The Poncelet Grid

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Abstract

Given a convex polygon P in the projective plane we can form a finite "grid" of points by taking the pairwise intersections of the lines extending the edges of P. When P is a Poncelet polygon we show that this grid is contained in a finite union of ellipses and hyperbolas and derive other related geometric information about the grid.

1 Introduction

Poncelet's porism is a classical result in algebraic geometry. Some good references for this result are [**B**], [**BKOR**], [**H**], [**GH**], and [**T**]. The porism deals with polygons P which are superscribed about a conic E_0 and simultaneously inscribed in another conic E_1. This is to say that the edges of P are all tangent to E_0 and the vertices of P are all contained in E_1. Figure 1.1 shows an example, in which E_0 is a circle and E_1 is an ellipse.

We call P a *Poncelet polygon* and we say that P is *defined relative to* (E_0, E_1). Poncelet's porism may be phrased as follows: P includes in a continuous family of Poncelet polygons, all defined relative to (E_0, E_1). Looking at Figure 1.1, we can imagine that P is able to "rotate around" continuously (and slightly changing shape as it "rotates") so as to remain superscribed about E_0 and inscribed in E_1. My Java applet [**S**] shows an animated version of this.

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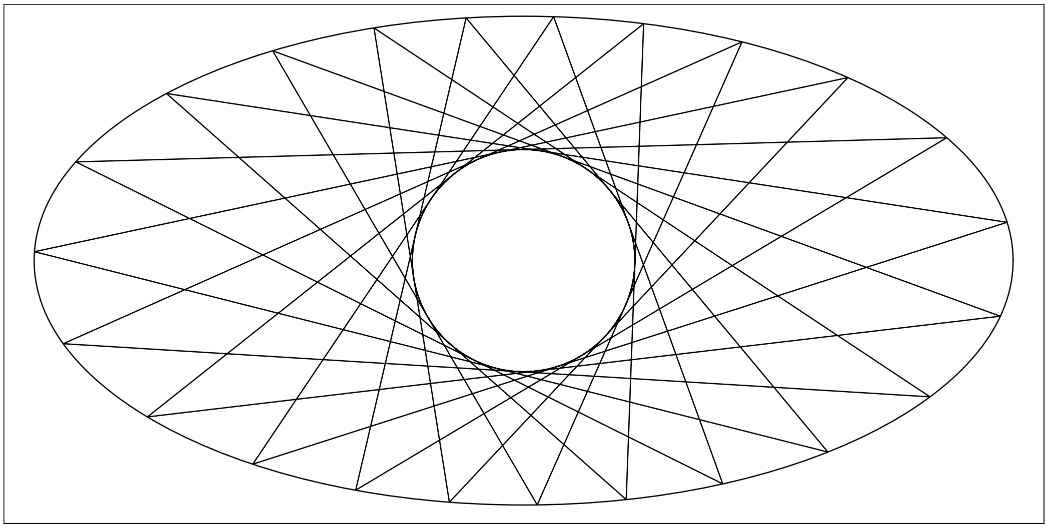

Figure 1.1

At first it might seem that Poncelet's porism is fairly trivial: Perhaps there is a 1-parameter subgroup of projective transformations which simultaneously stabilizes E_0 and E_1; then each polygon in the "rotating family" is in the orbit of each other one. What makes Poncelet's porism deep is that this is typically not the case. The well-known proof of Poncelet's porism involves the familiar but deep fact that elliptic curves over $\boldsymbol{C}$ may be uniformized so as to have flat Riemannian metrics. An excellent account, with historical references, is given in [**GH**]. In §2 we will sketch the main points of the proof. Compare [**T**, §5] for a different point of view.

One can define a certain collection of points in the projective plane based on P. Let $l_1, ..., l_n$ be the lines which contain the edges of P. We define the *Poncelet grid* to be the union G of points $\{l_i \cap l_j\}$. When $i = j$ we define $l_i \cap l_j$ to be the tangency point $l_i \cap E_0$. Many but not all of the points of G are shown in Figure 1.1. The purpose of this article is to investigate the properties of the Poncelet grid. For ease of exposition we only consider the case when $n \geq 3$ is odd. The even case is slightly different but similar results obtain.

Our main results refer to Figure 1.2, shown on the next page. We discovered these results experimentally, in part using Mathematica [**W**]. Figure 1.2 shows a decorated version of G. The outer conic E_1 is not shown; from our point of view, it represents an artificial truncation of the Poncelet grid. A network of ellipses and hyperbolas overlays G. Our main results is that

G is related to the network of ellipses and hyperbolas exactly as it appears to be. Each grid point is the intersection of an ellipse and a hyperbola. The obvious collection of quadrilaterals is colored [1] for a checkerboard effect.

