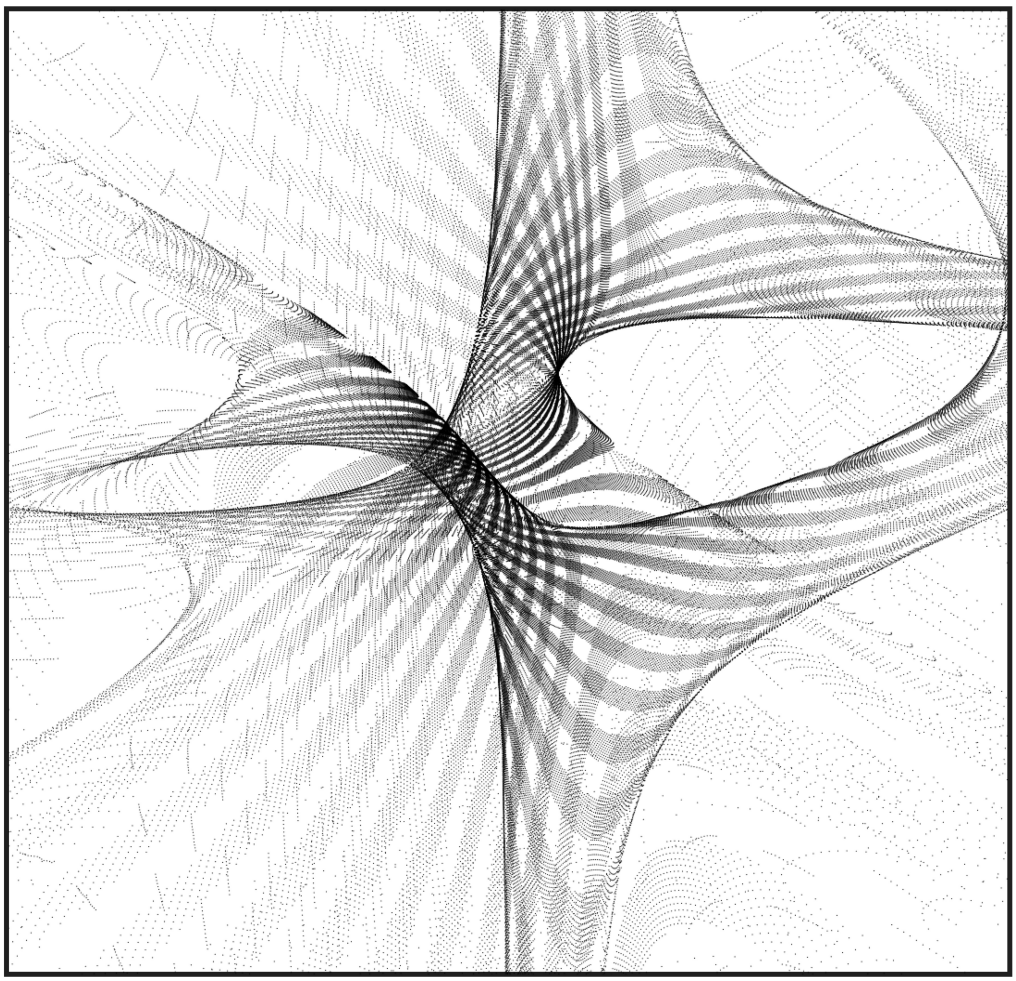
Lily's Book of Math



by Rich Schwartz (a.k.a. Pops)

Dear Lily,

Pops has written you a little book of math, mostly about algebra. Many math books ramble on and on and on, the author thinking that multiple and indirect exposures to the same concept will help you learn it. They are hundreds of pages long but their actual content would fit in a little pamplet. Reading those books is like walking through the same streets over and over again, in a dense fog.

Pops has a different idea. In this little book Pops will show how things are done, directly and precisely, but Pops is not going to bore you with explanations of *why* things are done a certain way. Furthermore, Pops won't explain things multiple times, because he believes that advanced math is made up of a bunch of simple math. So, as you go along and learn new things, you are automatically forced to re-practice what you have already learned. Also, Pops is lazy in some ways. In terms of content, Pop's book is much more compressed, but it still delivers the goods.

Pops won't waste your time asking you to explain how you got the right answer. This book is all business. If you repeatedly get the right answer, you know how to do it. Once you really know *how*, you will be able to figure out *why*.

