The Optimal Paper Moebius Band (Informally)

Richard Evan Schwartz *

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Abstract

This is a friendly, crisp, and slightly informal account of my solution of the 1977 Optimal Moebius Band Conjecture of B. Halpern and C. Weaver.

1 Introduction

You make a paper Moebius band by giving an odd number of twists to a $1 \times \lambda$ rectangular strip of paper and joining the ends together. If e.g. $\lambda = 6$, this is quite easy to do. If e.g. $\lambda = 2$ you can still do it, but you will find it more challenging. How small can you take λ? I guess that this question probably arose as soon as people started making paper Moebius bands, but in any case W. Wunderlich [**W**] discusses this question in the introduction to his 1962 paper. On the last line of their 1977 paper [**HW**], B. Halpern and C. Weaver conjecture that λ must be larger than $\sqrt{3}$. I first learned about this conjecture from the excellent survey article [**FT**, Chapter 14].

There is a nice physical experiment you can do which will probably lead you to conclude that $\lambda > \sqrt{3}$ is the right bound. Take a long rectangular strip of paper, give it one twist, and hold the ends together. Now slide the ends past each other, a move which has the effect of simulating a shorter rectangle. Keep sliding as much as you can. Eventually you will see a certain triangular pattern emerge. Figure 1 describes the limiting shape you will see if you do this experiment.

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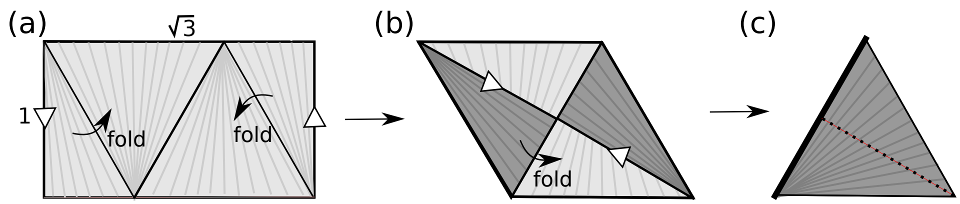

Figure 1: The Triangular Moebius Band

Figure 1(a) is a $1 \times \sqrt{3}$ rectangular strip that is lightly shaded on the side you can see and darkly shaded on the other side. Figure 1(b) shows the rhombus you get after folding over two flaps. Figure 1(c) shows what happens when you fold the rhombus in half like a taco. The folding pattern brings the two ends together with a twist. Figure 1(c) is the *triangular Moebius band*. The dotted line segment indicates the joined ends of the strip. The thick line segment indicates the "taco fold".

The triangular Moebius band, a completely planar object, does not quite count as a paper Moebius band. Technically, the triangular Moebius band is known as a *folded ribbon knot* [**DL**] whereas a paper Moebius band is defined as a smooth and locally isometric embedding of a flat Moebius band into space. (You need not understand this formal definition to follow this paper.) However, as is discussed in [**Sa**], [**HW**], and [**FT**, Chapter 14], you can approximate the triangular Moebius band as closely as you like with a *bona fide* paper Moebius band by taking a slightly longer strip of paper and rounding out the sharp folds. The physical experiment above essentially does this. This is why the Halpern-Weaver conjecture says that $\lambda > \sqrt{3}$ rather than $\lambda = \sqrt{3}$.

In a recent paper [**S**], I proved the Halpern-Weaver conjecture. I also provided a rigorous justification for the outcome of the physical experiment: If $\lambda \approx \sqrt{3}$, then the shape of your paper Moebius band must approximate the triangular Moebius band. In this paper I will give a friendly and somewhat informal account that I hope will reach a wider audience of scientists who might be interested in these results. The proof here is essentially the same as in [**S**] but it is presented in a different way, with an emphasis on crisp exposition rather than filling in every detail. I also omit a lot of the mathematical commentary, references, and acknowledgements found in [**S**].

The proof here starts with four easy optimization problems, the last of which I call the *Coupled Circuit Problem*. I then show how the Optimal Moebius Band Problem gives a special case of the Coupled Circuit Problem, and this solves the whole thing.