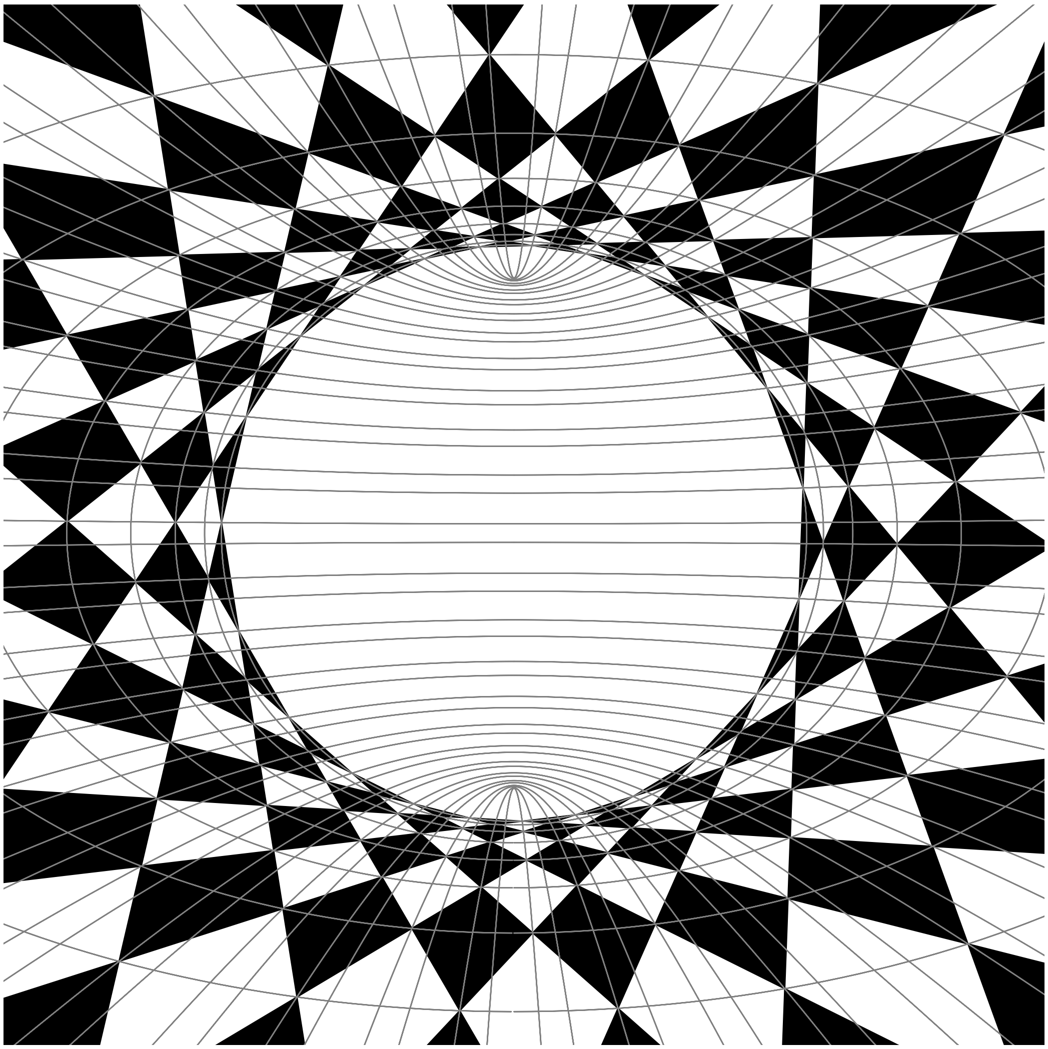

Figure 1.2

[1] One technical point about the coloring: In order to make the coloring consistent for the whole grid, the quadrilaterals which intersect the line at infinity need to be treated specially. The color in these quadrilaterals needs to switch from white to black, or from black to white, when the line at infinity is crossed. The reader can see this by experimenting, for instance, with the case $n = 3, 5$

It is convenient to think of our result in terms of an *angular part* and a *radial part*. The angular part was essentially known to Poncelet; we think that the radial part is new. The angular part of our result says that G is contained on $(n+1)/2$ ellipses, with n points per ellipse. The intersection $P_j = G \cap E_j$ is the vertex set of another Poncelet polygon, defined relative to the pair (E_0, E_j). Thus, the initial conic E_1 is nothing special. Any of the E_j could have been used to define G. As it turns out, G has a symmetry: For all i, j there is a projective transformation which carries P_i to P_j.

The *radial part* of our result says that G is contained in a finite union $\{H_j\}$ of hyperbolas. In the generic case there are n hyperbolas, each with $(n+1)/2$ points on them. This is the case we will discuss in detail. Let $Q_j = G \cap H_j$. We call Q_j a *radial arm*. It turns out that Q_j is the vertex set of another Poncelet polygon, even though Q_j has essentially half the number of vertices as P. Moreover, the Poncelet grid has a second symmetry. For all i, j there is a projective transformation which carries Q_i to Q_j, as long as neither collection of points is collinear. That is, all the radial arms of a Poncelet grid (generically) are projectively equivalent to each other.

The second symmetry has a neat extension: Recall that the Poncelet porism guarantees the existence of an infinite number of Poncelet polygons defined relative to the pair (E_0, E_1). Each one of these other polygons leads to a Poncelet grid G'. Typically G and G' are not projectively equivalent to each other. Nonetheless, each (generic) radial arm of G is projectively equivalent to each (generic) radial arm of G'. As the Poncelet grid "rotates around", the projective classes of its radial arms (generically) remain constant!

Here is a formal summary of our results:

Theorem 1.1 *Let G be the Poncelet grid associated to any Poncelet polygon defined in terms of two nested ellipses E_0 and E_1. There is a finite family $\{E_j\}$ of ellipses and a finite family $\{H_j\}$ of hyperbolas such that $G \subset \bigcup E_j$ and also $G \subset \bigcup H_j$. The intersections $P_j = E_j \cap G$ and $Q_j = H_j \cap G$ are vertex sets of new Poncelet polygons. All the Ps are projectively equivalent to each other and all the Qs are projectively equivalent to each other. Finally, there is a set S of 4 complex lines such that the complexified versions of E_j and H_j are tangent to the lines of S for all j.*

In §2 we will sketch the classic proof of Poncelet's porism. Following this, we will give some further analysis of the basic construction. In §3 we will prove Theorem 1.1, step by step.

2 Poncelet's Porism

2.1 Basic Ideas

Let E_0 and E_1 be two ellipses contained in $\boldsymbol{P}$, the real projective plane. Each ellipse bounds a disk on one side and a Mobius band on the other. We assume that the disk bounded by E_1 contains E_0 in its interior. We normalize by a projective transformation so that E_0 is the unit circle and

$$E_1 = \{(x, y) |\ (x/a)^2 + (y/b)^2 = 1\}, \tag{1}$$

where $a > b > 1$. We ignore the case $a = b$ because it is trivial.

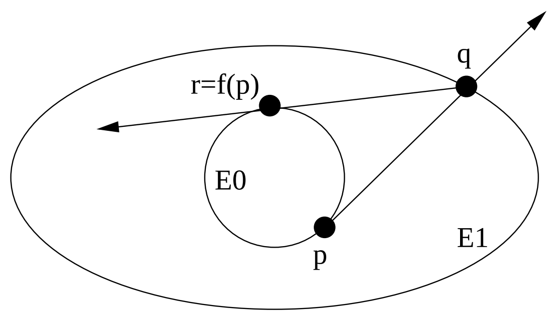

Figure 2.1

We give E_0 the counterclockwise orientation. At each point $p \in E_0$ the tangent ray to E_0 at p intersects E_1 in a point q. There is a unique point $r \in E_0$ such that $r \neq p$ and the ray $\overrightarrow{qr}$ contains the tangent ray to E_0 at r. We define $f(p) = r$. The point p is contained in an edge of a Poncelet n-gon if and only if $f^n(p) = p$. Poncelet's porism says, given two points $p, p' \in E_0$, we have $f^n(p) = p$ if and only if $f^n(p') = p'$. This result is proved by showing that f is conjugate to a rotation of a circle.

Here we sketch (a variant of) the classical argument. Let $\boldsymbol{P}(\boldsymbol{C})$ be the complex projective plane. Let $E_0(\boldsymbol{C})$ and $E_1(\boldsymbol{C})$ be the conics in $\boldsymbol{P}(\boldsymbol{C})$ which extend E_0 and E_1. There are 4 complex lines which are simultaneously tangent to $E_0(\boldsymbol{C})$ and $E_1(\boldsymbol{C})$.

