

A Textbook Case of Pentagram Rigidity

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Abstract

In this paper I will explain a rigidity conjecture that intertwines the deep diagonal pentagram maps and Poncelet polygons. I will also establish a simple case of the conjecture, the one involving the 3-diagonal map on a convex 8-gon with 4-fold rotational symmetry. This case involves a textbook analysis of a pencil of elliptic curves.

1 Introduction

1.1 Conjecture and Result

Let \mathbf{RP}^2 denote the real projective plane. A polygon in \mathbf{RP}^2 is *convex* if its image under a suitable projective transformation is a convex polygon in the standard affine patch of \mathbf{RP}^2 . See §2.1 for definitions. We call a convex polygon *Poncelet* if it is inscribed in one ellipse and circumscribed about another ellipse. More generally, a *Poncelet polygon* is one whose vertices lie in one conic section and whose edges lie in lines tangent to another conic section.

Let (n, k) be a pair of integers, not both even, with $n \geq 7$ and $k \in (2, n/2)$. Given an n -gon P_1 , we let $P_2 = T_k(P_1)$ be the n -gon obtained by intersecting the successive k -diagonals of P_1 . Figure 1 shows this for $(n, k) = (8, 3)$. The map T_k is generically defined and invertible. The same construction works in any field, but convexity is important for us here. The maps T_k and T_k^{-1} are always defined on convex n -gons, though the image of a convex n -gon under one of these maps need not be convex.

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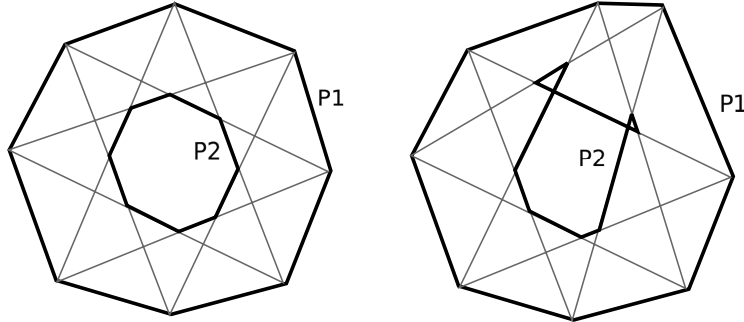


Figure 1: P_1 and $P_2 = T_3(P_1)$.

Figure 1 gives an example of where P_1 is convex but P_2 is not. Starting with P_0 we define the (n, k) -pentagram orbit $\{P_j\}$ where $P_j = T_k^j(P_0)$.

Conjecture 1.1 *Suppose that $\{P_j\}$ is an (n, k) -pentagram orbit and that P_j is convex for all $j \in \mathbf{Z}$. Then $\{P_j\}$ is convex Poncelet for all $j \in \mathbf{Z}$.*

I proved in [15] that if P_0 is a Poncelet polygon, $T(P_0)$ and P_0 are projectively equivalent. Thus, to prove the conjecture it is enough to prove that the hypotheses force P_0 to be convex Poncelet.

In this paper I will prove a simple but nontrivial case of the conjecture.

Theorem 1.2 *Suppose that P_0 is an 8-gon with 4-fold rotational symmetry and $\{P_j\}$ is the $(8, 3)$ -pentagram orbit. Then P_0 is regular if and only if P_j is convex for all j . More precisely*

1. P_0 has 8-fold dihedral symmetry, with symmetry lines containing the vertices, if and only if P_j is convex for all $j \geq 0$.
2. P_0 has 8-fold dihedral symmetry, with symmetry lines bisecting the edges, if and only if P_j is convex for all $j \leq 0$.

A version of Theorem 1.2 appears to be true when $(8, 3)$ is replaced by a general pair (n, k) when n is even and k is odd and the n -gons have $(n/2)$ -fold rotational symmetry. The proof should be similar. A much more interesting generalization is to the case of centrally symmetric octagons, because these include all Poncelet octagons. I have just finished proving this generalization in a much longer paper. See [16].

The conjecture is not true for n and k both even. In this case, T_k is 2-periodic (modulo scaling) when restricted to the space of n -gons with $(n/2)$ -fold rotational symmetry, and T_k has some non-regular convex fixed points modulo scale.

1.2 Context

The classic case of the pentagram map is $(n, 2)$ for $n \geq 5$. The case $n = 5$ has been studied e.g. by Clebsch in the 19th century and Motzkin [9] in middle of the 20th century. In 1992 I wrote a paper [13] defining the pentagram map for general n -gons and proving in the convex case that the forward orbit shrinks to a point. Very recently, M. Glick [2] found a kind of formula for this collapse point.