2 Four Optimization Problems

1. Triangle Problem: Let $\triangledown$ be a triangle with a horizontal base of length x, and a height of y. As is well known, the sum $\vee$ of the lengths of the two non-horizontal sides of $\triangledown$ is minimized when $\triangledown$ is isosceles. The Pythagorean Theorem computes this minimum as

$$\vee = \sqrt{x^2 + 4y^2}. \tag{1}$$

2. Planar Circuit Problem: Suppose $X = \overline{X_1X_2}$ and $Y = \overline{Y_1Y_2}$ are respectively horizontal and vertical line segments having lengths

$$|X| = x = \sqrt{1+t^2}, \qquad |Y| = y \geq 1. \tag{2}$$

Here and below $|\cdot|$ denotes arc length. For now, writing $x = \sqrt{1+t^2}$ is just a complicated way of saying that $x \geq 1$, but below t will mean more.

Suppose also that Y is contained in the open half-plane that lies beneach the horizontal line extending X. Finally, suppose γ is a continuous loop that successively connects the points X_1, Y_1, X_2, Y_2. Figure 2 shows all this.

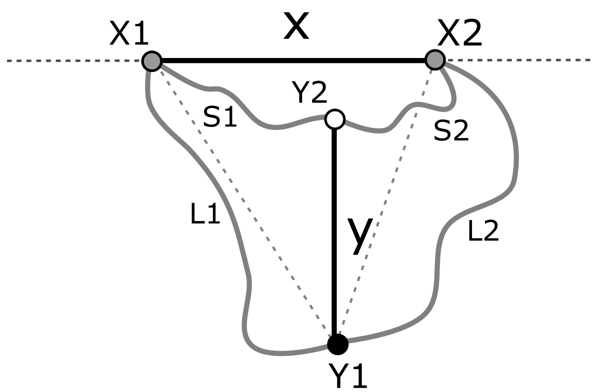

Figure 2: A circuit γ connecting the endpoints of X and Y.

The 4 arcs L_1, L_2, S_2, S_1 comprise γ. Let $|L| = |L_1| + |L_2|$ and likewise $|S| = |S_1| + |S_2|$. Each arc is at least as long as the straight line segment with the same endpoints. Hence $|S| \geq |X|$ and $|L| \geq \vee$, the sum of the lengths of the non-horizontal sides of the triangle $\triangledown$ with vertices X_1, X_2, Y_1. Since $\triangledown$ has base X and height greater than 1, Equations 1 and 2 give:

$$|L| > \sqrt{5+t^2}, \qquad |\gamma| = |S| + |L| > f(t) := \sqrt{1+t^2} + \sqrt{5+t^2} \tag{3}$$

3. Coupled Planar Circuit Problem: We keep the same problem but add a constraint that couples our loop γ to the segment X:

$$|S| = |L| - 2t. \tag{4}$$

Equations 3 (left) and 4 give another bound:

$$|\gamma| = |S| + |L| = 2|L| - 2t > g(t) := 2\sqrt{5+t^2} - 2t. \tag{5}$$

Combining Equations 3 and 5, we have $|\gamma| > h(t) := \max(f(t), g(t))$.

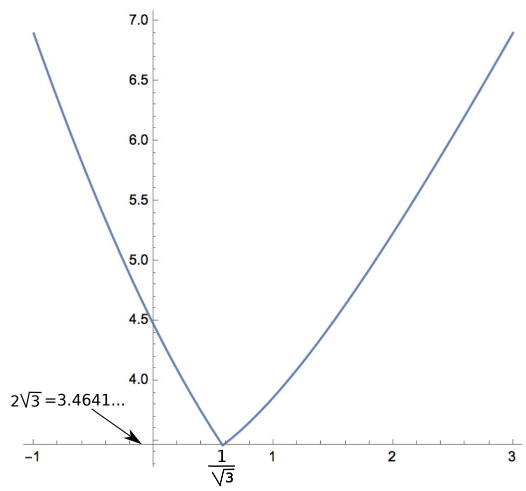

Figure 3: A plot of $h(t)$ for $t \in [-1, 3]$.

Figure 3 shows a plot of $h(t)$. The minimum occurs when $t = 1/\sqrt{3}$, and the minumum value is $2\sqrt{3}$. Therefore, $|\gamma| > 2\sqrt{3}$. Furthermore, if $|\gamma| \approx 2\sqrt{3}$ then $t \approx 1/\sqrt{3}$ and $x \approx 2/\sqrt{3}$, and $\bigtriangledown$ has height ≈ 1 and L_1, L_2, S_1, S_2 are all nearly line segments. Hence γ closely follows an equilateral triangle.