What will you need for this book? You will need a working brain, time, a pen, and a wheelbarrow full of paper. How will you learn from a book with no explanations? **Copy the methods until they are your own.** Also, of course, ask people for help when you need it.

Love and Good Luck,
Pops

1 Linear Equations

There is one thing you need to know to solve linear equations. Doing the same thing to both sides of an equation keeps it an equation. Here's how to solve the equation

$$3x + 7 = 22.$$

Subract 7 from both sides:

$$3x = 22 - 7 = 15.$$

Divide both sides by 3:

$$x = 15/3 = 5.$$

So, $x = 5$. That's it. Here is your first assignment. The problems start out easy and get harder, just like bucking bronco.

1. $x + 1 = 2$.
2. $2x + 5 = 9$.
3. $4x - 7 = 9$.
4. $3x - 1 = 8$.
5. $15x + 21 = 26$.
6. $-5x + 4 = 9$.
7. $\frac{1}{2}x + 7 = 11$.
8. $\frac{3}{7}x + \frac{2}{5} + \frac{1}{35}$.
9. $-12x + \frac{2}{7} = -11$.
10. $x + 2 = 5$.

Well, the last one is easy. I got sick of writing these equations. If you've seen one linear equation you've seen them all.

OK, Lily, now let me tell you something about your father that you didn't know. You father likes each section of his book to go all the way to the bottom of the page. In this case, there isn't quite enough material to fill a page, so I have to include some fluff. This paragraph doesn't have any information in it, but it does fill up the page very nicely.

2 Several Variables I

Here's another basic principle. If you add two equations together – I mean add left sides and then add right sides – you still have an equation. How do you solve the two equations

$$x + y = 5; \qquad x - y = 11 \quad ?$$

Solution: line up the equations like this and add

$$\begin{array}{rcl} x + y & = & 5 \\ + & & + \\ x - y & = & 11 \\ = & & = \\ 2x + 0y & = & 16 \end{array}$$

So, $2x = 16$. But that means $x = 8$. Now substitute $x = 8$ into one of the equations

$$8 + y = 5.$$

So, $y = -3$. The answer is:

$$x = 8; \qquad y = -3.$$

Here is your assignment. As before, they start out easy and get a bit harder. You have to use your brain for the last 3. In each case, find x and y.

1. $x + y = 4$ and $x - y = 5$.
2. $x + y = 14$ and $x - y = 11$.
3. $x + y = 2$ and $x - y = -6$.
4. $x + y = 3/4$ and $x - y = 7/8$.
5. $x + y = -17/4$ and $x - y = 3/5$.
6. $x - y = -5$ and $y + x = 7$. (Hint: $x + y = y + x$.)
7. $-x - y = 8$ and $y - x = -31$.
8. $x + 2y = 11$ and $x - y = 8$.

3 Several Variables II

Just remember that if you change both sides of an equation the same way you don't change the equation. Also, you can add (or subtract) two equations, left to left and right to right.

Here's how to solve the equations

$$2x - 4y = 7; \qquad 7x - 3y = 12.$$

Multiply the first one by 7 and the second one by 2

$$14x - 28y = 49; \qquad 14x - 6y = 24.$$

Now *subtract* left and right aides:

$$\begin{array}{rcl} 14x - 28y & = & 49 \\ - & & - \\ 14x - 6y & = & 24 \\ = & & = \\ 0x - 22y & = & 25 \end{array}$$

So $-22y = 25$, which means that

$$y = -25/22.$$

Plug this back into one of the equations

$$2x - 4 \times (-25/22) = 7.$$

Add $4 \times (-25/22)$ to both sides.

$$2x = 7 + 4 \times (-25/22) = 7 - 50/11 = 77/11 - 50/11 = 27/11.$$

Divide both sides by 2:

$$x = 27/22.$$

The answer is:

$$x = 27/22; \qquad y = -25/22.$$

Here is your assignment: Find x and y for these equations

1. $2x + 5y = 23$ and $6x - y = 21$.
2. $3x + 6y = 11$ and $2x - 5y = 12$.