It is nice to consider the map T_2 as defined on the space \mathcal{P}_n of n -gons modulo projective transformation. With the correct labeling, T_2 is the identity on \mathcal{P}_5 and 2-periodic on \mathcal{P}_6 . I observed experimentally that the orbits of T_2 in general seem to lie on tori. Motivated (for some reason) by the scattering transform for the KdV equation, I found [14] about n algebraically independent invariants for T_2 . These invariants are now called the *monodromy invariants* or the *pentagram integrals*.

In [10], V. Ovsienko, S. Tabachnikov, and I showed that T_2 has an invariant Poisson structure of corank 2 in the odd case and corank 4 in the even case, and that the monodromy invariants Poisson-commute with respect to this structure. This established the complete Arnold-Liouville integrability of the pentagram map on the larger space \mathcal{T}_n of so-called *twisted n -gons*. Essentially what this means is that the space \mathcal{T}_n , a space of dimension $2n$, has a singular foliation by manifolds of dimension about n such that the restriction of P_2 to each manifold is a translation in suitable coordinates. When these manifolds are compact they are necessarily finite unions of tori.

Subsequently, we proved in [11] that T_2 is Arnold-Liouville integrable on \mathcal{P}_n , which is naturally a codimension 8 subvariety of \mathcal{T}_n . For the subset of convex n -gons, the manifolds in the singular foliation are compact and hence finite unions of tori. At the same time, F. Soloviev [19] proved that the pentagram map is algebro-geometrically integrable on \mathcal{P}_n . This implies in particular that the torus foliation discussed above is naturally an abelian fibration, with the individual tori having natural descriptions as Jacobian varieties for certain Riemann surfaces. Very recently, M. Weinreich [20] proved that T_2 is algebro-geometrically integrable in any field of characteristic not equal to 2. In [3], M. Glick related the pentagram map to a cluster algebra.

By now there are many generalizations of the pentagram map, and also a number of ways to generate invariant functions and the invariant Poisson structure. In [1], M. Gekhtman, M. Shapiro, S. Tabachnikov, A. Vainshtein generalized the pentagram map to similar maps using longer diagonals, and

defined on spaces of so-called *corrugated polygons* in higher dimensions. The work in [1] also generalizes Glick's cluster algebra and establishes the complete integrability of these maps in some form. In [7], G. Mari-Beffa defines higher dimensional generalizations of the pentagram map and relates their continuous limits to various families of integrable PDEs. See also [8]. In [5], B. Khesin and F. Soloviev obtain definitive results about higher dimensional analogues of the pentagram map, their integrability, and their connection to KdV-type equations.

The little survey above is not meant to be complete. Now let me explain how these various results are related to the Pentagram Rigidity Conjecture above. First of all, the map T_k is the one used in [1]. It would be nice if one could conclude from [1] that T_k is completely integrable on \mathcal{P}_n , but this has not been directly worked out. The spaces of corrugated polygons are somewhat different than spaces of ordinary polygons, though in some sense closely related. Ordinary polygons are limits of corrugated polygons under a kind of flattening operation. Let me just leave it by saying that T_k is certainly *believed* to be completely integrable on \mathcal{P}_n in some sense.

The Pentagram Rigidity Conjecture is really about the global geometry of the torus foliation of \mathcal{P}_n (presumably) associated to T_k . The smaller space \mathcal{C}_n of convex n -gons modulo projective transformations is a subset of \mathcal{P}_n . For $k \in [3, n/2)$ the tori in this foliation probably are not contained in \mathcal{C}_n . So, if T_k is not the identity on one of these tori, and moreover the intersection of the torus with \mathcal{C}_n is not too large, then the orbit of T_k on this torus cannot stay in \mathcal{C}_n . (See Lemma 2.1 below.) This observation would prove the conjecture for the n -gons corresponding to this torus. The Poncelet polygons in \mathcal{C}_n also lie on these tori, but T is the identity there.

Motivated by the Pentagram Rigidity Conjecture, A. Izosimov [4] has recently proved that if n is odd and $P \in \mathcal{C}_n$ is a fixed point of T_2 then in fact P is a Poncelet polygon. So, we now can say that for n odd points in \mathcal{C}_n are fixed by T_2 if and only if they are Poncelet. The convexity is important here. Izosimov gave some easy examples of n -gons in \mathbf{CP}^2 that are fixed by T_2 but not Poncelet. The parity of n is also important. As I mentioned above, the result is not true when n is even. Presumably, Izosimov's result would also work for general pairs (n, k) where both numbers are not even.