Lemma 2.1 *The lines simultaneously tangent to E_0 and E_1 have the form $\pm icx \pm dy = 1$, where*

$$c^2 = \frac{b^2 - 1}{a^2 - b^2}; \qquad d^2 = c^2 + 1. \tag{2}$$

Proof: Let $\boldsymbol{P}^*(\boldsymbol{C})$ denote the dual projective plane. There is a projective duality $\delta : \boldsymbol{P}(\boldsymbol{C}) \to \boldsymbol{P}^*(\boldsymbol{C})$ which carries the line $ax + by = 1$, considered as a subset of $\boldsymbol{P}(\boldsymbol{C})$, to the point $(a, b) \in \boldsymbol{P}^*(\boldsymbol{C})$. Let $S(E_j)$ denote the set of lines tangent to $E_j(\boldsymbol{C})$. Then $\delta(S(E_0))$ is the conic having equation $x^2 + y^2 = 1$ and $\delta(S(E_1))$ is the conic having equation $(ax)^2 + (by)^2 = 1$. The intersection points of these dual conics are $(\pm ic, \pm d)$, where c and d are as in Equation 2. The result follows immediately. ♠

Let T denote the set of pairs (q, l) where $q \in E_1(\boldsymbol{C})$ and l is a complex line containing q and tangent to $E_0(\boldsymbol{C})$. We define $\pi : T \to E_0(\boldsymbol{C})$, by the equation $\pi(q, l) = l \cap E_0(\boldsymbol{C})$. The complex line l_p, tangent to $E_0(\boldsymbol{C})$ at p, intersects $E_1(\boldsymbol{C})$ in either 1 or 2 points, depending on whether or not l_p is also tangent to $E_1(\boldsymbol{C})$. Hence π is a double branched cover, branched at 4 points. This forces T to be a torus. The inclusion $T \hookrightarrow P(\boldsymbol{C})$ gives a complex structure on T in which π is a holomorphic homeomorphism.

Like all complex tori, T has a Euclidean metric, unique up to scale, in which all holomorphic and anti-holomorphic self-homeomorphisms are isometries. This is the uniformization theorem for elliptic curves. There are two natural involutions on T:

- We have $i_1(q, l) = (q, l')$, where (generically) l' is the other line through q which is tangent to $E_0(\boldsymbol{C})$. The map i_1 has 4 fixed points; these are pairs (q, l) where $q \in E_0(\boldsymbol{C}) \cap E_1(\boldsymbol{C})$.

- We have $i_2(q, l) = (q', l)$ where (generically) q' is the other point of $l \cap E_1(\boldsymbol{C})$. The map i_2 has 4 fixed points; these are the pairs (q, l) where l is tangent to both $E_0(\boldsymbol{C})$ and $E_1(\boldsymbol{C})$.

The fixed points of i_1 are completely distinct from the fixed points of i_2. Hence $\widetilde{f} = i_1 \circ i_2$ acts as a translation−i.e. with no fixed points.

$\pi^{-1}(E_0)$ consists of two circles, $\widetilde{E}_0$ and $\widetilde{E}_0'$. We label so that $\widetilde{E}_0$ consists of elements (q, l) where l contains a ray tangent to E_0, pointing in the counterclockwise direction, and q is on this ray. (Compare Figure 2.1.) The map π intertwines the action of $\widetilde{f}$ on $\widetilde{E}_0$ with the action of f on E_0. Complex conjugation preserves both our conics and hence induces an anti-holomorphic isometry of T. The fixed point set is exactly $\pi^{-1}(E_0)$. Since $\widetilde{E}_0$ is one component of the fixed point set of an isometry, $\widetilde{E}_0$ is a closed geodesic on T. All in all, $\widetilde{f}$ is a free isometry of $\widetilde{E}_0$, which is to say, a rotation. This proves that f is conjugate to a rotation.

2.2 Further Analysis

Let's sharpen the picture of T, the torus defined in the previous section. The two circles $\widetilde{E}_0$ and $\widetilde{E}'_0$ are obviously disjoint. Hence they bound a pair of annuli. These annuli are interchanged by the anti-holomorphic isometry induced from complex conjugation. Hence T is obtained by doubling an annulus across its geodesic boundary. An annulus with geodesic boundary can be obtained from a rectangle, with the top and bottom sides identified in the obvious way. Hence, T can be obtained by gluing together the opposite sides of a rectangle R, as shown in Figure 2.2.

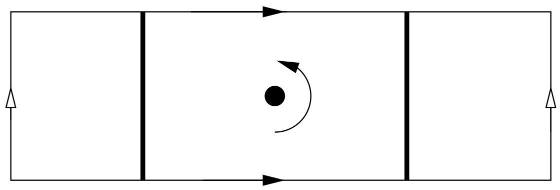

Figure 2.2

$\widetilde{E}_0$ and $\widetilde{E}'_0$ are the images of two evenly spaced vertical line segments, under the identification map. Since $\widetilde{f}$ preserves these sets, $\widetilde{f}$ is a vertical translation. The map π has the property that $\pi \circ \rho = \pi$, where ρ is the involution of T induced by the map which rotates 180 degrees about the center of R.

We normalize R to have height 1. The width of R determines the complex structure on T, and *vice versa*. At the same time, the complex structure on T is determined by the location of the branch points on $E_0(\boldsymbol{C})$. These branch points are determined by the value of c, given in Equation 2. We write $R = R(c)$ to denote the dependence of R on c.

Lemma 2.2 *The map $c \to R(c)$ is injective.*

Proof: Define

$$X = E_0(\boldsymbol{C}) \cap \Pi; \qquad \Pi = \{(ix, y)|\ x, y \in \boldsymbol{R}\}.. \tag{3}$$

The branch points of the map $T \to E_0(\boldsymbol{C})$ are $(\pm ic, \pm d) \in X$. There is a biholomorphic map $\beta : E_0(\boldsymbol{C}) \to \boldsymbol{C} \cup \infty$, the Riemann sphere. The complex structure on T is completely determined by the cross ratio of the branch points in $E_0(\boldsymbol{C})$. That is, the complex structure is determined by the cross ratio of the points $\beta((\pm ic, \pm d))$. We just need to show that this cross ratio is monotonic in c.

The map $r(z,w) = (\overline{z}, -\overline{w})$ is an anti-holomorphic automorphism of $E_0(\boldsymbol{C})$ which preserves X. Thus $\beta(X)$ is the fixed point set of the anti-holomorphic map $\beta \circ r \circ \beta^{-1}$. Hence $\beta(X)$ is a circle. Adjusting β, we can assume that $\beta(X)$ is the unit circle. The branch points of $E_0(\boldsymbol{C})$ are permuted by the dihedral group D whose elements have the form $(z,w) \to (\pm z, \pm w)$. Further adjusting β we can assume that β conjugates D to the dihedral dihedral group D_β generated by $z \to \overline{z}$ and $z \to -z$.