4 Square Roots I

Let's start off easy. The number $\sqrt{5}$ has the property that

$$\sqrt{5} \times \sqrt{5} = 5.$$

The number $\sqrt{9}$ has the property that

$$\sqrt{9} \times \sqrt{9} = 9.$$

Any idiot can see that $\sqrt{9} = 3$, because $3 \times 3 = 9$, but you have to be careful because

$$(-3) \times (-3) = 9$$

as well. So, actually $\sqrt{9} = 3$ and $\sqrt{9} = -3$. The first thing you need to know about the square-root symbol is that it always means two numbers rather than one, except for $\sqrt{0}$, which is just 0 no matter how you slice it. Here's your next assignment:

1. What two numbers are $\sqrt{1}$.
2. What two numbers are $\sqrt{4}$
3. What two numbers are $\sqrt{9}$.
4. What two numbers are $\sqrt{16}$.
5. What two numbers are $\sqrt{25}$.
6. What two numbers are $\sqrt{36}$.
7. What two numbers are $\sqrt{49}$.
8. What two numbers are $\sqrt{64}$.
9. What two numbers are $\sqrt{81}$.
10. What two numbers are $\sqrt{100}$.

If you can't complete this assignment you need to have your head examined.

Usually its hard to find the square root. The number might go on and on in some crazy decimal expansion. But, if you want to get a decent idea about the number you can just fool around with it until you know roughly the answer.

5 Square Roots II

Here's a problem. Find $\sqrt{5}$ to two decimal places. What does this mean? We want to say that $\sqrt{5}$ starts out as *something point something, something*. Solution: 2×2 is less than 5 and 3×3 is bigger than 5. So, obviously $\sqrt{5}$ starts out 2 point something, something. Now for some computing

- $2.1 \times 2.1 = 4.41$. This is less than 5.
- $2.2 \times 2.2 = 4.84$. Still less than 5.
- $2.3 \times 2.3 = 5.29$. Bigger than 5.

So, $\sqrt{5}$ is bigger than 2.2 and less that 2.3. That means that $\sqrt{5}$ starts out as 2.2 something. Let's repeat.

- $2.21 \times 2.21 = 4.8841$. Less than 5.
- $2.22 \times 2.22 = 4.92$. Less than 5.
- $2.23 \times 2.23 = 4.9729$. Jesus! Still less than 5.
- $2.24 \times 2.24 = 5.016$. Alrighty then.

So, $\sqrt{5}$ is bigger than 2.23 and less than 2.24. That means that $\sqrt{5}$ starts out 2.23. That's the answer. In case you care, it turns out that

$$\sqrt{5} = 2.23606797749978969640917366873127623544061835961152 57...$$

but, honestly, nobody knows how the digits go on. They are sort of random.

Here is your next assignment.

- Find $\sqrt{2}$ up to one decimal place.
- Find $\sqrt{3}$ up to two decimal places.

Notice that there aren't many problems on your assignment. These problems involve a lot of multiplication, so I'm not going to ask you to do a lot of them.

You could use the method above to find as many digits in the decimal expansion as you like, but the method is really painful to do by hand. Computers can do it quite quickly. That's how I was able to write down so many digits of $\sqrt{5}$. I asked the computer. The computer uses a more efficient method than the one I explained.

6 Quadratic Equations I

The symbol x^2 means x times x. So, $5^2 = 25$. What is the solution to the equation $x^2 = 9$? It's $\sqrt{9}$. But that means that $x = 3$ and $x = -3$. To save words, let's write $x = \pm 3$. This means, x is plus or minus 3.

What is the solution to the equation

$$x^2 + 6 = 11?$$

Answer: Subtract 6 from both sides:

$$x^2 = 5.$$

That means that $x = \sqrt{5}$. Sometimes, people write $x = \pm\sqrt{5}$ but, in my opinion, that is because they are idiots and don't remember that $\sqrt{5}$ isn't really a number at all, but rather stands for 2 numbers. But, what the hell, let's write $x = \pm\sqrt{5}$ just because everyone else does it.

What is the solution to the equation

$$(x + 2)^2 + 6 = 42.$$

Well, subtract 6 from both sides:

$$(x + 2)^2 = 36.$$

Take square roots of both sides:

$$x + 2 = \pm 6.$$

Add 2 to both sides

$$x = -2 \pm 6.$$

In other words

$$x = 4; \qquad x = -8$$

are both solutions.

