The Affine Shape of a Figure-Eight under the Curve Shortening Flow

Matei P. Coiculescu * and Richard Evan Schwartz †

July 22, 2022

Abstract

We consider the curve shortening flow applied to a class of figure-eight curves: those with dihedral symmetry, convex lobes, and a monotonicity assumption on the curvature. We prove that when (non-conformal) linear transformations are applied to the solution so as to keep the bounding box the unit square, the renormalized limit converges to a quadrilateral $\bowtie$ which we call a bowtie. Along the way we prove that suitably chosen arcs of our evolving curves, when suitably rescaled, converge to the Grim Reaper Soliton under the flow. Our Grim Reaper Theorem is an analogue of a theorem of S. Angenent in [2], which is proven in the locally convex case.

1 Introduction

We say that a smooth family $C : S^1 \times [0,T) \to \mathbf{R}^2$ of closed immersed plane curves is evolving according to *curve shortening flow* (CSF) if and only if for any point $(u,t) \in S^1 \times [0,T)$ we have

$$\frac{\partial C}{\partial t} = kN$$

where k is the curvature and N is the unit normal vector of the immersed curve $u \to C(u,t)$. We often abbreviate this curve as $C(t)$. In all cases, there is some time $T > 0$, called the *vanishing time*, such that $C(t)$ is defined for all $t \in (0,T)$ but not at time T.

*Supported by an N.S.F. Graduate Research Fellowship

†Supported by an N.S.F. Research Grant (DMS-1807320)

Some powerful results are known about this PDE. In [11], M. Gage and R. Hamilton prove that when $C(0)$ is convex the curve $C(t)$ (which remains convex) shrinks to a point as $t \to T$ and, moreover, there is a similarity S_t such that $S_t(C(t))$ converges to the unit circle. See also [12] and [13]. In [14], M. Grayson proves that if $C(0)$ is embedded then there is some time $t \in (0, T)$ such that $C(t)$ is convex. Thus, the combination of these two results says informally that the curve-shortening flow shrinks embedded curves to round points.

In [3] and [4], S. Angenent proves that if $C(0)$ is immersed and with finitely many self-intersections, then the number of self-intersections is monotone non-increasing with time. In the case of a Figure-8, a smooth loop with one self-intersection, M. Grayson proves two things [15]:

- If one of the two lobes of the figure-eight has smaller area than the other, then this lobe shrinks to a point before the vanishing time. Then the flow can be continued through the singularity and it turns into the embedded case.

- If the lobes have equal area, the double point does not disappear before the vanishing time T, and the isoperimetric ratio of $C(t)$ tends to ∞ as $t \to T$.

Grayson conjectures [15] that in the second case, the figure-8 converges to a point under the curve-shortening flow, but this is as yet unresolved. In case $C(0)$ has 2-fold rotational symmetry, it follows from Corollary 2 of [8] that $C(t)$ does shrink to a point (the double point) as $t \to T$. In a related direction, the papers [1], [9], and [16] discuss self-similar solutions to the CSF. These shrink to a point and retain their shape.

We work with figure-8 curves that have convex lobes and 4-fold dihedral symmetry. We normalize so that the coordinate axes are the symmetry axes and that the x-axis intersects the curve in 3 points. Thus, our curves look like ∞ symbols. Angenent proves in [3] and [4] that if $C(0)$ has convex lobes then so does $C(t)$ for all $t \in (0, T)$.

Let $C_+(t)$ denote the righthand lobe of $C(t)$. We define $\kappa(\theta, t) > 0$ to be the curvature of $C_+(t)$ at the point where the tangent line makes an angle θ with the x-axis. We measure this angle in such a way that the top half of $C_+(t)$ is parametrized by $\theta \in (-\alpha(t), \pi/2]$, where $\alpha(t)$ is the tangent angle at the origin. Let $\kappa_\theta = \partial\kappa/\partial\theta$, etc. Computing the time evolution of κ, we have

$$\kappa_t = \kappa^2(\kappa + \kappa_{\theta\theta}). \tag{1}$$

See [11] for a proof. We note that the curve satisfying $\kappa(\theta) = \sin(\theta)$, for $\theta \in (0, \pi)$ is a stationary solution for Equation 1. Up to isometries of the plane, this curve is called the *Grim Reaper Soliton*. It evolves by translation under the curve shortening flow.

Definition: $C(0)$ is a *monotone* figure-eight curve if and only if

- $C(0)$ is real analytic.
- $C(0)$ has 4-fold dihedral symmetry.
- $C(0)$ has convex lobes.
- $\kappa_\theta(\theta, 0) > 0$ for $\theta \in (-\alpha(0), \pi/2)$
- $\kappa_{\theta\theta}(\pi/2, 0) \neq 0$.
- The signed curvature of $C(0)$, as a function of arc length, does not vanish to second order at the double point.

The Lemniscate of Bernoulli is an example of a monotone figure 8 curve. The first condition is not much of a restriction because the curve shortening flow instantly turns curves real analytic. The last two conditions are nondegeneracy conditions included to simplify our arguments. In §2 we prove that the curve-shortening flow preserves monotonicity: if $C(0)$ is monotone, then $C(t)$ is monotone for all $t < T$. The proof is basically an application of the maximum principle and the so-called Sturmian principle for various strictly parabolic equations.

Define

$$F(\theta, t) = \frac{\kappa(\theta, t)}{\kappa(\pi/2, t)}. \tag{2}$$

Here F is a rescaled version of κ. In §3 we prove the following result.

Theorem 1.1 (Grim Reaper)**.** *Assume that $C(0)$ is monotone. Let $J \subset (0, \pi)$ be an arbitrary closed interval. Let $\epsilon > 0$ be given. For t sufficiently close to T, we have*

$$\sup_{\theta \in J} |F(\theta, t) - \sin(\theta)| < \epsilon, \qquad \sup_{\theta \in J} |F_\theta(\theta, t) - \cos(\theta)| < \epsilon.$$

The Grim Reaper Theorem is the analogue of Theorem D in [2]. In [2], S. Angenent also makes the monotonicity assumption when applying his Theorem D to specific curves. The result implies that a suitably rescaled copy of the arc of $C(t)$ corresponding to $\theta \in (0, \pi)$ converges to the Grim Reaper curve. The arc in question is the one between the two dots in Figure 1. Our proof departs from that in [2] because we are not working with locally convex curves as in [2].

The *bounding box* of a compact set in the plane is the smallest rectangle, with sides parallel to the coordinate axes, that contains the set. The main goal of the paper is to understand the limit of the curves $\{L_t(C(t)\}$ where L_t is the positive diagonal matrix such that $L_t(C(t))$ has the square $[-1,1]^2$ for a bounding box. Even though affine transformations do not interact in a nice way with the curve shortening flow, nothing stops us from looking at a solution and applying affine transformations afterwards.

The *bowtie* is the quadrilateral whose vertices are

$$(-1,-1), \quad (1,1), \quad (1,-1), \quad (-1,1)$$

in this cyclic order. It is shaped like this: $\bowtie$. The *Hausdorff distance* between two compact subsets of the plane is the smallest ϵ such that each set is contained in the ϵ-neighborhood of the other. This distance makes the set of compact planar subsets into a metric space. Here is our main result.

Theorem 1.2 (Bowtie)**.** *Suppose that $C(0)$ is monotone. As $t \to T$, the curves $L_t(C(t))$ converge in the Hausdorff metric to the bowtie.*

Figure 1 shows a picture of a numerical simulation of the curve shortening flow. The curve on the left is $L_0(C(0))$ where $C(0)$ is the Leminscate of Bernoulli. The black curve on the right is $L_t(C(t))$ for some later time t. The blue curve on the right is $\Gamma(t) = C(t)/X(t)$, the rescaled version of $C(t)$ whose bounding box has width 2. The black and white dots correspond to where $\theta = 0$ and $\theta = \pi$ respectively. Figure 1 shows some hints of the bowtie forming.

