX + WX + VY}{(U + W)(X - Y)}. \tag{5.16}$$

$$c = \chi(F_{-4}) = \chi(P_{-11}, P_{-7}, L_{-9}L_{-1}, L_{-9}L_3) = \frac{X - Y}{X - Z} \tag{5.17}$$

$$d = \chi(F_{-2}) = \chi(P_5, P_1, L_3L_{-5}, L_3L_{-9}) = d = \frac{X}{X + Y + Z}. \tag{5.18}$$

Our result is immediate from these formulas and from equation (5.15). $\square$

Lemma 5.6. *$a = u^s + O(u^{s+1})$, where $s = (n - 4)(n - 3)/2$.*

Proof. We have

$$a = \chi(F_{-8}) = \chi(P_{-15}, P_{-11}, L_{-13}L_{-5}, L_{-13}L_{-1}) = \chi(A, B, C, D). \tag{5.19}$$

We will estimate a by considering the middle coordinate of equation (5.7). Calculations similar to the ones above give

$$A = (0, 1, 1) + O(u), \quad B = (0, 1, 1) + O(u),$$

$$C = (1, 0, 0) + O(u), \quad D = (1, 1, 1) + O(u). \tag{5.20}$$

Hence

$$(A \times C)_2 = +1 + O(u); \quad (B \times D)_2 = +1 + O(u); \quad (C \times D)_2 = -1 + O(u). \tag{5.21}$$

Recall that

$$P_{-15} = P_{-15+4n}; \qquad P_{-11} = P_{-11+4n}. \tag{5.22}$$

According to equation (5.11), we have

$$A \times B = -(x_1 x_3, ...x_{2n-9}) L_{-13+2n} =$$
$$-u_2 u_4 ... u_3 u_1 L_{-13+2n} = -u^s L_{-13+4n}. \tag{5.23}$$

But

$$L_{-13+4n} = (0, 1, -1) + O(u). \tag{5.24}$$

Therefore

$$(A \times B)_2 = -u^s + O(u^{s+1}).$$

Looking at the signs in equation (5.21), we see that $a = u^s + O(u^{s+1})$. □

5.4 The tangent space

Recall that $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n-8}$ is the projection which strips off the outer 4 coordinates. Let $\pi^{\perp}$ be as in the Tangent Estimate Lemma 4.18. Recall that $\{v_k\}$ is the special basis of $T_P(C)$ such that $\pi(v_k) = e_k$ for $k = 5, ..., 2n-4$.

Lemma 5.7. *The following holds concerning the coordinates of $\pi^{\perp}(v_j)$:*

- *Coordinates $2, 4, 5, 7$ of $\pi^{\perp}(v_j)$ have size $O(1)$*
- *Coordinates $3, 6$ have size $O(u)$.*

Proof. As above, we will just consider coordinates $2, 3, 4$. The other cases follow from symmetry.

We refer to the quantities used in the proof of Lemma 5.5. Each of these quantities is a polynomial in the coordinates, depending only on n. Hence dX/dt, etc., are all of size at most $O(1)$. Moreover, the denominators on the right hand sides of equations (5.16), (5.17), and (5.18) are all $O(1)$ in size. Our first claim now follows from the product and quotient rules of differentiation.

For our second claim, we differentiate equation (5.17):

$$\frac{dc}{dt} = \frac{X'(Y-Z) - X(Y'-Z') + ZY' - YZ'}{(X+Z)^2} =^*$$
$$Y' - Z' + O(u) = \frac{d}{dt}(-x_0 x_1) O_3^{3+k}. \tag{5.25}$$

The starred equality comes from the fact that $(X, Y, Z) = (0, 1, 1) + O(u)$. The claim now follows from the fact that $x_0(0) = u$ and $x_1(0) = u^2$ and $(O_3^{3+k})'(0) = O(1)$. □

Lemma 5.8. *The following holds concerning the coordinates of $\pi^{\perp}(v_j)$:*

- *When j is not a middle index, coordinates 1 and 8 of are of size $O(u^{\delta+1})$.*
- *When j is the first middle index, coordinate 1 equals $u^{\delta}(1+O(u))$ and coordinate 8 is of size $O(u^{\delta+1})$.*
- *When j is the second middle index, coordinate 8 equals $u^{\delta}(1+O(u))$ and coordinate 1 is of size $O(u^{\delta+1})$.*

Proof. We will just deal with coordinate 1. The statements about coordinate 8 follow from symmetry.

Let us revisit the proof of Lemma 5.6. Let $f=-(A\times B)_2$. We have $a=fg$, where

$$g=-\frac{(C\times D)_2}{(A\times C)_2(B\times D)_2}. \tag{5.26}$$

We imagine that we have taken some variation, and all these quantities depend on t.