Your assignment is to find x in each of these equations

1. $(x + 3)^2 = 36.$
2. $(x - 1)^2 = 11.$
3. $(x - 3)^2 = 7.$

7 Quadratic Equations II

Observe that

$$(x+2)^2 =$$
$$(x+2) \times (x+2) =$$
$$x \times (x+2) + 2 \times (x+2) =$$
$$(x^2 + 2x) + (2x+4) =$$
$$x^2 + 4x + 4.$$

Here is another one.

$$(3x+2)^2 =$$
$$(3x+2) \times (3x+2) =$$
$$3x \times (3x+2) + 2 \times (3x+2) =$$
$$(9x^2 + 6x) + (6x+4) =$$
$$9x^2 + 12x + 4.$$

This is called *expanding out*. Your assignment is to expand out the following:

1. $(x+5)^2$
2. $(x-3)^2$
3. $(2x-7)^2$.
4. $(5x+11)^2$.

If you really want to go crazy, observe that

$$(x+3/2)^2 =$$
$$(x+3/2) \times (x+3/2) =$$
$$x \times (x+3/2) + 3/2 \times (x+3/2) =$$
$$(x^2 + 3x/2) + (3x/2 + 9/4) =$$
$$x^3 + 3x + 9/4.$$

Remember this one. We'll see it again in the next section.

8 Quadratic Equations III

Let's solve the equation

$$x^2 + 2x + 1 = 7.$$

We get lucky, because

$$(x^2 + 2x + 1) = (x + 1)^2.$$

So,

$$(x + 1)^2 = 7.$$

The answer is

$$x = -1 \pm \sqrt{7}.$$

We got lucky, but in general you have to make your own luck. Here's how to solve the equation

$$4x^2 + 12x - 8 = 0.$$

Divide both sides by 4:

$$x^2 + 3x - 2 = 0.$$

Now the the big idea: Add and subtract the same number to the left side:

$$(*) \qquad \underline{x^2 + 3x + 9/4} - 9/4 - 2 = 0.$$

It's still the same equation. Remember from the last section that

$$\underline{x^2 + 3x + 9/4} = (x + 3/2)^2.$$

Substitute $(x + 3/2)^2$ for the underlined part of $(*)$:

$$(x + 3/2)^2 - 9/4 - 2 = 0.$$

Add $9/4 + 2$ to both sides

$$(x + 3/2)^2 = 9/4 + 2 = 9/4 + 8/4 = 17/4.$$

Take square roots

$$x + 3/2 = \pm\sqrt{17/4} = \pm\sqrt{17}/\sqrt{4} = \pm\sqrt{17}/2$$

Subtact $3/2$ from both sides.

$$x = -3/2 \pm \sqrt{17}/2.$$

Here is your assignment: Solve $2x^2 - 6x - 10 = 0$.

9 Quadratic Equations IV

There seems to be no number x such that $x^2 = -1$. Let's make up a new number, and call it i. The number i is just declared by law to have the property that $i^2 = -1$. So, you might say that $i = \sqrt{-1}$. This isn't quite right, because the square root always represents two numbers. So, the right thing to write is

$$\sqrt{-1} = \pm i.$$

Observe that

$$(5i)^2 = (5i) \times (5i) = 5 \times 5 \times i \times i = -25i.$$

So,

$$\sqrt{-25} = \pm 5i.$$

In the same way

$$\sqrt{-7} = \pm\sqrt{7}i.$$

Let's solve the equation

$$x^2 - x + 2 = 0.$$

Add and subtract a good choice of number to the left hand side:

$$(*) \qquad \underline{x^2 - x + 1/4} - 1/4 + 2 = 0.$$

Observe that

$$\underline{x^2 - x + 1/4} = (x - 1/2)^2.$$

Substitute $(x - 1/2)^2$ for the underlined part of $(*)$:

$$(x - 1/2)^2 - 1/4 + 2 = 0.$$

Subtract $-1/4 + 2$ to both sides:

$$(x - 1/2)^2 = 1/4 - 2 = -1/4 + 8/4 = -7/4.$$

Take square roots

$$x - 1/2 = \sqrt{-7/4} = \sqrt{-7}\sqrt{4} = \pm\frac{\sqrt{7}i}{2}.$$

Add 1/2 to both sides

$$x = \frac{1}{2} \pm \frac{\sqrt{7}i}{2}.$$

10 Polynomials I

A polynomial is something like

$$8x^4 - 3x^2 - 11x + 12.$$

The letter x is the variable. A polynomial could also be something like $4t^3 + 11t - 7$. In this case t is the variable. Mostly, I'll stick to using x, but you should be aware that any letter will do.