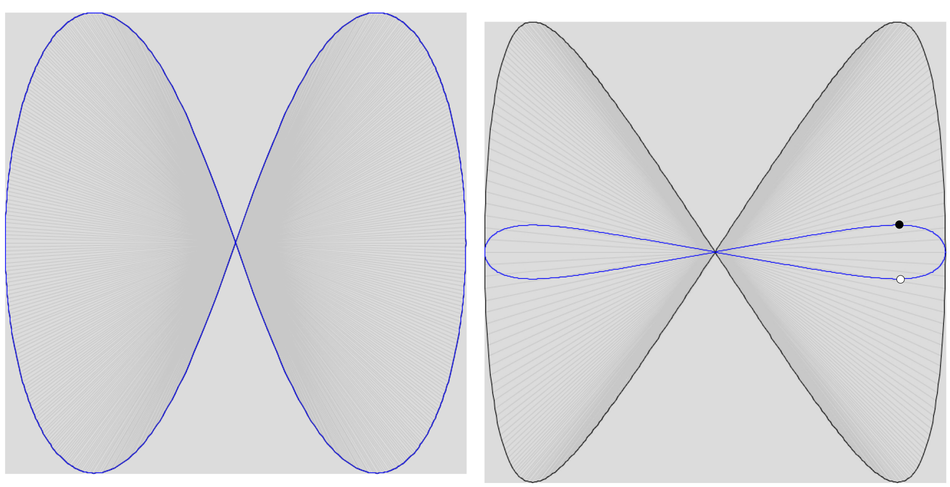

Figure 1: A hint of the bowtie.

Now we sketch the proof of the Bowtie Theorem. Our argument fleshes out the outline proposed in [7]. Let $A(t)$ be the area of the region – i.e., the two lobes – bounded by $C(t)$. The main thrust of our proof is establishing the following three formulas.

$$\lim_{t\to T}\frac{A(t)}{T-t}=2\pi, \quad \liminf_{t\to T}\kappa(\pi/2,t)Y(t)\geq\pi/2, \quad \liminf_{t\to T}\frac{X(t)}{(T-t)\kappa(\pi/2,t)}\geq 2 \quad (3)$$

The first of these formulas is essentially the same as the bound in [15], but sharpened by the fact, which we prove, that the angle at the double point tends to 0 as $t\to T$. The second formula is an easy application of the Grim Reaper Theorem. The third formula follows from a well-chosen rescaling argument combined with the Sturmian Principle. These formulas combine to give the upper bounds

$$\limsup_{t\to T}\frac{A(t)}{X(t)Y(t)}\leq 2, \qquad \limsup_{t\to T}\text{area}(L_t(C(t)))\leq 2 \quad (4)$$

The first bound immediately implies the second. On the other hand, it follows from convexity that $L_t(C(t))$ has area at least 2. We conclude that

$$\lim_{t\to T}\text{area}(L_t(C(t)))=2, \quad \lim_{t\to T}\kappa(\pi/2,t)Y(t)=\pi/2, \quad \lim_{t\to T}\frac{X(t)}{(T-t)\kappa(\pi/2,t)}=2 \quad (5)$$

Similar asymptotic results are obtained for everywhere locally convex curves in [5] and [6]. We use the middle equation in Equation 5, the Y bound, to prove:

Lemma 1.3 (Migration)**.** *The point on $L_t(C(t))$ having positive first coordinate and largest second coordinate converges to $(1,1)$ as $t\to T$.*

Now, suppose that there is a sequence $\{t_n\}$ for which $L_{t_n}(C(t_n))$ does not converge in the Hausdorff topology to the bowtie. We pass to a further subsequence so that $\{L_{t_n}(C(t_n))\}$ has some limit point (x,y) in the positive quadrant that lies outside the region bounded by the bowtie. Let Ψ be the polygonal figure 8, with 4-fold dihedral symmetry, whose right lobe is the convex hull of points

$$(0,0), \qquad (1,\pm 1), \qquad (x,\pm y)$$

Ψ has area greater than 2. By symmetry and convexity, the region bounded by $L_{t_n}(A(t_n))$ contains a subset which converges to Ψ in the Hausdorff metric. But then the area of $L_{t_n}(A(t_n))$ cannot converge to 2. This contradicts Equation 5.

In §2 we prove that the flow preserves monotonicity. In §3 we establish some basic asymptotic facts about $C(t)$ as $t \to T$, such as the decay of the angle at the double point and the area estimate. In §4 we prove the Grim Reaper Theorem. In §5 we establish the second and third formulas in Equation 3. In §6 we prove the Migration Lemma, thereby completing the proof of the Bowtie Theorem.

Acknowledgements: We thank Peter Doyle and Mike Gage for some helpful conversations. We thank Brown University and the National Science Foundation for their support. The first author thanks Princeton University for its support. The second author thanks the Simons Foundation, in the form of a Simons Sabbatical Fellowship, and the Institute for Advanced Study for a year-long membership funded by a grant from the Ambrose Monell Foundation. We would also like to thank the anonymous referees for their insightful comments.

2 Preservation of Monotonicity

2.1 Strictly Parabolic Equations

In this chapter we prove that the curve shortening flow preserves the monotonicity condition. We begin with a discussion of strictly parabolic equations and two of their basic properties. We follow the notation in [10] and [2].

Let U be an open interval containing $[x_0, x_1]$. We suppose that $u : U \times [0, \tau]$ satisfies the equation

$$u_t = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u. \tag{6}$$

This equation is called *strictly parabolic* if and only if $a(x,t)$, $b(x,t)$, and $c(x,t)$ are smooth and $a(x,t) > 0$. We assume that u satisfies a strictly parabolic PDE. Here is the well-known *Maximum Principle.*

Theorem 2.1 (The Maximum Principle)**.** *Suppose that $u \neq 0$ on $\{x_0, x_1\} \times [0, \tau]$ and also nonzero on $[x_0, x_1] \times \{0\}$. Then u is nonzero on $[x_0, x_1] \times [0, \tau]$.*

Geometrically we are looking at the behavior of u on a rectangle. If we know that u is nonzero on 3 sides of ∂R then we know u is nonzero on all of R. The side $[x_0, x_1] \times \{0\}$ is the bottom side of R and the side $[x_0, x_1] \times \{\tau\}$ is the top. Here we are picturing time as running vertically and space as running horizontally.

Here is the well-known Sturmian Principle.

Theorem 2.2 (The Sturmian Principle)**.** *Suppose u is nonzero on $\{x_0, x_1\} \times [0, \tau]$. Then the number N_t of times $u(*, t)$ vanishes on (x_0, x_1) is non-increasing with time. Moreover, if $u(*, t)$ vanishes to second order somewhere on (x_0, x_1) then $N_{t'} < N_t$ for all $t' \in (t, \tau]$.*

C. Sturm discovered this principle in 1836. see [17]. The proof of the above version of the Sturmian Principle may be found in [3]. For more references about these theorems, see [10] or [3]. Note that if u, v solve the same strictly parabolic equation then so does $w = u - v$. This yields the following corollary.

Corollary 2.3. *Suppose w is nonzero on $\{x_0, x_1\} \times [0, \tau]$. Then the number N_t of zeroes for $w(*, t)$ on (x_1, x_2) is finite and non-increasing. Moreover, at any time t when $w(*, t)$ vanishes to second order, we have $N_{t'} < N_t$ for all $t' > (t, \tau]$.*

Curvilinear Domains: Rectangular domains are too restrictive for one of our purposes. The same principles work when the rectangle in question is replaced by a piecewise analytic quadrilateral $\mathcal{Q}$ with the following two properties:

1. The top and bottom sides are line segments, with the bottom one corresponding to time 0 and the top one corresponding to time τ.

2. The function w does not vanish on the other two sides.

The other two sides play the role of $\{x_0\} \times [0, \tau]$ and $\{x_1\} \times [0, \tau]$. The main issue is that the non-horizontal sides prevent zeros from "leaking in or out".