4. Coupled Circuit Problem: This is almost the same problem as the previous one. This time we think of our plane as the XY-plane sitting in space. We keep the segments X and Y as before but this time we allow our loop γ to move in space, above and below the XY-plane. All we require is that the endpoints of the 4 arcs L_1, L_2, S_2, S_1 comprising γ are again X_1, Y_1, X_2, Y_2. Figure 2 again depicts the situation, except that you should imagine you are looking down on the XY plane from space. This problem has the same analysis as the planar version.

3 Recognizing a Coupled Circuit

Let M be a paper Moebius band based on a $1 \times \lambda$ rectangular strip, and let γ be its boundary loop. Note that $|\gamma| = 2\lambda$. We aim to recognize γ as a loop that arises in the Coupled Circuit Problem.

A *bend* is a straight line segment in M which cuts across M and has its endpoints in γ. A T*-pattern* is a pair of coplanar bends which point in perpendicular directions. The dotted segment and the bold segment in Figure 1(c) make a T-pattern. I'll prove below that M has a T-pattern. We can rotate M in space so that the bends X, Y of the T-pattern, and the loop γ, are situated just as in the Coupled Circuit Problem. X and Y cut across M and so have length at least 1. This gives Equation 2.

To derive Equation 4 we cut open M along Y and flatten it out in the plane. We get a symmetric trapezoid τ. Figure 4 shows one of several possible ways τ could look, depending on how X and Y slant. The labels match Figure 2. (The repeat of Figure 2, included for convenience, is not quite drawn to scale.)

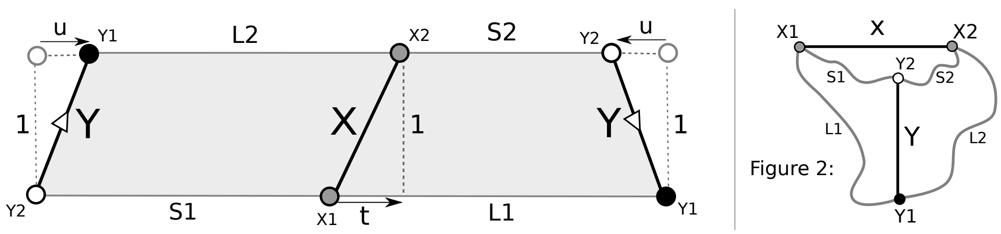

Figure 4: The symmetric trapezoid τ.

The left and right sides of τ get the same labels because on M they are joined together. Since $|X| = \sqrt{1+t^2}$ the Pythagorean Theorem tells us that t equals the horizontal displacement of the endpoints of X. Therefore

$$|S_1| + t = |L_2| + u, \qquad |S_2| + u = |L_1| - t. \tag{6}$$

We get Equation 4 by adding these equations together and simplifying. For the other possible pictures of τ, in which X and/or Y slant the other way, the signs of t and/or u change but we get Equation 4 in all cases.

Having recognized γ as a loop that arises in the Coupled Circuit problem, we get $|\gamma| > 2\sqrt{3}$ and $\lambda > \sqrt{3}$. This proves the Halpern-Weaver Conjecture. Furthermore, if $\lambda \approx \sqrt{3}$ then we have $|\gamma| \approx 2\sqrt{3}$ and, as we remarked above, γ closely follows an equilateral triangle. Hence the triangular Moebius band is the only limit of a sequence of paper Moebius bands having $\lambda \to \sqrt{3}$.

4 Finding a T Pattern

The only piece of unfinished business is finding a T-pattern in our Moebius band M. The proof here benefitted from insightful comments by Matei Coiculescu and Jeremy Kahn.

The Bend Partition: Examining a paper Moebius band, you can see that it has a continuous partition into bends that sweep through it. The pinstriping in Figure 1 is the analog of the bend partition for the triangular Moebius band. (For the triangular Moebius band, we don't quite get a partition because some bends have a common endpoint.) You could take the partition as being part of the definition of a paper Moebius band, but in [**S**, §4] we start with the formal definition of a paper Moebius band given in the introduction and show that the bend partition exists. This is a classical result.

Parametrizing Bends: The bend partition of M is parametrized by the circle $\boldsymbol{R}/2\pi$. In $\boldsymbol{R}/2\pi$ two values are the same if they differ by an integer multiple of 2π. Each $\theta \in \boldsymbol{R}/2\pi$ corresponds to a bend X_θ in the partition.