What is

$$(3x^2 - 4x + 6) + (x^3 + 2x^2 - 5x).$$

Solution: put in zeros as place holders, line up the terms, and add term by term.

$$\begin{array}{cccc} 0x^3 & 3x^2 & -4x & 6 \\ + & + & + & + \\ x^3 & 2x^2 & -5x & 0 \\ = & & & \\ x^3 & 5x^2 & -9x & 6 \end{array}$$

The answer is

$$x^3 + 5x^2 - 9x + 6.$$

Here is your assignment. In all cases, add the polynomials together. Well, in the last cases, you have to subtract polynomials. In this case, the method is the same as above, except that you subtract rather than add.

1. $(x^2 + x + 6) + (3x - 4)$.
2. $(x^4 - x^2) + (2x^3 - 12)$.
3. $(x^2 + x + 1) + (5x^2 - 3x + 7)$.
4. $(4x^5 + 3x^2 + 11x) + (x^5 - x^4 - 2x^2 + 34)$.
5. $(3t + 7) + (-3t^2 - 3t + 11)$.
6. $(4t^3 + 7) - (t^3 + t + 11)$.
7. $(x^2 - x - 6) - (x^3 - 2x^2)$.

Don't be afraid of trying something new (like subtracting polynomials) before learning what it is. Your best guess will probably be right.

11 Polynomials II

What is

$$(3x^2 - 4x + 6) \times (x^3 + 2x^2 - 5x)?$$

Before I answer this, I want to say that this is also written as

$$(3x^2 - 4x + 6)(x^3 + 2x^2 - 5x)$$

Solution: Make a grid and fill it in:

	$3x^2$	$-4x$	6
x^3	$3x^5$	$-4x^4$	$6x^3$
$2x^2$	$6x^4$	$-8x^3$	$12x^2$
$-5x$	$-15x^3$	$20x^2$	$-30x$

Add all the terms in the boxes that you filled in:

$$3x^5+$$

$$-4x^4 + 6x^4+$$

$$6x^3 - 8x^3 - 15x^3+$$

$$12x^2 + 20x^2+$$

$$-30x$$

Simplify to get the final answer:

$$3x^5 + 2x^4 - 17x^3 + 32x^2 - 30x.$$

Here is your assignment:

1. What is $(x + 1)(x + 2)$.
2. What is $(x^3 + 1)(x^3 - 1)$.
3. What is $(x^2 + 1)(3x - 5)$.
4. What is $(3x^3 - 5x + 1)(4x^2 - 7)$.
5. What is $(1 - x)(x^4 + x^3 + x^2 + x)$.

The last one is actually supposed to be fun.

12 Euclidean Algorithm

What is the largest number that divides evenly into 4 and 6? It's 2 obviously. What is the largest number that divides evenly into 14 and 22. Here's the trick: replace the big number by the big one minus the small one. Then repeat until the two numbers are the same.

1. Start with $14, 22$.
2. Then you get $8, 14$.
3. Then you get $6, 8$.
4. Then you get $2, 6$.
5. Then you get $2, 4$.
6. Then you get $2, 2$.

You end in 2, so 2 is the greatest number that goes evenly into both 14 and 22. The number 2 is called the *greatest common divisor* of 14 and 22. Usually, people say *GCD* for short.

You can use the GCD to put fractions in lowest terms. What is 14/22 in lowest terms? Solution: First find the GCD. It's 2. Now divite top and bottom by the GCD:

$$\frac{14}{22} = \frac{14/2}{22/2} = 7/11.$$

Here is your assignment.

1. What is the GCD of 34 and 56?
2. What is the GCD of 124 and 768?
3. What is the GCD of 99 and 1999?
4. Write 485/995 in lowest terms.
5. Write 162/222 in lowest terms.
6. Write 217/287 in lowest terms.

13 End of Book Test

1. Solve $3x + 7 = 11$ for x.
2. Solve $2x - 6y = 1/2$ and $7x - 4y = 1$ for x and y.
3. $4x^2 + 5x - 7 = 0$ for x.
4. Find $\sqrt{11}$ to 2 decimal places.
5. What is $(3x^2 + x - 7) + (x^2 + x + 1)$?
6. What is $(3x^2 + x - 7)(x^3 - x^2 - 11)$?
7. Find the greatest common divisor of 335 and 95.

That's it. If you can solve these equations, and others like them, you have learned about half of what you learn in a year-long algebra class. There's other important stuff to learn in an algebra class, but most of the other stuff has to do with the *interpretation* of the basic equations above. Maybe we'll get to some of that later on.

If you really want to master the material in this little book, you should give yourself other equations to solve and try to solve them.