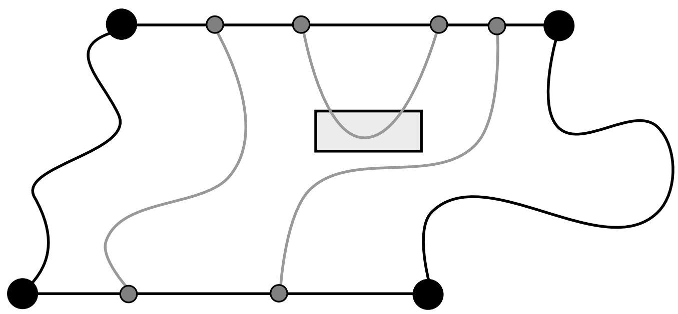

Figure 2: The Curvilinear case

Let us explain why the rectilinear principle implies the curvilinear principle. Suppose we have a situation where w has m zeros on the bottom of $\mathcal{Q}$ and $n > m$ on the top of $\mathcal{Q}$. Let I be the set of times where w has more than m zeros. Let $t = \inf I$. The zeros of w at times converging to t cannot converge to the non-horizontal sides of the domain. Hence at least two of them must coalesce. But then we can find a small rectangle $R \subset \mathcal{Q}$ which surrounds these coalescing points. See the small shaded rectangle in Figure 2. (If more points coalesce, the picture would look more complicated.) This gives a contradiction to the rectilinear principle.

2.2 Evolution Equations

The evolution equation for κ is given in Equation 1. Here it is again.

$$\kappa_t = (\kappa^2)\kappa_{\theta\theta} + (0)\kappa_\theta + (\kappa^2)\kappa. \tag{7}$$

This equation is strictly parabolic.

Let $u = \kappa_\theta$. Differentiating Equation 1 with respect to θ we get the evolution equation for u:

$$u_t = (\kappa^2)u_{\theta\theta} + (2\kappa u)u_\theta + (3\kappa^2)u \tag{8}$$

All we need to know about this equation is that it is strictly parabolic. Also, we only need this equation in this chapter.

Let $y(x,t)$ be the evolution of the height of the curve $C(t)$. Let

$$k(x,t) := k(x, y(x,t)) \tag{9}$$

denote the signed curvature at the point $(x, y(x,t))$. For fixed t, the curve is defined in terms of x and y and the curvature is given in terms of partial derivatives of y with respect to x holding t fixed. Note that the domain for x is shrinking to a point. The following evolution equation for $\mu = k_x$ is derived in [14]:

$$\mu_t = (\zeta)\mu_{xx} + (-2y_x y_{xx}\zeta^2)\mu_x + (3k^2)\mu, \qquad \zeta = \frac{1}{1+y_x^2}. \tag{10}$$

Again, all we need to know about this equation is that it is strictly parabolic and we only need this equation in this chapter.

Equations 7 and 8 are valid on the domain

$$\mathcal{D} = \bigcup_{t\in[0,T)} (-\alpha(t), \pi + \alpha(t)) \times \{t\}. \tag{11}$$

Equation 10 is valid away from places where our curve has vertical tangents. In particular on any time range $[0,t]$ for $t < T$ it is valid on each strand in some fixed neighborhood of the double point.

2.3 Monotonicity

Now we prove that the curve shortening flow preserves the monotonicity property. We assume that $C(0)$ is monotone.

Lemma 2.4. *If $\kappa_\theta(\theta,t) > 0$ for all $\theta \in (0,\pi/2)$ and all $t \leq t_0$, then*

1. *$k_x(0,t) > 0$ for all $0 \leq t \leq t_0$.*
2. *$\kappa_{\theta\theta}(\pi/2,t) \neq 0$ for all $0 \leq t \leq t_0$.*

Proof. For the first statement, it suffices to prove that $k_x(0,t_0) > 0$. Not first that the last monotonicity property implies that $k_x(0,0) > 0$. The reason is that near the double point the x-coordinate is a smooth invertible function of arc length. We apply the Maximum principle to a rectangle of the form $[-\epsilon,\epsilon] \times [0,t_0]$ and we get a contradiction.

For the second statement, it suffices to prove that $\kappa_{\theta\theta}(\pi/2,t_0) \neq 0$. This is an application of the Sturmian Principle for κ_θ with respect to a rectangle of the form $[\pi/2-\epsilon, \pi/2+\epsilon] \times [0,t_0]$. We are assuming that $\kappa_{\theta\theta}(\pi/2,0) \neq 0$. Hence $\kappa_\theta(*,0)$ only vanishes to first order at $\pi/2$. Also κ_θ does not vanish on the vertical sides of the rectangle, by symmetry. □

Lemma 2.5. *$\kappa_\theta(*,t) > 0$ on $(-\alpha(t), \pi/2)$ for all $t < T$.*

Proof. Recall that $u = \kappa_\theta$. We need to show is that $u(\theta,t) > 0$ on the domain $\mathcal{D}$. Suppose this fails. Let I denote the set of times t' for which $u(*,t')$ vanishes somewhere. Let $t_0 = \inf I$. There are several cases to consider.

Suppose first that $t_0 \in I$. Then there is some $(\theta, t_0) \in \mathcal{D}$ such that $u(\theta, t_0) = 0$ but $u(*,t) > 0$ for all $t \in [0, t_0)$. In this case we get a contradiction by applying the Maximum Principle to u on a rectangle $[\theta - \epsilon, \theta + \epsilon] \times [t_0 - \epsilon, t_0]$. For sufficiently small ϵ this rectangle belongs to $\mathcal{D}$. Since u is analytic we can further choose ϵ so that $u(\theta \pm \epsilon, t_0) > 0$. We now contradict the Maximum Principle. Hence $t_0 \notin I$.

Let (θ_n, t_n) be a sequence of points in $\mathcal{D}$ such that $u(\theta_n, t_n) = 0$ and $t_n \to t_0$. Since $t_0 \notin I$, we must have (after using symmetry and passing to a subsequence) either $\theta_n \to -\alpha(t)$ or $\theta_n \to \pi/2$. By Lemma 2.4 we know that $k_x(0, t_0) > 0$ and $\kappa_{\theta\theta}(\pi/2, t_0) \neq 0$. Consider the cases.

- Suppose $\theta_n \to -\alpha(t_0)$. By the Chain rule, $k_x(x_n, t_n) = 0$ for a sequence $x_n \to 0$. But then $k_x(0, t_0) = 0$ by continuity. This is a contradiction.

- Suppose $\theta_n \to \pi/2$. Since we are now in the interior of the domain $\mathcal{D}$ and u is a smooth function, we have

  $$\kappa_{\theta\theta}(\pi/2, t_0) = \lim_{n\to\infty} \frac{u(\pi/2, t_n) - u(\theta_n, t_n)}{\pi/2 - \theta_n} = 0.$$

  This is a contradiction.

This completes the proof. □

Let us now check that $C(t)$ is monotone for any $t < T$.

- The curve shortening flow preserves analyticity, hence $C(t)$ is analytic.

- The curve shortening flow respects symmetry, so $C(t)$ has 4-fold symmetry.

- As we mentioned in the introduction, Angenent proves in [3] and [4] that $C(t)$ has convex lobes for all $t < T$.

- Lemma 2.5 says exactly that $\kappa_\theta(\theta, t) > 0$ for $\theta \in (-\alpha(t), \pi/2)$.

- Lemma 2.4 shows that $\kappa_{\theta\theta}(\pi/2, t) \neq 0$.

- Lemma 2.4 shows that the signed curvature of $C(t)$, as a function of arc length, does not vanish to second order at the double point.

3 Some Asymptotic Results

Recall that $[-X(t), X(t)] \times [-Y(t), Y(t)]$ is the bounding box of $C(t)$.

Lemma 3.1 (Bounding Box)**.** $\lim_{t\to T} Y(t)/X(t) = 0$.

Proof. The perimeter of $C(t)$ and the area of the region bounded by $C(t)$ are respectively within a factor of 2 of the perimeter and area of the bounding box of $C(t)$. Thus, Grayson's isoperimetric result tells us that the aspect ratio of the bounding box tends to 0. This means that either $Y(t)/X(t) \to 0$ or $Y(t)/X(t) \to \infty$ as $t \to T$.

We established in §2.3 that $C(t)$ is monotone for all $t \in (0, T)$. Given that $X_t(t) = -\kappa(\pi/2, t)$ and $Y_t(t) = -\kappa(0, t)$, we have

$$Y(t) = \int_t^T \kappa(0, u)\ du < \int_t^T \kappa(\pi/2, u)\ du = X(t).$$

This rules out the second option above. □

The remainder of our estimates, and also some arguments in subsequent chapters, use the well-known Tait-Kneser Theorem from differential geometry. One can find a proof in practically any book of differential geometry.