In the case I consider, that of T_3 acting on 8-gons with 4-fold rotational symmetry, there is just a single invariant for the map, and its level curves are nonsingular elliptic curves except when they contain points corresponding to octagons having 8-fold dihedral symmetry. These are the curves I analyze

in §2.2. These elliptic indeed stretch outside \mathcal{C}_8 in the appropriate sense, and this is enough to prove Theorem 1.2. See Figure 2 in §3.4.

The various invariant-generating machines for the pentagram map probably would turn up the invariant I found, but these machines are better developed for T_2 than they are for T_3 . I just guessed the invariant for T_3 by looking at the picture, and then checked algebraically that it works. I will explain in §3.2 what led me to the invariant.

There are two other connections I want to make between the Pentagon Rigidity Conjecture and other areas of mathematics. When I originally thought of this conjecture, about 30 years ago, I had imagined it as a projective geometry analogue of the *circle packing rigidity* theorem [12] of B. Rodin and D. Sullivan. Much more recently, it occurred to me that the conjecture is something like a discrete analogue of the Birkhoff-Poritsky Conjecture about billiards in strictly convex ovals. This conjecture says roughly that if a neighborhood of the boundary of the (cylindrical) billiard phase space is foliated by invariant curves (corresponding to caustics) then the oval is an ellipse.

1.3 Organization

In §2 I will give some background information about projective geometry and also analyze the family of elliptic curves that arises in the proof of Theorem 1.2. I will also present a few well-known results about complex tori. In §3 I give the proof of Theorem 1.2.

The interested reader can download the computer program I wrote, which does experiments with the 3-diagonal map on centrally symmetric octagons. The location of the program is

<http://www.math.brown.edu/~res/Java/OCTAGON.tar>:

1.4 Acknowledgements

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2 Preliminaries

2.1 Projective Geometry

The *real projective plane* \mathbf{RP}^2 is the space of lines through the origin in \mathbf{R}^3 . Equivalently it is the space of scale-equivalence-classes of nonzero vectors in \mathbf{R}^3 . Points in \mathbf{RP}^2 will be denote by $[x : y : z]$. This point represents the line through the origin and (x, y, z) . The quotient map $\mathbf{R}^3 \rightarrow \mathbf{RP}^2$ is called *projectivization*.

There is a natural inclusion $\mathbf{R}^2 \rightarrow \mathbf{RP}^2$ given by

$$(x, y) \rightarrow [x : y : 1]. \quad (1)$$

The image of this inclusion is known as the *standard affine patch*. I often identify \mathbf{R}^2 with its image under this inclusion, and when speaking about points in the affine patch I will often write (x, y) for $[x : y : 1]$. The inclusion in Equation 1 has an inverse, given by

$$[x : y : z] \rightarrow (x/z, y/z). \quad (2)$$

The *line at infinity* is the subset of \mathbf{RP}^2 outside the standard affine patch. The line at infinity consists of points of the form $[x : y : 0]$. More generally, a *line* in \mathbf{RP}^2 is the set of members represented by lines in a 2-dimensional subspace of \mathbf{R}^3 . We can also represent lines by triples $[a : b : c]$. This point represents the linear subspace given by the equation $ax + by + cz = 0$. Conveniently, the line through 2 points is represented by the cross product of the corresponding vectors. Likewise, the intersection of 2 lines is given by the cross product of the corresponding vectors. These facts make computations with the pentagram map very easy.

The *dual projective space* \mathbf{RP}_*^2 is the space of lines in \mathbf{RP}^2 . As our notation suggests, there is an isomorphism between \mathbf{RP}^2 and \mathbf{RP}_*^2 . The isomorphism sends the point represented by $[u : v : w]$ to the line represented by $[u : v : w]$. This isomorphism sends collinear points to coincident lines.

Note that all the same words apply with the field \mathbf{R} replacing the field \mathbf{C} . Thus \mathbf{CP}^2 is the *complex projective plane*. A *projective variety* in \mathbf{CP}^2 is the projectivization of the set $V(x, y, z) = 0$ where $V(x, y, z)$ is a homogeneous polynomial in 3 variables. This variety is called *nonsingular* if the (formal) gradient ∇V is everywhere nonzero on the set $V(x, y, z) = 0$. When V is cubic and nonsingular the corresponding projective variety is a smooth Riemann surface of genus 1, also known as a complex torus. See [18].