The set of branch points $\beta((\pm ic, \pm d))$ is invariant under D_β and hence must be a rectangle R_c' whose sides are parallel to the coordinate axes. The cross ratio of the vertices of R_c' is monotonic in the modulus–i.e. the width-to-length ratio. But the modulus of R_c' is monotonic in c because each choice of c gives rise to a different rectangle R_c'. ♠

We define $\Delta = \{(a,b) |\ a > b > 1\}$, and think of c as a function from Δ to $\boldsymbol{R}^+$. From Equation 2 we see that the fiber of c in the (a,b)-plane is the hyperbola

$$h_c = \{(a,b) |\ (c^2+1)b^2 - c^2a^2 = 1\}. \tag{4}$$

h_c contains the point $(1,1)$, and has asymptotes $b = \pm ka$ with $|k| < 1$. Here $k = c^2/(c^2+1)$. Hence

- If $c(a,b) = c(a',b')$ then $a - a'$ and $b - b'$ have the same sign.
- Each fiber of c contains pairs (a,b) arbitrarily close to the pair $(1,1)$.
- Each fiber of c contains pairs (a,b) arbitrarily close to the pair (∞,∞).

Interpreting these statements geometrically, and using the fact that each fiber of c is a smooth curve, we get the following result:

Lemma 2.3 *For each $c \in \boldsymbol{R}^+$, there is a foliation P_c of the outside of the unit disk. Each leaf of P_c is an ellipse E_1. The rectangle $R = R(E_1)$ is independent of the choice of E_1 within the foliation. Moreover, if E_1 and E_1' are two ellipses such that $R(E_1) = R(E_1')$, then there is a constant c such that E_1 and E_1' both belong to the foliation P_c.*

Proof: The foliaion P_c consists precisely in the ellipses E_1, given by Equation 1, where $a, b > 1$ lie on the hyperbola h_c. The rest of the lemma follows from the properties of h_c mentioned above. ♠

3 The Grid

3.1 Some General Constructions

By *dihedral group* we mean the group D whose elements have the form $(x,y) \to (\pm x, \pm y)$. Let μ be a D-invariant unit length Riemannian metric on E_0. Let $p \in E_0$ and let $n \geq 3$ be some odd integer. There is a unique collection $p_1, ..., p_n = p$ of points which are evenly spaced with respect to μ. Let l_j be the line in $\boldsymbol{P}$ which is tangent to E_0 at p_j. We take indices cyclically, so that $l_{n+k} = l_k$. We define

$$G(\mu, n, p) = \bigcup_{i,j} l_i \cap l_j. \tag{5}$$

When $i = j$ we define $l_i \cap l_j$ to be the tangency point p_i. Thus G consists of $n(n+1)/2$ points.

We distinguish two kinds of subsets of G.

$$P_j = \bigcup_{i=1}^{n} l_i \cap l_{i+j}; \qquad Q_j = \bigcup_{i=1}^{n} l_{j-i} \cap l_{j+i}. \tag{6}$$

Our definitions have some redundancy. First, $P_j = P_{n-j}$ for all j. Second, Q_j only has $(n+1)/2$ distinct points.

Let D_0 be the unit disk. Given $q \in \boldsymbol{P} - D_0$, there are two lines l_1 and l_2 containing q and tangent to E_0. These two lines are tangent to E_0 at points r_1 and r_2. The complement $E_0 - r_1 - r_2$ consists of two arcs I_1 and I_2. Let $|I_j|$ be the μ-length of I_j and let m_j be the μ-midpoint of I_j.

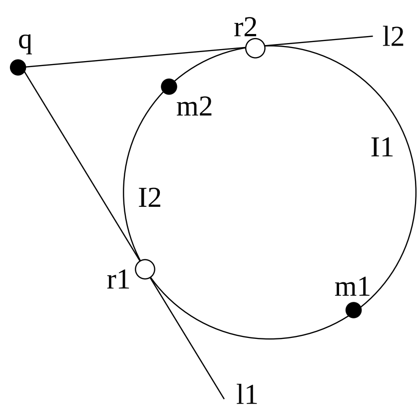

Figure 3.1

(In Figure 3.1 the points m_j are drawn off-center to emphasize the fact that the metric μ might not be the rotationally symmetric one.) Based on the constructions in Figure 3.1 we can define two natural maps on $\boldsymbol{P} - D_0$:

$$\lambda_1(q) = \min(|I_1|, |I_2|); \qquad \lambda_2(q) = \{m_1, m_2\}. \tag{7}$$

The range of λ_1 is $[0, 1/2]$. The range of λ_2 is the set $E_0/\pm$ of pairs of antipodal points on E_0. Both λ_1 and λ_2 extend continuously to E_0.

If E is any fiber of λ_1 then E is simple closed curve. We call E an *angular fiber*. There is a unique Riemannian metric on E which makes the map $\lambda_2 : E \to E_0/\pm$ a local isometry. We call this metric the *angular metric*. If H is any fiber of λ_2 then H is a simple arc with endpoints on E_0. We call H a *radial fiber*. There is a unique Riemannian metric on H which makes the map $\lambda_1 : H \to [0, 1/2]$ an isometry away from the points $\lambda_1^{-1}(\{0, 1/2\})$. We call this the *radial metric* on H.

We can associate some distinguished fibers to our grid G. The map λ_1 is constant on the sets P_j. We define E_j to be the fiber of λ_1 which contains P_j. The points of P_j are evenly spaced in the radial metric on E_j. The map λ_2 is constant on the sets Q_j and we define H_j to be the fiber of λ_2 which contains Q_j. The points of Q_j are evenly spaced in the Riemannian metric on H_j, as long as they are sufficiently far away from the endpoints of H_j.

3.2 Poncelet Metrics

As in §2 we have the branched covering $\pi : T \to E_0(\boldsymbol{C})$, which depends on the parameters $a > b > 1$. We define $\mu(a, b)$ to be the unit length analytic Riemannian metric on E_0 which makes π an isometry between $\widetilde{E}_0$ and E_0. We call $\mu(a, b)$ a *Poncelet metric* on E_0. Let $c = c(a, b)$ be as in Equation 2.

Lemma 3.1 *$\mu(a, b) = \mu(a', b')$ if and only if $c(a, b) = c(a', b')$.*

Proof: Suppose that $c(a, b) = c(a', b')$. Then $R(a, b) = R(a', b')$. Here R is the rectangle considered in §2.2. We can identify the tori T and T'. The maps π and π' have the same domain and satisfy the equations $\pi \circ \rho = \pi$ and $\pi' \circ \rho = \pi'$. This implies that $g = \pi' \circ \pi^{-1}$ is a well-defined holomorphic self-map of T. By symmetry g lies in the normalizer of the dihedral group. This forces g to be one of finitely many maps. Since g varies continuously with the parameters, g must be the identity. Hence $\pi = \pi'$ and $\mu(a, b) = \mu(a', b')$.

Suppose that $\mu(a,b) = \mu(a',b')$. Let t be the translation length of $\widetilde{f} : T \to T$ the holomorphic map considered in §2. Likewise define t'. By varying E_1 within the foliation F_c, an operation which does not change μ, we can arrange that $t = t'$. But then f and f' are the same rotation with respect to the common metric $\mu(a,b) = \mu(a',b')$. Hence $f = f'$. Let $p \in E_0$.