Theorem 3.2 (Tait-Kneser)**.** *Suppose γ is a curve of strictly monotone increasing curvature. Then the osculating disks of γ are strictly nested. The largest one is at the initial endpoint and the smallest one is at the final endpoint. In particular, γ lies inside the osculating disk at its initial endpoint and outside the osculating disk at its final endpoint.*

Lemma 3.3 (Curvature Blowup)**.** $\lim_{t\to T} \kappa(\theta, t) = \infty$ *for any* $\theta \in (0, \pi/2]$.

Proof. Let $\Gamma(t) = C(t)/X(t)$. This is a rescaled version of $C(t)$ whose bounding box has width 2. The height of the bounding box tends to 0 by Lemma 3.1. Let

$$K(\theta, t) = X(t)\kappa(\theta, t)$$

be the curvature of $\Gamma(t)$ at the point where the tangent angle is θ. Since we have $\lim_{t\to T} X(t) = 0$, it suffices to prove that $K(\theta, t) \geq 2\sin(\theta)$.

Let $\Delta = \Delta(\theta, t)$ be the osculating disk to $\Gamma(t)$ at $\Gamma(\theta, t)$. By the Tait Kneser Theorem, the arc of $\Gamma(t)$ connecting the origin to $\Gamma(\theta, t)$ lies outside Δ. This forces $\partial\Delta$ to cross the horizontal line L through $\Gamma(\theta, t)$ twice inside the bounding box and in the positive quadrant. The intersection $L \cap \Delta$ has length at most 1 and the angle between L and $\partial\Delta$ at the intersection points is θ. It follows from trigonometry that Δ has radius at most $1/(2\sin\theta)$. Hence $K(\theta, t) \geq 2\sin(\theta)$. □

Lemma 3.4 (Angle Decay)**.** $\lim_{t \to T} \alpha(t) = 0$.

Proof. Let $\Gamma(t)$ be as in the previous lemma. Suppose that there is a sequence of times $t_n \to T$ such that $\alpha(t_n) > \delta > 0$ for some constant δ. Let L be the line through the origin which makes an angle of $\delta/2$ with the x-axis. Again, the height of the bounding box for $\Gamma(t_n)$ tends to 0 as $n \to \infty$. Hence, L hits the top of the bounding box at a point whose distance to the origin tends to 0 as $n \to \infty$.

By construction $\Gamma(t_n)$ starts out from the origin lying to the left of L. Since $\Gamma(t_n)$ lies inside its bounding box, we see that $\Gamma(t_n)$ crosses L at some point p_n such that $\|p_n\| \to 0$. The total variation of the tangent angle of $\Gamma(t)$ along the arc connecting $(0, 0)$ to p_n is, by convexity, at least $\delta/2$. Since the length of this arc tends to 0, and since the curvature is monotone increasing, the curvature of Γ_n at p_n is eventually at least 4.

By the Tait-Kneser Theorem the arc of $\Gamma(t)$ connecting p_n to $(1, 0)$ is trapped in a disk of radius $1/4$ which contains p_n in its boundary. This is a contradiction. □

Corollary 3.5 (Area Asymptotics)**.** *The first formula in Equation 3 is true.*

Proof. Let $k(s, t)$ denote absolute value of the curvature as a function of arc length and time. Consider the two curves $C(t)$ and $C(t + \delta)$ for some very small δ. At any given point (s, t) on $C(t)$ the distance from $C(t)$ to $C(t + \delta)$ equals $\kappa(s, t)\delta$. up to order $(\delta)^2$. So, up to order δ^2 the total change in area is

$$\int k(s, t)ds = 2\pi + 2\alpha(t).$$

By the definition of the derivative, we therefore have $A_t(t) = -2\pi - 2\alpha(t)$. Hence

$$\lim_{t \to T} A_t(t) = -2\pi - \lim_{t \to T} \alpha(t) = -2\pi.$$

We set $B(t) = T - t$. Since $\lim_{t \to T} A(t) = \lim_{t \to T} B(t) = 0$ we have

$$\lim_{t \to T} \frac{A(t)}{T - t} = \lim_{t \to T} \frac{A(t)}{B(t)} = \lim_{t \to T} \frac{A_t(t)}{B_t(t)} = \frac{-2\pi}{-1} = 2\pi$$

by L'Hôpital's rule. □

4 The Grim Reaper Theorem

4.1 Counting Zeros

Our first lemma has nothing to do with the flow. A very similar principle is used in [2]. Let $J \subset \mathbf{R}$ be some interval. Call a function $g : J \to \mathbf{R}$ *small* if

$$\sup_J g^2 + (g')^2 < 1. \tag{12}$$

Call J *small* if it has length at most π. Every small interval is contained in a closed interval of length π. Closed intervals of length π count as being small.

Lemma 4.1. *If g is a small function and J is a small interval then the difference $w(x) = g(x) - \sin(x)$ vanishes at most twice on J, counting multiplicity.*

Proof. Let $f(x) = \sin(x)$. We note the crucial property that

$$f^2 + (f')^2 = 1 > g^2 + (g')^2.$$

Let F and G respectively denote the graphs of F and G. These graphs must be transverse wherever they intersect. Otherwise we would have $g^2 + (g')^2 = 1$ at an intersection point. This is impossible. We show that $f = g$ at most twice. Given the transversality just mentioned, this is equivalent to the statement that $w = g - f$ vanishes at most twice on J, counting multiplicity.

As usual in calculus, say that $x \in J$ is an *extreme point* if $f'(x) = 0$. The only way that J can contain two extreme points is if J has length π, and the endpoints are the two extreme points, and $|f| = 1$ at these endpoints. In this case $f \neq g$ at the endpoints because $|g| < 1$. So, even in this case, we can replace J by a smaller interval which contains all the points where $f = g$. Thus, we can assume without loss of generality that J contains at most one extreme point.

Suppose first that J has no extreme points. Then f is either monotone increasing on J or monotone decreasing. Consider the case when f is monotone increasing. Suppose it happens that there are two consecutive points $x_1, x_2 \in J$ where f and g agree. The portion of G between $(x_1, g(x_1))$ and $(x_2, g(x_2))$ either lies above F or below. In the first case we have $g'(x_1) > f'(x_1)$, which is a contradiction. In the second case we have $g'(x_2) > f'(x_2)$ and we have the same contradiction. Hence $f(x) = g(x)$ for at most one point $x \in J$. The same argument works when f is monotone decreasing on J.

Now consider the case when J has exactly one extreme point. In this case we can write $J = J_1 \cup J_2$ where f is monotone on each J_i. In this case, the same argument above, applied to each of these sub-intervals, shows that they each have at most one point where $f = g$. Hence J has at most 2 such points. $\square$

4.2 The Sine Lemma

Here is the crucial step in the proof of the Grim Reaper Theorem. This section is devoted to proving the following result.

Lemma 4.2 (Sine)**.** *Let J be any closed interval contained in $(0, \pi)$. Let $\epsilon > 0$ are given. If t is sufficiently close to T then*

$$\left|\frac{\kappa_\theta(\theta, t)}{\kappa(\theta, t)} - \frac{\cos(\theta)}{\sin(\theta)}\right| < \epsilon,$$

for all $\theta \in J$.

We will assume for the sake of contradiction that there is a sequence of times $t_n \to T$ and a sequence $\{\theta_n\} \in J$ such that

$$\left|\frac{\kappa_\theta(\theta_n, t_n)}{\kappa(\theta_n, t_n)} - \frac{\cos(\theta_n)}{\sin(\theta_n)}\right| > \epsilon. \tag{13}$$

Passing to a subsequence we can assume that $\theta_n \to \theta_0 \in J$. By compactness of J we can choose a constant $\Sigma = \Sigma(J, \epsilon) > 0$ so that

$$\left|\frac{\cos(\phi + \theta_n)}{\sin(\phi + \theta_n)} - \frac{\cos(\theta_n)}{\sin(\theta_n)}\right| > \epsilon \qquad \Longrightarrow \qquad |\phi| > \Sigma, \tag{14}$$

as long as $\phi + \theta_n \in (0, \pi)$.