2.2 A Family of Cubics

In this section we study the solutions to the equation

$$\frac{(x-y)(x^2+y^2-1)}{xy} = \lambda, \quad (x-y)(x^2-y^2-1) - \lambda xy = 0. \quad (3)$$

The second equation is a rearrangement of the first one. To bring this equation into the form we mentioned at the end of the last section, we expand it out and then homogenize it by padding the z -variable. This gives us the equation

$$V(x, y, z) = x^3 - y^3 - x^2y + xy^2 - xz^2 + yz^2 - \lambda xyz = 0. \quad (4)$$

Let E_λ denote the complex projective variety corresponding to $V = 0$. Let ρ be reflection in the line $\{y = -x\}$. Call this line L . Call a subset of \mathbf{RP}^2 *bounded* if it lies in \mathbf{R}^2 and otherwise *unbounded*. Below, I will prove two results:

1. For all $\lambda \neq 0, \pm 2, \pm 4i\sqrt{2}$ the variety E_λ is nonsingular, and hence a complex torus.
2. When $\lambda \in \mathbf{R} - \{-2, 0, 2\}$, the set $E_\lambda \cap \mathbf{RP}^2$ consists of 2 smooth loops, both ρ -invariant, one bounded and one unbounded. The bounded loop intersects L twice and the unbounded loop intersects L once.

Figure 2 in the next chapter shows a rough but topologically accurate picture of $E_\lambda \cap \mathbf{R}^2$ for $\lambda \in (0, 2)$.

First Statement: We want to see that the gradient never vanishes on the level set $V = 0$. We compute

$$\nabla V = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \begin{bmatrix} -\lambda yz + 3x^2 - 2xy + y^2 - z^2 \\ -\lambda xz - x^2 + 2xy - 3y^2 + z^2 \\ 2z(y-x) - \lambda xy \end{bmatrix} \quad (5)$$

To analyze this, let us first consider the points in the line at infinity that belong to $V = 0$. Independent of λ , these are the 3 points

$$[1 : 1 : 0], \quad [i : 1 : 0], \quad [-i : 1 : 0].$$

Since both $x, y \neq 0$ and $z = 0$ for these points, we see from the equation that the third coordinate of ∇V is nonzero. This takes care of these points.

To consider the remaining points of V we can set $z = 1$. If $\nabla V = 0$ we have $V_x + V_y = 0$. This gives one of two equations:

$$y = -x, \quad y = \frac{2x - \lambda}{2}. \quad (6)$$

When $y = -x$ we have

$$\nabla V = \begin{bmatrix} \lambda x + 6x^2 - 1 \\ -\lambda x - 6x^2 + 1 \\ x(\lambda x - 4) \end{bmatrix} \quad (7)$$

This can only vanish when $x = 4/\lambda$. But then

$$V\left(\frac{4}{\lambda}, -\frac{4}{\lambda}, 1\right) = \frac{256}{\lambda^3} + \frac{8}{\lambda},$$

and this vanishes only if $\lambda = \pm 4i\sqrt{2}$.

When $y = (2x - \lambda)/2$ we have

$$V(x, y, 1) = \frac{\lambda(4 - \lambda^2)}{8}.$$

This can only vanish when $\lambda = 0, \pm 2$. ♠

Second Statement: Let $\lambda \in \mathbf{R} - \{-2, 0, 2\}$. The set $E_\lambda \cap \mathbf{RP}^2$ is a finite disjoint union of smooth loops, permuted by ρ .

If C is a bounded component and $\rho(C) \neq C$ then $C \cup \rho(C)$ would intersect some line 4 times, a contradiction. Hence ρ preserves each bounded component, and each bounded component intersects L twice at right angles. Since E_λ intersects $\mathbf{CP}^2 - \mathbf{C}^2$ three times, and exactly one of these intersection points, namely $[1 : 1 : 0]$, lies in \mathbf{RP}^2 , we see that $E_\lambda \cap \mathbf{RP}^2$ has one unbounded component.

Note that $E_\lambda \cap L$ always consists of 3 points, namely $(x, -x)$ for $x = 0$ and

$$x = \frac{-\lambda \pm \sqrt{\lambda^2 + 32}}{8} \neq 0.$$

We conclude that $E_\lambda \cap \mathbf{RP}^2$ must have exactly two components, both ρ -invariant, one bounded and one unbounded, and the intersections are as claimed. ♠

2.3 Uniformization

Let $E \subset \mathbf{CP}^2$ be nonsingular cubic variety. As we mentioned above, E is a complex torus. Let $f : E \rightarrow E$ be some birational map which is also invertible. The birational nature of f implies that all the singularities of f on E are removable. This means that f is a biholomorphic map of E and orientation preserving.