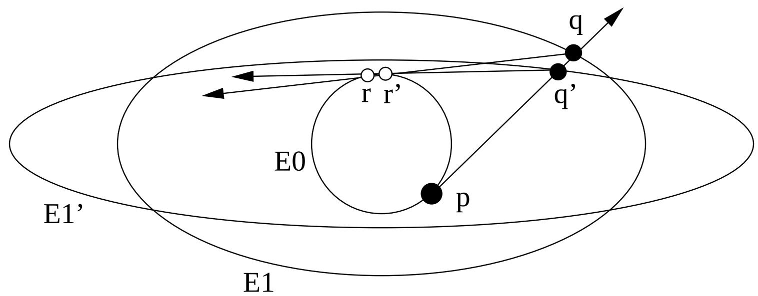

Figure 3.2

Let $q \in E_1$ and $q' \in E_1'$ be as in Figure 3.2. Then $r = f(p) = f'(p) = r'$. For typical point p this forces $q = q'$. Hence E_1 and E_1' have many intersections. Hence $E_1 = E_1'$. Hence $(a,b) = (a',b')$. ♠

Henceforth we write $\mu(c)$ instead of $\mu(a,b)$.

Lemma 3.2 *Suppose that E_0 is equipped with $\mu(c)$. Then the fibers of λ_1 are precisely the ellipses in the foliation P_c.*

Proof: Let E_1 be a generic ellipse in P_c. Let $f : E_0 \to E_0$ be the map from §2, defined relative to E_0 and E_1. Then f is an isometry relative to $\mu(c)$. If we take E_1 to be generic then f has infinite order and every orbit of f is dense. By construction λ_1 is constant on a dense set of points of E_1. hence λ_1 is constant on E_1. Hence E_1 is a fiber of λ_1. Since the fibers of λ_1 are disjoint and foliate the outside of E_0 we see that the foliation by fibers of λ_1 must equal P_c. ♠

Now we can relate our grids to Poncelet polygons. Let $\mu = \mu(c)$ and Let $G(\mu, n, p)$ be a grid based on a Poncelet metric μ. Then the set P_j defined in Equation 6 lies in a fiber E_j of λ_1, and the points of P_j are evenly spaced in the angular metric on E_j. Hence P_j is a Poncelet n-gon defined relative to (E_0, E_j). Independent of j we get the same grid. Conversely, and Poncelet n-gon defined relative to a pair (E_0, E_1), with E_1 in the foliation P_c, sets up a Poncelet grid of the form just discussed, for $\mu = \mu(c)$.

3.3 Symmetry of Angular Fibers

We let $\mu = \mu(c)$ be as in the previous section and we consider a grid $G(\mu, n, p)$. Let μ_j be the angular metric on E_j. In particular, $\mu = \mu_0$. Say that a linear map is *diagonal* if it is represented by a diagonal matrix, and *positive diagonal* if it is represented by a diagonal matrix with positive entries. There is a unique positive diagonal map $\delta_j : E_j \to E_0$.

Lemma 3.3 *$\delta_j(\mu_j) = \mu$.*

Proof: Without loss of generality we take $j = 1$. We set $\delta = \delta_1$. We use the notation from §2. Let $\widetilde{\mu}$ be the metric on $\widetilde{E}_0$, induced from the flat metric on T. The map $\pi : \widetilde{E}_0 \to E_0$ carries $\widetilde{\mu}$ to μ, the Poncelet metric on E_0. Let $\mu' = \delta(\mu_1)$. We would like to show that $\mu' = \mu$.

Recall that T is the set of pairs (q, l) where $q \in E_1(\boldsymbol{C})$ and l is a line containing q and tangent to $E_0(\boldsymbol{C})$. We define $\sigma : T \to E_1(\boldsymbol{C})$ by the formula

$$\sigma(q, l) = q. \tag{8}$$

An argument like one given in §2.1 shows that σ is a holomorphic double branched covering from T to $E_1(\boldsymbol{C})$, such that $\sigma(\widetilde{E}_0) = E_1$. The points of any Poncelet polygon defined relative to (E_0, E_1) are evenly spaced with respect to μ_1. This means that $\mu_1 = \sigma(\widetilde{\mu})$. Therefore

$$\mu' = \pi'(\widetilde{\mu}); \qquad \pi' = \delta \circ \sigma \tag{9}$$

The maps π and π' are both holomorphic branched coverings from T to $E_0(\boldsymbol{C})$. Both maps carry $\widetilde{E}_0$ to E_0. This means that $\pi' = \phi_2 \circ \pi \circ \phi_1$ where ϕ_1 is a vertical translation of T and ϕ_2 is a complex projective automorphism of $E_0(\boldsymbol{C})$. Now ϕ_1 is an isometry in the flat metric on T. That is $\phi_1(\widetilde{\mu}) = \widetilde{\mu}$. Hence $\pi'(\widetilde{\mu}) = \phi_2(\pi(\mu))$. Since $\phi_2(\pi(\mu))$ and $\pi(\mu)$ both have dihedral symmetry, ϕ_2 must act isometrically on E_0. Hence $\pi'(\widetilde{\mu}) = \pi(\widetilde{\mu})$. ♠

The following result summarizes the symmetry of the angular fibers of the Poncelet grid:

Lemma 3.4 *Let $\{E_i\}$ denote the collection of ellipses determined the grid $G(\mu, n, p)$, where μ is a Poncelet metric. Then the positive diagonal map δ_{ij} which carries E_i to E_j also carries P_i to P_j.*

Proof: We order the points of P_j so that P_j is a convex polygon inscribed in E_j. Then the points of P_j are evenly spaced in the angular metric on E_j. Likewise, the points of P_i are evenly spaced in the angular metric on E_i. Note that $\delta_{ij} = \delta_j^{-1}\delta_i$. From Lemma 3.3, the map δ_{ij} is an isometry relative to the angular metrics on E_i and E_j. This means that $\delta_{ij}(P_i)$ and P_j are equivalent to each other by an isometric rotation. Let $\tau(p)$ denote the translation length of this isometric rotation. If we vary p, the polygons P_i and P_j rotate isometrically along E_i and E_j respectively. Hence $\tau(p)$ is independent of $p \in E_0$. When $p = (1,0)$ we see, by symmetry, that both E_i and E_j contain a point on the x-axis. Therefore $\tau((1,0)) = 0$. Hence $\tau(p) = 0$ for all p. Therefore $\delta_{ij}(P_i) = P_j$. ♠

3.4 Symmetry of Radial Arms and Fibers

Let G be a Poncelet grid, defined relative to a Poncelet metric μ_0 on E_0. We call the sets Q_j, defined in Equation 6, the *radial arms*. Each Q_j includes in a radial fiber H_j. We say that a radial arm is generic if it is not contained in one of the coordinate axes.