Call the non-horizontal sides of our domains the *sidewalls*. Thanks to Lemma 3.4 we can omit the initial portion of our evolution and arrange that

$$\sup_{t \in [0,T)} \alpha(t) < 10^{-100}\Sigma. \tag{15}$$

We are making the horizontal displacement of the sidewalls of $\mathcal{D}$ extremely small in comparison to the other relevant quantities that arise below. We don't need the factor of 10^{-100}; we add it for emphasis.

Let

$$C = \sup_{\theta \in [0,\pi]} \kappa^2(\theta, 0) + \kappa_\theta^2(\theta, 0), \qquad B_n = \kappa^2(\theta_n, t_n) + \kappa_\theta^2(\theta_n, t_n). \tag{16}$$

By Lemma 3.3 there is some n such that $B_n > C$. Our notivation for taking $B_n > C$ is the following corollary of Lemma 4.1.

Corollary 4.3. *Suppose*

$$\sup_{\theta\in J} \kappa^2(\theta,0)+\kappa_\theta^2(\theta,0)\leq C.$$

Let $S(\theta)=\sqrt{B}\sin(\phi+\theta)$ *for any value* ϕ*. If* $B>C$ *then* $w(*)=\kappa(*,0)-S(*)$ *vanishes at most twice on* J*, counting multiplicity.*

Proof. This follows from Lemma 4.1 by symmetry and scaling. □

We fix n for which $B_n>C$. We set $B=B_n$ and $t=t_n$. There is some angle ϕ such that

$$\frac{\kappa_\theta(\theta_n,t)}{\kappa(\theta_n,t)}=\frac{\cos(\phi+\theta_n)}{\sin(\phi+\theta_n)}.$$

For this choice of ϕ we have

$$S(\theta_n)=\sqrt{B}\sin(\phi+\theta_n)=\kappa(\theta_n,t),\qquad S_\theta(\theta_n)=\sqrt{B}\cos(\phi+\theta_n)=\kappa_\theta(\theta_n,t). \quad (17)$$

Our function S determines a unique interval I of length π such that $S>0$ on the interior of I and $\theta_n\in I$. Note also that $S=0$ on ∂I. Let $\Omega=I\times[0,t]$. This is exactly the domain considered in [2], but now our proof departs from [2].

Lemma 4.4. *One sidewall of* Ω *is disjoint from the closure of* $\mathcal{D}$ *and the other sidewall of* Ω *lies in* $\mathcal{D}$*.*

Proof. The properties of S imply the following:

$$\left|\frac{\cos(\phi+\theta_0)}{\sin(\phi+\theta_0)}-\frac{\cos(\theta_0)}{\sin(\theta_0)}\right|=\left|\frac{\kappa_\theta(\theta_0,t)}{\kappa(\theta_0,t)}-\frac{\cos(\theta_0)}{\sin(\theta_0)}\right|>\epsilon. \quad (18)$$

By equation 14, we have $|\phi|>\Sigma$.

If we had $\phi=0$ we would have $I=[0,\pi]$. As it is, we have $|\phi|>\Sigma$. This shifts I and Ω by at least Σ to the left or to the right. Given our bound on the horizontal displacement of the sidewalls of $\mathcal{D}$, this shift causes one sidewall or the other to stick out completely. See Figure 3 below. At least one point of I lies in $(0,\pi)$ and the total width of I is π. Hence I cannot both contain points less than 0 and greater than π. This means that the other sidewall lies inside $\mathcal{D}$. □

We now create a new domain $\mathcal{Q}$ by intersecting Ω with $\mathcal{D}$ and pushing in the curvilinear sidewall a bit. We treat the case when Ω sticks out on the left. The other case is essentially the same.

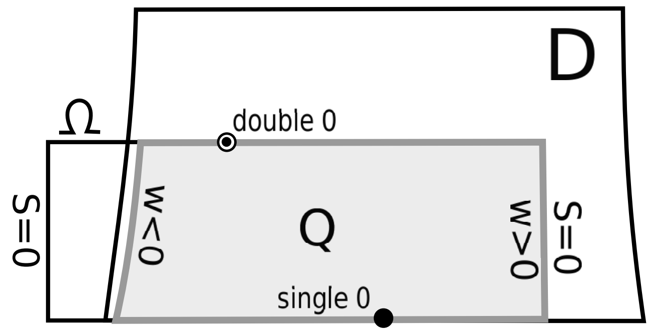

Figure 3: The new domain $\mathcal{Q}$, shaded.

Define

$$w(\theta,t) = \kappa(\theta,t) - S(\theta). \tag{19}$$

The function S is a stationary solution to Equation 1, meaning that $S_t = 0$. This means that w is exactly the sort of difference of solutions to which the Sturmian Principle applies. Let us examine the behavior of w on the boundary of $\mathcal{Q}$.

Left: Since κ limits to 0 on the sidewalls of $\mathcal{D}$ and $S > 0$ on the left sidewall of $\mathcal{D}$, we can by compactness make the perturbation small enough so that $w < 0$ on the left sidewall of $\mathcal{Q}$.

Right: The right sidewall of $\mathcal{Q}$ lies in $\mathcal{D}$. Since $S = 0$ on the right sidewall of $\mathcal{Q}$ and $\kappa > 0$ everywhere in $\mathcal{D}$, we have $w > 0$ on the right sidewall of $\mathcal{Q}$.

Bottom: Applying Corollary 4.3 to the bottom side J of $\mathcal{Q}$, we see that $w(*,0)$ vanishes at most twice on J counting multiplicity. Since w has opposite signs on the sidewalls of $\mathcal{Q}$ the number of zeros of w on J is odd, counting multiplicity. Since this number is at most 2, it must be exactly 1. In short, w vanishes exactly once on the bottom side of $\mathcal{Q}$, counting multiplicity.

Top: On the top side J' of $\mathcal{Q}$ we have arranged that w and w_θ vanish at (θ_0,t). This means that w vanishes at least twice, counting multiplicity, on J'. We have shown this double point in Figure 3. Since w has opposite signs on the sidewalls of $\mathcal{Q}$ the number of zeros of w on J' is odd, counting multiplicity. Since this number is at least 2 it is actually at least 3. In short, w vanishes at least 3 times on the top side of $\mathcal{Q}$ counting multiplicity.

The above properties violate the Sturmian Principle for (Equation 1, $\mathcal{Q}$, w). This completes the proof of the Sine Lemma.

4.3 The End of the Proof

In this section we prove the Grim Reaper Theorem.

Corollary 4.5. *Let $\epsilon > 0$ be given and let $J \subset (0, \pi)$ be any closed interval. We have*

$$\sup_{\theta \in J} \left| \frac{F_\theta(\theta,t)}{F(\theta,t)} - \frac{\cos(\theta)}{\sin(\theta)} \right| < \epsilon,$$

for t sufficiently close to T.

Proof. We can replace κ by F because for each time these functions are constant multiples of each other. □

Consider the new function

$$G(\theta,t) = \frac{F(\theta,t)}{\sin(\theta)}. \tag{20}$$

Using Lemma 4.5 we have the following result:

$$|G_\theta| = \frac{|F_\theta(\theta,t)\sin(\theta) - F(\theta,t)\cos(\theta)|}{\sin^2(\theta)} < \frac{\epsilon F(\theta,t)\sin(\theta)}{\sin^2(\theta)} = \epsilon G, \tag{21}$$

This holds for all $\theta \in J$ provided that we take t sufficiently close to T. The last calculation shows that the logarithmic derivative G_θ/G is nearly 0 on J. Hence G is nearly constant on J. But $G(\pi/2,t) = 1$. Hence G is nearly 1 on J. This proves that $F(\theta,t)$ converges uniformly to $\sin(\theta)$ for $t \in J$. But this combines with Corollary 4.5 to show that $F_\theta(\theta,t)$ converges uniformly to $\cos(\theta)$ for $t \in J$. This completes the proof of the Grim Reaper Theorem.

5 Asymptotic Formulas

5.1 The Y Bound

In this section we deduce the middle bound in Equation 3 from the Grim Reaper Theorem, namely

$$\liminf_{t \to T} Y(t)\kappa(\pi/2, t) \geq \pi/2.$$

The key is to get a nice integral formula for this expression.