Each of the factors in the equation for g has derivative of size $O(1)$. Moreover, the denominator in g has size $O(1)$. From this, we conclude that

$$g(0)=1+O(u); \qquad g'(0)=O(1). \tag{5.27}$$

It now follows from the product rule that

$$\frac{da}{dt}=\frac{df}{dt}(1+O(u)). \tag{5.28}$$

Equations 5.21 and 5.23 tell us that

$$f(t)=(x_1x_3,...,x_{2n+9})\lambda(t); \qquad \lambda(t)=(L_{-13+2n})_2. \tag{5.29}$$

By equation (5.10), we have

$$\lambda(0)=1+O(u); \qquad \lambda'(0)=O(1). \tag{5.30}$$

Hence, by the product rule,

$$\frac{da}{dt}=\frac{d}{dt}(x_1x_3...x_{2n+9})(1+O(u)). \tag{5.31}$$

Using the variables

$$x_1=u,...,x_j=u^j+t,x_{j+1}=u^{j+1},... \tag{5.32}$$

we get the result of this lemma as a simple exercise in calculus. □

The results above combine to prove the Tangent Space Lemma.

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References

[1] V.I. Arnold, Mathematical methods of classical mechanics, Springer-Verlag, New York, 1989.

[2] V. Fock, A. Marshakov, in preparation.

[3] S. Fomin, A. Zelevinsky, *Cluster algebras. I. Foundations.* J. Amer. Math. Soc. **15** (2002), 497–529.

[4] E. Frenkel, N. Reshetikhin, M. Semenov-Tian-Shansky, *Drinfeld-Sokolov reduction for difference operators and deformations of W-algebras. I. The case of Virasoro algebra*, Comm. Math. Phys. **192** (1998), 605–629.

[5] M. Gekhtman, M. Shapiro, A. Vainshtein, Cluster algebras and Poisson geometry. Amer. Math. Soc., Providence, RI, 2010.

[6] M. Gekhtman, M. Shapiro, S. Tabachnikov, A. Vainshtein, *Higher pentagram maps, weighted directed networks, and cluster dynamics*, Electron. Res. Announc. Math. Sci. **19** (2012), 1–17.

[7] M. Glick, *The pentagram map and Y-patterns*, Adv. Math. **227** (2011), 1019–1045.

[8] M. Glick, *On singularity confinement for the pentagram map*, arXiv:1110.0868

[9] B. Khesin, F. Soloviev, in preparation.

[10] I. Krichever, *Analytic theory of difference equations with rational and elliptic coefficients and the Riemann-Hilbert problem*, Russian Math. Surveys **59** (2004), 1117–1154.

[11] G. Mari Beffa, *On generalizations of the pentagram map: discretizations of AGD flows*, arXiv:1103.5047.

[12] S. Morier-Genoud, V. Ovsienko, S. Tabachnikov, 2-*frieze patterns and the cluster structure of the space of polygons*, Ann. Inst. Fourier, to appear.

[13] Th. Motzkin, *The pentagon in the projective plane, with a comment on Napiers rule*, Bull. Amer. Math. Soc. **52** (1945), 985–989.

[14] V. Ovsienko, R. Schwartz, S. Tabachnikov, *Quasiperiodic motion for the pentagram map*, Electron. Res. Announc. Math. Sci. **16** (2009), 1–8.

[15] V. Ovsienko, R. Schwartz, S. Tabachnikov, *The pentagram map: a discrete integrable system*, Comm. Math. Phys. **299** (2010), 409–446.

[16] V. Ovsienko, S. Tabachnikov, Projective differential geometry old and new, from Schwarzian derivative to the cohomology of diffeomorphism groups, Cambridge Univ. Press, Cambridge, 2005.

[17] R. Schwartz, *The pentagram map*, Experiment. Math. **1** (1992), 71–81.

[18] R. Schwartz, *The pentagram map is recurrent*, Experiment. Math. **10** (2001), 519–528.

[19] R. Schwartz, *Discrete monodromy, pentagrams, and the method of condensation*, J. Fixed Point Theory and Appl. **3** (2008), 379–409.

[20] R. Schwartz, S. Tabachnikov, *Elementary surprises in projective geometry*, Math. Intelligencer **32** (2010), 31–34.

[21] R. Schwartz, S. Tabachnikov, *The pentagram integrals on inscribed polygons*, Electronic J. of Combinatorics, Electronic J. of Combinatorics **18** (2011), P171.

[22] F. Soloviev, *Integrability of the Pentagram Map*, arXiv:1106.3950.

[23] A. Tongas, F. Nijhoff, *The Boussinesq integrable system: compatible lattice and continuum structures*, Glasg. Math. J. **47** (2005), 205–219.

[24] Lobb, S. B.; Nijhoff, F. W. *Lagrangian multiform structure for the lattice Gel'fand-Dikii hierarchy*, J. Phys. A **43** (2010), no. 7., 11 pp.

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