Lemma 3.5 *Suppose Q_i and Q_j are generic radial arms. Then there is a diagonal map U_{ij} such that $U_{ij}(Q_i) = Q_j$.*

Proof: Let $Q_i(0), Q_i(1), ...$ be the points of Q_i, listed in the linear order they inherit from their inclusion in H_i. We label so that $Q_i(0) \in E_0$. For any $k \in \{1, ..., (n+1)/2\}$ the two points $Q_i(k)$ and $Q_j(k)$ are contained in the ellipse E_k of the foliation P_c. Here c is such that $\mu_0 = \mu(c)$. Let μ_k be the angular metric on E_k.

Let X denote the positive x-axis in $\boldsymbol{R}^2$. Lemmas 3.3 and 3.4 together imply that the (counterclockwise) μ_k-distance from $E_k \cap X$ to $Q_i(k)$ coincides with the (counterclockwise) μ_0-distance from $E_0 \cap X$ to $Q_i(0)$. Hence $\delta_k(Q_i(k)) = Q_i(0))$. Here δ_k is as in Lemma 3.3. Our result is independent of the index i. Since neither $Q_i(0)$ nor $Q_j(0)$ is contained in the coordinate axes, there is a diagonal map U such that $U(Q_i(0)) = Q_j(0)$. Since U and δ_k commute,

$$U(Q_i(k)) = U\delta_k^{-1}(Q_i(0)) = \delta_k^{-1}(U(Q_i(0)) = \delta_k^{-1}(Q_j(0)) = Q_j(k). \qquad (10)$$

Since this works for all k we have $U(Q_i) = Q_j$. ♠

Lemma 3.6 *Suppose that H_1 and H_2 are generic radial fibers. Then there is a diagonal map T such that $T(H_1) = H_2$. Moreover, T is an isometry relative to the two radial metrics.*

Proof: For each n we choose a Poncelet grid $G_n = G(n, p, \mu)$. We always choose $p = H_1 \cap E_0$. We let $Q_{1,n}$ be the radial arm of G_n contained in H_1. We let $Q_{2,n}$ be the radial arm of G_n closest to H_2. We let $H_{2,n}$ be the radial fiber containing $Q_{2,n}$. There is a linear map T_n such that $T_n(Q_{1,n}) = Q_{2,n}$. This map is determined by where it sends q, and $\lim_{n\to\infty} T_n(q) = H_2 \cap E_0$. Therefore, the limit $T = \lim T_n$ exists. Note that $Q_{1,n}$ becomes arbitrarily dense in H_1, and the points of $Q_{1.n}$ are evenly spaced in the radial metric on H_1. Likewise $Q_{2,n}$ becomes arbitrarily dense in $H_{2,n}$ and the points of $Q_{2,n}$ are evenly spaced in the radial metric on $H_{2,n}$. Furthermore $H_{2,n} \to H$. These facts all combine to show that $T(H_1) = H_2$ and that T is an isometry relative to the two metrics. ♠

Corollary 3.7 *Suppose that G and G' are two Poncelet grids defined relative to μ and having the same number of points. Then each generic radial arm of G is equivalent to each generic radial arm of G' via a diagonal map.*

Proof: This follows from Lemma 3.6, from the fact that the points of a radial arm are evenly spaced relative to the radial metric of the corresponding radial fiber, and from the fact that both radial arms contain a point on E_0. ♠

3.5 Radial Arms are Hyperbolas

As above, all our constructions are done relative to a Poncelet metric $\mu = \mu(c)$ on E_0. Let H be a generic radial fiber. Let X and Y be the coordinate axes. We parametrize H as

$$H(t) = (\alpha(t), \beta(t)),$$

where t is the radial distance–i.e. the distance in the radial metric–from $H(t)$ to $H \cap E_0$. Let E_t be the ellipse in the foliation P_c which contains $H(t)$. Let $a(t) > b(t) > 1$ be the two constants defining E_t.

Lemma 3.8 *There exist constants c' and c'' (depending on H) such that $c'\alpha(t) = a(t)$ and $c''\beta(t) = b(t)$.*

Proof: First we deal with $\alpha(t)$. Let $\{H'_n\}$ be a sequence of radial fibers of μ, contained in the positive quadrant, such that $H'_n \to X$. By Lemma 3.6 there is a positive diagonal map T'_n such that $T'_n(H) = H'_n$. Since T'_n is an isometry relative to the radial metrics, we have $T'_n(H(t)) \to E_t \cap X = a(t)$. One of the diagonal entries of the matrix representing T'_n tends to 0 while the other one tends to $a(t)/\alpha(t) = c'$. The maps T'_n are independent of t. Hence c' is independent of t. That is, $c'\alpha(t) = a(t)$. The proof for $\beta(t)$ is the same, except we use Y in place of X. ♠

Looking at Equation 2 we see that the curve $t \to (a(t), b(t))$ lies in a hyperbola H_0. Lemma 3.8 now says that there is a diagonal map T_H such that

$$T_H(a(t), b(t)) = (\alpha(t), \beta(t)) \qquad \forall t. \tag{11}$$

Hence H is contained in the hyperbola $T_H(H_0)$.

3.6 Radial Arms are Poncelet Polygons

We continue the notation from the previous section. In particular, H is a generic radial fiber. Let $E_H \subset \boldsymbol{RP}^2$ be the hyperbola which contains H. Then there are 3 components of $E_H - E_0$. One of the noncompact components is H. Without loss of generality we suppose that H has nontrivial intersection with the positive quadrant. Then $H \cap \boldsymbol{R}^2$ has two components, one of which is contained in the positive quadrant and one of which is contained in the negative quadrant.

$E_H \cap E_0$ consists of 4 points. Let $\mathcal{H}$ denote the set of real conics H' such that $E_H \cap H' = E_H \cap E_0$. In other words, all the conics in $\mathcal{H}$ contain the same 4 points of E_0. Some members of $\mathcal{H}$, such as E_0, are ellipses. The other members of $\mathcal{H}$ are hyperbolas.

We now make some constructions based on some $H' \in \mathcal{H}$. Some of these constructions turn out to be independent of H' and others do not. Let $E_H(\boldsymbol{C})$ denote the complexification of E_H. Let $T(H')$ to be the set of pairs (q, l) where $q \in E_H(\boldsymbol{C})$ and l is a line containing q and tangent to $H'(\boldsymbol{C})$. We have the double branched covering

$$\sigma : T(H') \to E_H(\boldsymbol{C}); \qquad (q, l) \to q.$$

The branch points are $E_H \cap H'$ and do not depend on $H' \in \mathcal{H}$. Since the branch points do not depend on H', the isometry type of $T(H')$ is independent of H'.