Lemma 5.1.

$$Y(t)\kappa(\pi/2, t) = \int_0^{\pi/2} \frac{\sin(\phi)}{F(\phi, t)} d\phi. \tag{22}$$

Proof. Let s_0 and s_1 respectively denote the arc-length parameters that correspond to $\theta_0 = 0$ and $\theta_1 = \pi/2$. On the level of 1-forms:

$$dy = -ds \sin\theta, \qquad \kappa(\theta, t)ds = d\theta.$$

(The minus sign appears because y decreases as s increases.)

$$Y(t) = \int_0^{Y(t)} dy = -\int_{s_1}^{s_0} \sin(\theta)ds = \int_{s_0}^{s_1} \sin(\theta)ds = \int_0^{\pi/2} \frac{\sin(\theta)}{\kappa(\theta, t)} d\theta. \tag{23}$$

Multiplying through by $\kappa(\pi/2, t)$, we get Equation 22. □

Letting $\delta > 0$ be arbitrary, we have

$$Y(t)\kappa(\pi/2, t) = \int_0^{\delta} \frac{\sin(\phi)}{F(\phi, t)} d\phi + \int_{\delta}^{\pi/2} \frac{\sin(\phi)}{F(\phi, t)} d\phi > \int_{\delta}^{\pi/2} \frac{\sin(\phi)}{F(\phi, t)} d\phi \tag{24}$$

By the Grim Reaper Theorem, the integrand in the last integral to 1 as $t \to T$. Hence the right hand side is at least $\pi/2 - 2\delta$ once t is sufficiently close to T. This establishes our bound.

Remark: The Y bound in Equation 3 is weaker than the Y bound in Equation 5 and one might wonder about a direct proof of the stronger result. It is difficult to conclude directly that the first integral in Equation 24 converges to 0 as $\delta \to 0$ because the integrand could potentially blow up near $\theta = 0$. The issue is that in the Grim Reaper Theorem we only get convergence on the open interval $(0, \pi)$. Our indirect argument for the bound in Equation 5, which uses convexity and all the inequalities in Equation 3 together, avoids this difficulty.

5.2 The X Bound

The rest of the chapter is devoted to proving the third bound in Equation 3. A similar asymptotic result is proven in [5], albeit for everywhere locally convex curves.

Define

$$\beta(t) := \frac{X(t)}{(T-t)\kappa(\pi/2,t)} > 0. \tag{25}$$

It suffices to show that $\beta(t) > 2$ for t sufficiently close to T.

Our argument in this section gives a clear reason why this should be the case, but there is one technical detail which takes a rather long time to prove. Here we give the main argument.

Define

$$\ell(t) := \log(X(t)) - \frac{1}{2}\log(T-t). \tag{26}$$

Lemma 5.2. *$\ell_t(t) > 0$ if and only if $\beta(t) > 2$.*

Proof. This is just a calculation. We have $X_t(t) = -\kappa(\pi/2,t)$. Therefore,

$$2\ell_t(t) = \frac{2X_t(t)}{X(t)} + \frac{1}{T-t} = -\frac{2\kappa(\pi/2,t)}{X(t)} + \frac{1}{T-t} = \frac{1}{T-t} \times \left(1 - \frac{2}{\beta(t)}\right). \tag{27}$$

Hence $\ell_t(t) > 0$ if and only if $\beta(t) > 2$. □

Lemma 5.3. $\lim_{t\to T} \ell(t) = +\infty$.

Proof. This is equivalent to the statement that

$$\lim_{t\to T} \frac{X(t)}{\sqrt{T-t}} \to \infty.$$

Consider the rescaled curve $C(t)/\sqrt{T-t}$. The area of this curve converges to 2π as $t \to T$ and the aspect ratio converges to 0. Hence the rightmost point, namely $X(t)/\sqrt{T-t}$, converges to ∞. □

Since $\ell(t) \to \infty$ as $t \to T$, there is a sense in which $\ell_t(t) > 0$ much more often than $\ell_t(t) \leq 0$. However, we don't know *a priori* that the sign does not switch infinitely often as $t \to T$. This is the technical detail. The rest of the chapter is devoted to showing that ℓ_t changes sign at most finitely many times as $t \to T$. This combines with Lemma 5.3 to show that $\ell_t(t) > 0$ once t is sufficiently close to T. Lemma 5.2 then tells us that $\beta(t) > 2$ for t sufficiently close to T.

5.3 The Support Function

As a prelude to showing that ℓ_t changes sign finitely many times, we discuss some of the geometry of the curve $C(t)$.

We introduce the *support function*

$$p(\theta, t) = C(\theta, t) \cdot \mathbf{n}(\theta), \qquad \mathbf{n}(\theta) = (\sin(\theta), \cos(\theta)). \tag{28}$$

Note that the normal vector $\mathbf{n}$ is independent of the time variable since we are parametrizing in terms of the tangent angle θ.

Lemma 5.4.

$$C(\theta, t) = p(\theta, t)\mathbf{n}(\theta) + p_\theta(\theta, t)\mathbf{n}_\theta(\theta). \tag{29}$$

Moreover $\mathbf{n}$ *is the outward normal vector field with respect to* $C(t)$*.*

Proof. We suppress the dependence on the t variable. Let Q denote the curve on the right hand side of Equation 29. We want to see that $C = Q$. This follows from the uniqueness of first order differential equations and three facts:

1. C and Q have the same dot product with $\mathbf{n}$ at each point. This comes from the fact that $\mathbf{n}$ and $\mathbf{n}_\theta$ form an orthonormal basis.

2. The tangent lines to C and Q are parallel at each point. To see this, we suppress the θ variable and compute

$$Q_\theta = p_\theta \mathbf{n} + p\mathbf{n}_\theta + p_{\theta\theta}\mathbf{n}_\theta + p_\theta \mathbf{n}_{\theta\theta} = (p + p_{\theta\theta})\mathbf{n}_\theta.$$

   The reason for the cancellation of two terms is that $\mathbf{n}_{\theta\theta} = -\mathbf{n}$.

3. $C(\pi/2) = Q(\pi/2)$, giving the same initial condition for a first order ODE defining these curves. To see this equality, note that $p(\pi/2) = X(t)$ and $p_\theta(\pi/2) = 0$ by symmetry. Hence

$$Q(\pi/2) = X(t)(1, 0) = (X(t), 0) = C(\pi/2).$$

   Here $X = X(t)$ is half the width of the bounding box for $C(t)$, as before.

We have already computed that the tangent line to $C(\theta)$ is parallel to $\mathbf{n}_\theta$. Hence $C(\theta)$ is normal to $\mathbf{n}$ at each point. Since $\mathbf{n}(\pi/2) = (1, 0)$, we have the outward normal rather than the inward normal. □

5.4 The Parabolic Rescaling

The method here is an adaptation of an idea in Angenent's paper [4]. We introduce another new variable τ, which is related to t as follows:

$$\tau = \log \frac{1}{T - t}, \qquad t = T - e^{-\tau}. \tag{30}$$

Note that $\tau \to +\infty$ corresponds to $t \to T$.

We introduce the parabolic rescaling:

$$D(\theta, \tau) = e^{\tau/2} C(\theta, T - e^{-\tau}) \tag{31}$$

Next, we introduce the *node function*

$$\nu(\theta, \tau) = D_\tau(\theta, \tau) \cdot \mathbf{n}(\theta). \tag{32}$$

This quantity measures the component of the velocity of the curve $\tau \to D(\theta, \tau)$ in the normal direction. Angenent calls points where $\nu(\theta, \tau) = 0$ *nodes* and proves results about how the number of such is monotone non-increasing with time. We take the same approach.

Lemma 5.5. *For corresponding times t, τ, we have $\ell_t(t) = 0$ iff $\nu(\pi/2, \tau) = 0$.*

Proof. We compute

$$D(\theta, \tau) = e^{\tau/2}(C(\theta, t)) = \frac{1}{\sqrt{T - t}} C(\theta, t), \qquad t = T - e^{-\tau}. \tag{33}$$

Hence $D(\tau)$ is a reparametrization for $C(t)/\sqrt{T - t}$.