Let $\mathcal{S}$ be the space of ordered pairs $(X_{\theta_1}, X_{\theta_2})$ with $\theta_1 \neq \theta_2$. Let S^2 denote the unit sphere, with north pole P_+ and south pole P_-. We can identify $\mathcal{S}$ with a $S^2 - P_\pm$ as follows. The pair $(X_{\theta-t}, X_{\theta+t})$ corresponds to the point in S^2 having longitude $\theta \in \boldsymbol{R}/2\pi$ and latitude $t \in (0, \pi)$. Here t measures the angle between the point on S^2 and P_+.

The *antipodal* map A on S^2 interchanges each point with the diametrically opposite point. With our identification, the action on $\mathcal{S}$ is given by $A(X, Y) = (Y, X)$. A function g on S^2 is *odd* if $g \circ A = -g$.

Two Geometric Functions: Each bend X has 2 unit vectors $\pm\overrightarrow{X}$ parallel to it. Given a choice $\overrightarrow{X}_{\theta_1}$, we choose $\overrightarrow{X}_{\theta_2}$ so that, as θ moves forwards in $\boldsymbol{R}/2\pi$ from θ_1 to θ_2, the choice $\overrightarrow{X}_\theta$ varies continuously. We write $\overrightarrow{X}_{\theta_1} \rightsquigarrow \overrightarrow{X}_{\theta_2}$.

Let $(X, Y) = (X_{\theta_1}, X_{\theta_2})$. Let m_X and m_Y be the midpoints of X and Y. Using the dot product $(\cdot)$ and the cross product $(\times)$ we define

$$g_1(X, Y) = \overrightarrow{X} \cdot \overrightarrow{Y}, \qquad g_2(X, Y) = (m_X - m_Y) \cdot (\overrightarrow{X} \times \overrightarrow{Y}). \tag{7}$$

Here $\overrightarrow{X} \rightsquigarrow \overrightarrow{Y}$. Since $-\overrightarrow{X} \rightsquigarrow -\overrightarrow{Y}$, we would get the same values starting with $-\overrightarrow{X}$. If $g_1(X, Y) = 0$ then $\overrightarrow{X} \perp \overrightarrow{Y}$. If $g_2(X, Y) = 0$ then X and Y are coplanar. So, any common zero of g_1 and g_2 gives a T-pattern.

Odd Extension: Given the continuous nature of the bend partition, g_1 and g_2 are continuous on $\mathcal{S}$. Since $\overrightarrow{X} \rightsquigarrow \overrightarrow{Y}$ and we are on a Moebius band, we have $\overrightarrow{Y} \rightsquigarrow -\overrightarrow{X}$. Hence

$$g_1(Y,X) = \overrightarrow{Y} \cdot (-\overrightarrow{X}) = -\overrightarrow{X} \cdot \overrightarrow{Y} = -g_1(X,Y),$$

$$g_2(Y,X) = (m_Y - m_X) \cdot (\overrightarrow{Y} \times (-\overrightarrow{X})) = (m_Y - m_X) \cdot (\overrightarrow{X} \times \overrightarrow{Y}) = -g_2(X,Y).$$

When (X,Y) is near $P_\pm$ we have $\overrightarrow{Y} = \pm\overrightarrow{X}$. For this reason, g_1 and g_2 both extend continuously to S^2 once we define $g_1(P_\pm) = \pm 1$ and $g_2(P_\pm) = 0$. Thus g_1 and g_2 extend to give continuous odd functions on S^2.

The Borsuk Ulam Theorem says these two continuous odd funtions on S^2 have a common 0. Since $g_1(P_\pm) \neq 0$ our common 0 lies in $\mathcal{S}$. We're done.

Borsuk-Ulam Theorem: Here is a proof of the Borsuk-Ulam Theorem, tailored to our situation. The map $G : \mathcal{S} \to \boldsymbol{R}^2$ given by $G = (g_1, g_2)$ satifies $G \circ A = -G$. Suppose there is no $p \in S^2$ such that $G(p) = (0,0)$. Let ℓ be any line of longitude and consider the image $G(\ell)$. This continuous path runs from $(1,0)$ to $(-1,0)$ and misses $(0,0)$. If you stand at $(0,0)$ and watch $G(\ell)$ as it moves from $(1,0)$ to $(-1,0)$ your neck will twist a half-integer number $N(\ell)$ times. The function N is a continuous function of ℓ and therefore constant. However, consider $\ell' = A(\ell)$. We have $G(\ell') = -G(\ell)$. This gives $N(\ell') = -N(\ell)$, a contradiction.

5 References

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