Let $\widetilde{H} = \sigma^{-1}(H)$. The set $\widetilde{H}$ is fixed under the antiholomorphic isometry $(z,w) \to (\overline{z},\overline{w})$. Hence $\widetilde{H}$ is a geodesic. H is homeomorphic to a line segment, and the endpoints of H are two points of $E_0 \cap H$. The map $\sigma : \widetilde{H} \to H$ is a double branched cover, branched over the endpoints. Thus $\widetilde{H}$ is topologically a circle and topologically σ folds $\widetilde{H}$ in half to produce H. We say that the *Poncelet metric* on H is the metric which makes the map $\sigma : \widetilde{H} \to H$ a local isometry away from the branch points. Given that the isometry type of $T(H')$ does not depend on H', the poncelet metric on H is independent of H'.

We say that a *critical Poncelet polygon* on H is a set of the form $\sigma(\widetilde{Q})$ where $\widetilde{Q} \in \widetilde{H}$ is the orbit of one of the critical points of σ under $\widetilde{f}$. As we will discuss below, this situation arises when H' is a hyperbola but not when H' is an ellipse. Recall that H also has a radial metric, defined in terms of the Poncelet metric μ on E_0.

Lemma 3.9 *On H, the poncelet and radial metrics coincide.*

Proof: Since the Poncelet metric on H is independent of $H' \in \mathcal{H}$ we will take $H' = E_0$. Figure 3.3 shows a picture. We shall work with a Poncelet grid $G = G(\mu,p,n)$ which is invariant under the map $(x,y) \to (-x,y)$. We can arrange this by taking $p \in Y$, the y-axis.

Half of E_H is shown. The shaded disk is the one bounded by E_0. Let $Q_0, Q_1, \ldots$ be the points on $Q = G \cap H$, labelled so that successive points move away from E_0. The first $n/4$ of these points lie in the same component of $H \cap \boldsymbol{R}^2$. Let $Q^k = Q_0 \cup Q_2 \cup \ldots Q_{2k}$.

Let $k = \text{Floor}(n/8)$. We define $g : Q^{k-1} \to Q^k$ by $g(Q_i) = Q_{i+2}$. Looking at Figure 3.3, and recalling the definition of $\widetilde{f}$ we see that σ conjugates g either to $\widetilde{f}^2$ or to $\widetilde{f}^{-2}$. This means that the points $Q_0, Q_2, \ldots, Q_k$ are evenly spaced with respect to the Poncelet metric on H. But these points are also evenly spaced with respect to the radial metric on H. By increasing the number of points in G (and adjusting H if necessary) we can make the points of Q^k as dense as we like in the relevant connected component of $H \cap \boldsymbol{R}^2$. Hence, the two metrics agree on an open subset of H. Being analytic, they agree everywhere on H. ♠

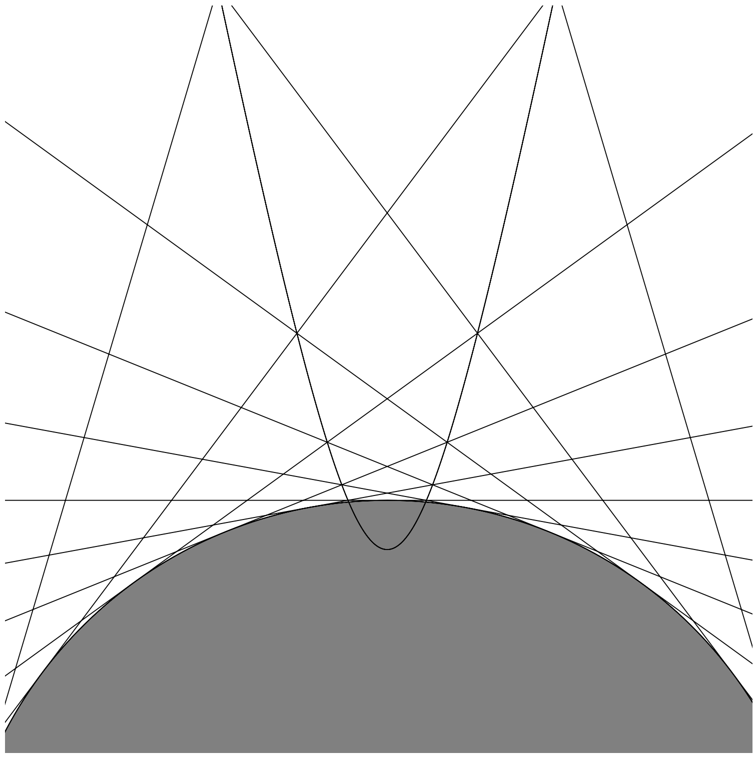

Figure 3.3

Remark: We had to use the map $\widetilde{f}^2$ (or $\widetilde{f}^{-2}$) because $\widetilde{f}$ does not preserve $\widetilde{H}$. The point here is that the corresponding map $f : E_H \to E_H$ swaps the left and right components of branch of the hyperbola shown in Figure 3.3. (We are only interested in the portion of this hyperbola outside the shaded disk.) The right branch is one of the components of $H \cap \boldsymbol{R}^2$ but the left branch is not a subset of H. On the other hand, when H' is a hyperbola then $\widetilde{f}$ does preserve $\widetilde{H}$. We can see this by applying a projective transformation to the whole picture, so that both H and H' are transformed into ellipses, as shown in Figure 3.4. Here H is the topmost component of $E_H - H'$. This is the reason why, for our next construction, we need to consider members of $\mathcal{H}$ rather than just E_0.

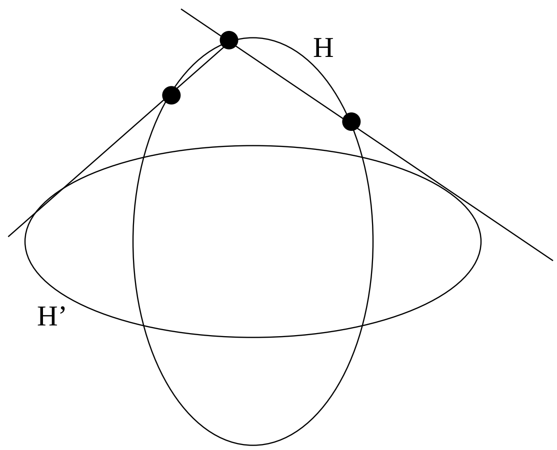

Figure 3.4

Now we know that the radial and Poncelet metrics on H coincide. Given $H' \in \mathcal{H}$ we normalize our metric on $T(H')$ so that $\widetilde{H} = \sigma^{-1}(H)$ has length 1. We can choose H' so that the translation length of $\widetilde{f}$ is any desired $r \in (0, 1/2)$. Now suppose that G is a Poncelet grid, Q is a radial arm and H is the radial fiber which contains Q. We can choose a hyperbola $H' \in \mathcal{H}$ so that the translation length of $\widetilde{F}$ coincides with the spacing between the successive points of Q. Let $x \in H'$ be the branch point such that $\sigma(x) = Q \cap E_0$. Let $\widetilde{Q}$ be the orbit of x under $\widetilde{f}$. By construction $\sigma(\widetilde{Q}) = Q$. Thus Q is a critical Poncelet polygon.