The support function for D is

$$P(\theta, \tau) = e^{\tau/2} p(\theta, t), \qquad t = T - e^{-\tau}. \tag{34}$$

The function $P_\tau(\pi/2, \tau)$ describes the velocity of the point $D(\pi/2, \tau)$. This is zero if and only if the velocity of the point

$$\frac{1}{\sqrt{T - t}} C(\pi/2, t) = \frac{X(t)}{\sqrt{T - t}}$$

is zero, because D and C are just reparametrizations of each other and the time variables are related by an orientation-preserving diffeomorphism. Finally, note that $\nu(\pi/2, \tau) = P_\tau(\pi/2, \tau)$. □

5.5 Finitely Many Sign Changes

There are two things that we need to know about the node function ν. We let $K = K(\theta, \tau)$ be the curvature of $D(\theta, \tau)$. Then:

$$\nu = \frac{P}{2} - K. \tag{35}$$

$$\nu_\tau = K^2 \nu_{\theta\theta} + (K^2 + 1/2)\nu. \tag{36}$$

We will derive these in the next section. Equation 36 is a strictly parabolic equation in the sense of Equation 6. In particular, the Sturmian Principle applies to it. To apply the Sturmian Principle effectively, we need to know that nodes cannot "leak in" from the boundary of the spacetime domain. Let us finish the proof assuming that $\nu < 0$ near the spacetime boundary. Then we will justifiy this claim.

Since D is analytic, and ν is negative near the boundary of the domain, there is a finite number $N(\tau)$ of points where $\nu(*, \tau)$ vanishes. By the Sturmian Principle, applied to domains whose vertical sides are contained entirely in the regions near the boundary where $\nu < 0$, we see that $N(\tau)$ is non-increasing with time and $N(\tau)$ drops by at least 2 at any time τ where the function $\nu(*, \tau)$ vanishes to at least second order at some point. The function $\nu(*, \tau)$ is invariant with respect to the reflection $\theta \to \pi - \theta$. Hence if this function vanishes at $\pi/2$, it vanishes to at least second order. This means that $N(\tau)$ drops whenever $\nu(\pi/2, \tau) = 0$. Hence this can happen at most finitely many times. Lemma 5.5 now tells us that $\ell_t(t)$ can vanish at most finitely many times as $t \to T$.

Now we justify the negativity claim.

Lemma 5.6. *For any value of τ, we have $\nu(\theta, \tau) < 0$ as long as θ is sufficiently close to $-\alpha(t)$ or $\pi + \alpha(t)$. Here t and τ are related as above.*

Proof. By symmetry it suffices to analyze the situation near $-\alpha(t)$. We fix τ and suppress it from our notation. Let s denote the arc length along D chosen so that $s = 0$ corresponds to $\theta = -\alpha$. Let T_s denote the unit tangent vector to D at s, chosen so that the first coordinate is positive.

Consider the situation at a point corresponding to $s < \sqrt{2}$ along D. Once τ is sufficiently large, the corresponding arc will lie in the positive quadrant. Let a denote the angle between T_s and $D(s)$. We estimate P. Note that $P > 0$ by convexity. We have

$$P(s) = D(s) \cdot \mathbf{n}(s) = \|D(s)\| \sin(a) < s \sin(a) < sa,$$

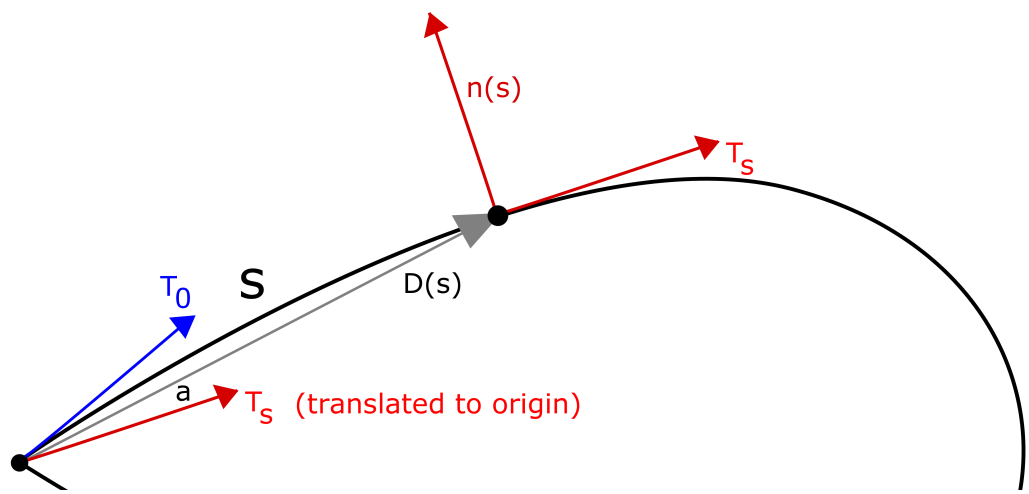

Figure 4: The relevant vectors

By convexity, the vector $D(s)$ lies in the sector bounded by T_0 and T_s. Hence a is less than the angle $\overline{a}$ between T_0 and T_s. At the same time, $\overline{a}$ is exactly the integral of the curvature with respect to arc length along the arc joining $D(0)$ to $D(s)$:

$$a < \overline{a} = \int_0^s K(\sigma) d\sigma < sK(s).$$

The last inequality comes from the fact that K is monotone increasing as a function of arc length. Putting our two estimates together we have

$$P(s) < s^2 K(s).$$

Hence

$$\nu(\theta, \tau) = \frac{P(s)}{2} - K(s) < \frac{s^2}{2} K(s) - K(s) < 0.$$

The last inequality comes from taking $s < \sqrt{2}$. □

Remark: Lemma 5.6 makes good sense geometrically. As τ increases, the curves $D(\tau)$ are becoming long and thin. So, inevitably, this curve must have at least one node in $(-\alpha(t), \pi/2)$. Here t and τ are corresponding times. This picture is consistent with ν being negative at one end and positive at the other.

5.6 Derivations

In this section we derive Equations 35 and 36. We need to compute some auxiliary quantities along the way.

Lemma 5.7.

$$P_\tau = \frac{P}{2} - K. \tag{37}$$

Proof. Using the fact that **n** does not depend on time, and is the outward normal, we compute

$$p_t = \frac{d}{dt}\bigg(C(t)\cdot \mathbf{n}\bigg) = -\kappa \mathbf{n}\cdot\mathbf{n} = -\kappa. \tag{38}$$

Now we set $p = p(T - e^{-\tau})$ and use the product and chain rule to compute

$$P_\tau = \frac{d}{d\tau}\bigg(e^{\tau/2}p\bigg) = (1/2)e^{\tau/2}p - e^{-\tau}e^{\tau/2}\kappa = \frac{P}{2} - e^{-\tau/2}\kappa = \frac{P}{2} - K.$$

This does it. □

Lemma 5.8 (Equation 35)**.**

$$\nu = \frac{P}{2} - K.$$

Proof. Suppressing the arguments, we have

$$D = P\mathbf{n} + P_\theta \mathbf{n}_\theta. \tag{39}$$

Hence

$$\nu = D_\tau \cdot \mathbf{n} = (P_\tau \mathbf{n} + P_{\theta\tau}\mathbf{n}_\theta)\cdot \mathbf{n} = P_\tau = \frac{P}{2} - K.$$

This does it. □

Lemma 5.9.

$$P + P_{\theta\theta} = \frac{1}{K}. \tag{40}$$

Proof. Here is the formula for the signed curvature of a parametrized plane curve.

$$K = \pm\frac{D_\theta \times D_{\theta\theta}}{\|D_\theta\|^3} \tag{41}$$

The ambiguity in the sign comes from the fact that K is alway taken to be positive. Using this equation for the parametrization given in Equation 39, we get Equation 40 up to sign. To get the sign in Equation 40 we note that the sign is correct in the special case D is the unit circle, parametrized in a clockwise way. But then, since we are parametrizing D in a clockwise way, the sign is correct in the arbitrary case. □

Lemma 5.10.

$$K_\tau = -\frac{K}{2} + K^2 K_{\theta\theta} + K^3. \tag{42}$$

Proof. Using the chain rule, and setting $\kappa = \kappa(T - e^{-\tau})$ we have

$$K_\tau = \frac{d}{d\tau} e^{-\tau/2}\kappa =$$

$$-(1/2)e^{-\tau/2}\kappa - e^{-\tau/2}e^{-\tau}\kappa_t =$$

$$-(1/2)K + e^{-3\tau/2}(\kappa^2\kappa_{\theta\theta} + \kappa^3) =$$

$$-\frac{K}{2} + K^2 K_{\theta\theta} + K^3.$$

This does it. □

Lemma 5.11 (Equation 36)**.**

$$\nu_\tau = K^2\nu_{\theta\theta} + (K^2 + 1/2)\nu.$$

Proof. Therefore, we have

$$\nu_\tau = \frac{P_\tau}{2} - K_\tau =$$

$$(P/4 - K/2) - (-K/2 + K^2 K_{\theta\theta} + K^3) =$$

$$-K^2 K_{\theta\theta} - K^3 + \frac{\nu + K}{2} \tag{43}$$

Therefore

$$\nu_{\theta\theta} + \nu = \frac{P_{\theta\theta} + P}{2} - \big(K_{\theta\theta} + K\big) = \frac{1}{2K} - \big(K_{\theta\theta} + K\big)$$

Multiplying through by K^2 we get

$$-K^2 K_{\theta\theta} - K^3 + \frac{K}{2} = K^2(\nu + \nu_{\theta\theta})$$

Thus:

$$\nu_\tau = -K^2 K_{\theta\theta} - K^3 + \frac{\nu + K}{2} = K^2(\nu_{\theta\theta} + \nu) + \frac{\nu}{2} = K^2\nu_{\theta\theta} + (K^2 + 1/2)\nu.$$

This does it. □

6 The Bowtie Theorem

The only detail missing in the proof of the Bowtie Theorem is the Migration Lemma, which we now prove.

Let $\Gamma(t) = C(t)/X(t)$. The bounding box for $\Gamma(t)$ is

$$[-1,1] \times [-H(t), H(t)], \qquad H(t) = \frac{Y(t)}{X(t)}.$$

Let $x(P)$ and $y(P)$ respectively denote the x and y coordinates of a point P.

Lemma 6.1. *Let $\delta > 0$ be given. Then $\lim_{t\to T} x(\Gamma(\delta,t)) = 1$.*

Proof. By convexity, the arc of $\Gamma(t)$ connecting $\Gamma(\delta,t)$ to $\Gamma(\pi/2,t) = (1,0)$ lies inside the solid right triangle bounded by the x-axis, the tangent line to $\Gamma(t)$ at $\Gamma(\delta,t)$, and the vertical line through $\Gamma(\delta,t)$. But the horizontal side of this triangle has length $H(t)/\tan(\delta)$. Hence

$$x(\Gamma(\delta,t)) > 1 - \frac{H(t)}{\tan(\delta)},$$

a quantity which tends to 1 as $t \to T$. □

Lemma 6.2. *Let $\delta_t > 0$ denote the value such that $\Gamma(\delta_t,t)$ has y-coordinate equal to $H(t)/2$. Then $\lim_{t\to T} x(\Gamma(\delta_t,t)) = 1$.*

Proof. The Grim Reaper Curve $G = G(\theta)$ has maximum curvature 1, and it occurs at $G(\pi/2)$, a point on the x-axis. The total height of G is π. Hence there is some value $\delta > 0$ such that

$$y(G(\delta)) = \pi/3.$$

Define the rescaling

$$G^*(t) = C(t) \times \frac{\pi/2}{Y(t)} = \Gamma(t) \times \frac{\pi/2}{H(t)}.$$

By the middle formula in Equation 5 and the Grim Reaper Theorem together, $G^*(t)$ converges uniformly to G (modulo horizontal translations) on $[\delta,\pi/2]$. Hence

$$\lim_{t\to T} y(G^*(\delta,t)) = \pi/3, \qquad \lim_{t\to T} \frac{y(G^*(\delta,t))}{\pi/2} = 2/3.$$

The second equation is just a reformulation of the first. Recaling, we have

$$\lim_{t\to T} \frac{y(\Gamma(\delta,t))}{H(t)} = 2/3.$$

But then $\delta_t > \delta$ for t sufficiently close to T. Now apply the previous result. □

If the Migration Lemma is false, there is some $\eta > 0$ and a sequence of times $t_n \to T$ such that

$$\lim_{n\to\infty} L_{t_n}(C(0,t_n)) = (1-\eta, 1). \tag{44}$$

Let $\delta_n = \delta_{t_n}$ be the constant from Lemma 6.2. Let $H_n = H(t_n)$. If we scale the y-coordinate by $1/H_n$ (and do nothing to the x-coordinate) we map $\Gamma(t_n)$ to $L_{t_n}(C(t_n))$. Therefore, Lemma 6.2 implies that

$$\lim_{n\to\infty} L_{t_n}(C(\delta_n,t_n)) = (1, 1/2). \tag{45}$$

Combining Equation 44, Equation 45, convexity, and symmetry, we see that the right lobe of $L_{t_n}(C(t_n))$ bounds a convex polygon which converges in the Hausdorff metric to the polygon with vertices

$$(0,0), \quad (1-\eta,1), \quad (1,1/2), \quad (1,-1/2), \quad (1-\eta,-1).$$

But this polygon has area $1+(\eta/2)$. This contradicts the fact that the area of the region bounded by the right lobe of $L_{t_n}(C(t_n))$ converges to 1.

References

[1] Abresch, U., Langer, J. *The Normalized Curve Shortening Flow and Homothetic Solutions*. J. Diff. Geo. 23. (1986) 175-196.

[2] Angenent, S. *On the Formation of Singularities in the Curve Shortening Flow*. J. Diff. Geo. 33. (1991) 601-633.

[3] Angenent, S. *The Zero Set of a Solution of a Parabolic Equation*. J. Reine Angew. Math. 390. (1988) 79-96

[4] Angenent, S. *Parabolic Equations for Curves on Surfaces (Part II)*. Annals of Math. Vol. 133. No.1. pp. 171-215. (1991).

[5] Angenent, S. Velazquez, J.J. *Asymptotic Shape of Cusp Singularities in Curve Shortening*. Duke Math. J. Vol. 77. No. 1. (1995).

[6] Friedman, A, and McLeod, B *Blow-up solutions of nonlinear degenerate parabolic equations*, Arch. Rational. Mech. Anal **96** (1986) 55-80

[7] Coiculescu, M.P. *Some New Results in Geometric Analysis*. Mathematics Theses and Dissertations. Brown Digital Repository. Brown University LIbrary. (2021) https://doi.org/10.26300/6fjz-ax05

[8] Drugan, G., He, W., Warren, M.W. *Legendrian curve shortening in* $\mathbf{R}^3$ Commun. Anal. Geo. Vol. 26. No. 4. (2018) pp. 759-785.

[9] Epstein, C. L. and Weinstein, M. I., *A Stable Manifold Theorem for the Curve Shortening Equation*, Communications in Pure and applied Mathematics, **40** (1) (1987) pp. 119-139

[10] Evans, L.C. *Partial Differential Equations*. 2nd Edition. Graduate Studies in Mathematics. Vol. 19. American Mathematical Society. (2010)

[11] Gage, M., Hamilton, R.S. *The Heat Equation Shrinking Convex Plane Curves*. J. Diff. Geo. 23. (1986) 69-96.

[12] M. Gage, *An Isoperimetric inequality with applications to curve shortening* Duke Math J. **50** no. 4 (1983) pp 1225 1229

[13] M. Gage, *Curve Shortening Makes Convex Curves Circular*, Invent Math. **76** (1984) 357-364.

[14] Grayson, M.A. . *The Heat Equation Shrinks Embedded Curves to Round Points.* J. Diff. Geo. Vol. 26, No.2. (1987)

[15] Grayson, M.A. *The Shape of a Figure-Eight under the Curve Shortening Flow.* Invent. Math. 96, (1989) 177-180.

[16] Halldorsson, H.P. *Self-Similar Solutions to the Curve Shortening Flow*, Transactions of the American Mathematical Society. Vol. 364. No. 10. (2012)

[17] Sturm, C. *Mémoire sur une classe d'équations à différences partielles.* J. Math. Pures Appl. (1836) 373-444.