Remark: There are $(n+1)/2$ points in a radial arm which is based on the grid $G(n, p, \mu)$. One might ask why there aren't n points. The idea is that the map σ is two-to-one on the set $\widetilde{Q} - x$ and one-to-one on x. So, $\widetilde{Q}$ has n points whereas its image has only $(n+1)/2$ points.

3.7 The Double Foliation

If G is a Poncelet grid then there is a finite union $\{E_j\}$ of ellipses and a finite union $\{H_j\}$ of hyperbolas such that $G \subset \bigcup E_j$ and $G \subset \bigcup H_j$. There is some constant $c \in (0, \infty)$ such that E_j is an angular fiber relative to the Poncelet metric $\mu(c)$. At the same time, H_j is a radial fiber relative to the same metric. As we have already seen in §3.2 the foliation P_c is the set of all angular fibers. We let Q_c denote the set of all radial fibers.

Let S_P denote the set of 4 complex lines satisfying $\pm icx \pm dy = 1$, where c and d are as in Equation 2. We have already seen that every ellipse in P_c

(when complexified) is tangent to the 4 lines in S_P. Now we establish the same result for the hyperbolas in Q_c.

Lemma 3.10 *There is a set S_Q of 4 complex lines such that every hyperbola in Q_c (when complexified) is tangent to the 4 lines of S_Q.*

Proof: Let H be a generic fiber. Let $T(H)$ be the complex torus defined relative to the pair (E_0, H), as in §2.1. We have the covering map $T(H) \to E_0$, which is branched over the 4 points of E_0. The complex structure on $T(H)$ is determined by the 4 branch points on E_0. Let $S(H)$ be the set of 4 lines tangent to E_0 at the 4 branch points. By construction H is tangent to the 4 lines of $S(H)$. We already know that the radial fibers, equipped with their radial metrics, are (generically) isometric to each other via diagonal maps. The same argument as in Lemma 3.1 shows that the complex structure on $T(H)$ is independent of H. The complex structure on $T(H)$ is determined by the points on E_0 over which π is branched. Hence $S(H)$ is independent of H. Setting $S_Q = S(H)$, we get the result of this lemma. ♠

Lemma 3.11 $S_Q = S_P$.

Proof: Both S_P and S_Q are dihedrally invariant sets. Thus, it suffices to show that the line $ixc + dy = 1$ lies in S_Q. This line contains the points $(-i/c, 0)$ and $(0, 1/d)$. Both these points are contained in the plane Π, given in Equation 3.

Each radial fiber H is given by an equation of the form $-ex^2 + fy^2 = 1$ where $e, f > 0$. From this fact it is easy to see that $H(\boldsymbol{C}) \cap \Pi$ is an ellipse. Both $S_Q \cap \Pi$ and $H(\boldsymbol{C}) \cap \Pi$ have dihedral symmetry. Figure 3.5 shows the only possibility. The 5 ellipses drawn show different choices of H. As H converges to one of the coordinate axes, the ellipse $H(\boldsymbol{C}) \cap \Pi$ degenerates either to a horizontal line segment or to a vertical line segment. To figure out the lines in $S_Q \cap \Pi$ we just have to find the endpoints of the degenerate line segments. We will show that the endpoints of the horizontal degenerate line segment are $(\pm i/c, 0)$ and the endpoints of the vertical degenerate line segment are $(0, \pm 1/d)$. This fact establishes our claim that the line $icx + dy = 1$ lies in S_Q.

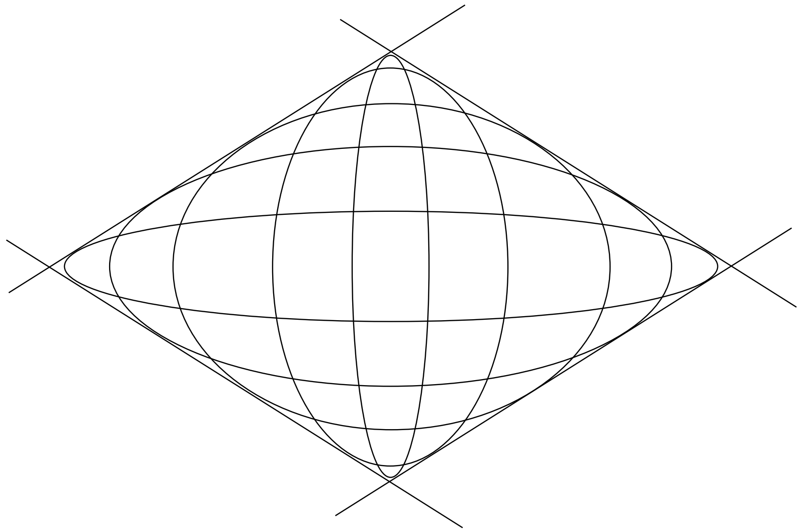

Figure 3.5

Let T_H and H_0 be as in Equation 11. We refer to the notation used in the proof of Lemma 3.8, and suppose that H is a radial fiber close to the x-axis. In this case, $\alpha(t)$ is close to $a(t)$ and $\beta(t)$ is close to 0. This means that

$$T_H \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

where $\sim$ is an approximation which improves as H converges to the x-axis.

Lemma 3.12 *The ellipse $H_0(\boldsymbol{C}) \cap \Pi$ intersects the horizontal axis in the point $(\pm i/c, 0)$.*

Proof: To find the intersection points we set $b = 0$ in Equation 2 and then solve for a. ♠

Given our lemma, we know that $H(\boldsymbol{C}) \cap \Pi$ very nearly intersects the horizontal axis in the points $(\pm ic, 0)$. Taking the limit we see that $(\pm i/c, 0)$ are the endpoints of the degenerate horizontal line segment, as desired. The same argument shows that $(0, \pm i/d)$ are the endpoints of the degenerate vertical line segment. ♠

Setting $S = S_P = S_Q$ we have the set of 4 lines advertised in §1. We have now verified all the claims made in Theorem 1.1.

4 References

[**B**] M. Berger, *Geometry*, Springer-Verlag 1987

[**BKOR**] H.J.M. Bos, C. Kers, F. Oort, D.F. Raven, *Poncelet's Closure Theorem*, Expo. Math **5** (1987) pp 287-364

[**GH**] P. Griffiths and J. Harris, *a Poncelet Theorem in Space*, Comment Math Helvetici, **52** (1977) pp 145-160

[**H**] D Hilbert and S. Cohn-Vosson, *Geometry and the Imagination*, Chelsea Publishing Company, 1950

[**S**] R. Schwartz, Java Applet 37
http://www.math.umd.edu/~res/Java/App37/test1.html

[**T**] S. Tabachnikov, *Geometry and Billiards*, preliminary textbook, 2005

[**W**] S. Wolfram, *Mathematica: A System for Doing Mathematics by Computer*, Wolfram Press (2000)