As is well known, there is also a biholomorphic map $\phi : E \rightarrow \mathbf{C}/\Lambda$ where $\Lambda \subset \mathbf{C}$ is a lattice. The map ϕ conjugates f to an isometry of \mathbf{C}/Λ . Thus, we may simply equip E with the coordinates coming from ϕ and treat E as a flat torus and $f : E \rightarrow E$ as an orientation preserving isometry. We call this the *flat structure* on E .

2.4 Minor Subsets of Tori

In this section I will prove a general lemma about flat tori. The only case required for the proof of Theorem 1.2 is that of the circle \mathbf{R}/\mathbf{Z} , but the general case might be useful for a more general version of the conjecture. The general case rather quickly reduces to the circle case anyway.

Say that a subset $S \subset \mathbf{R}^n/\mathbf{Z}^n$ is *minor* if there is some translation ϕ of $\mathbf{R}^n/\mathbf{Z}^n$ such that $\phi(S) \subset (0, 1/2)^n$. In general, say that a subset S of a flat torus is minor if an affine isomorphism from the flat torus to $\mathbf{R}^n/\mathbf{Z}^n$ carries S to a minor subset.

Lemma 2.1 *Suppose $p \in S \subset Y$ where S is a minor subset of the flat torus Y . Suppose that $f : Y \rightarrow Y$ is a nontrivial translation. Then the forward orbit $\{f^k(p) \mid k > 0\}$ is not contained in S .*

Proof: By affine symmetry it suffices to prove this when $Y = \mathbf{R}^n/\mathbf{Z}^n$. The translation f has a nontrivial action in at least one coordinate. Let $f : \mathbf{R}^n/\mathbf{Z}^n \rightarrow \mathbf{R}^n/\mathbf{Z}^n$ be the projection onto this coordinate. By construction $f(S)$ is minor in \mathbf{R}/\mathbf{Z} and f covers a nontrivial translation of \mathbf{R}/\mathbf{Z} . This reduction shows that it suffices to prove our result for $Y = \mathbf{R}/\mathbf{Z}$. This is what we do. If $f(p) \notin S$ then we are done. Otherwise $|p - f(p)| < 1/2$. But then the forward orbit of p is at least $1/4$ -dense. This means that every point of \mathbf{R}/\mathbf{Z} is within $1/4$ of some point in the forward orbit. In particular, the point ζ diametrically opposed from the midpoint of S has this property. But then the orbit point that is within $1/4$ of ζ is disjoint from S . ♠

3 Proof of the Result

3.1 Formulas

Let \mathcal{X} denote the space of labeled 8-gons with 4-fold rotational symmetry modulo similarities in the plane. We normalize so that the 4-fold symmetry in question is the map

$$\rho([x : y : z]) = [-y : x : z]. \quad (8)$$

This map fixes the origin $(0, 0)$ in the affine patch and preserves the whole affine patch. It is just rotation by 90 degrees counterclockwise.

One possibility is that ρ cycles the vertex labels by 2 and the other possibility is that ρ cycles the vertex labels by -2 . We only consider the first possibility; the second possibility is essentially treated by symmetry. For the purpose of getting formulas, we ignore for now the members of \mathcal{X} which have points on the line at infinity. We call the remaining members *finite*. In other words, the finite members lie entirely in the standard affine patch.

The Map: Every finite member of \mathcal{X} has a canonical representative $P(x, y)$ with vertices

$$(1, 0), (x, y), (0, 1), (-y, x), (-1, 0), (-x, -y), (0, -1), (y, -x). \quad (9)$$

Here (x, y) is really $[x : y : 1]$, etc.

Expressed in these coordinates, and with a suitable labeling scheme, the map T_3 is given by $T_3(x, y) = (x', y')$, where

$$x' = -Ax(B - 2xy), \quad y' = +Ay(B + 2xy), \quad (10)$$

$$A = \frac{\alpha_{10} + \alpha_{20}}{(1 + \alpha_{10})(\alpha_{20} + 2\alpha_{30} + \alpha_{40} - \alpha_{11} - 2\alpha_{12} + \alpha_{22})}, \quad \alpha_{ij} = x^i y^j + y^i x^j.$$

$$B = \beta_{10} + 2\beta_{20} + \beta_{30} + \beta_{12}, \quad \beta_{ij} = x^i y^j - y^i x^j. \quad (11)$$

In other words, T_3 sends the polygon $P(x, y)$ to the polygon $P(x', y')$. I computed this map (and everything else in the paper) using Mathematica [6].

The Invariant: Define the function

$$\Psi(x, y) = \frac{(x - y)(x^2 - y^2 - 1)}{xy}. \quad (12)$$

A direct calculation shows that $\Psi \circ T_3 = \Psi$. This is the invariant mentioned in the introduction. In the next section I will explain where it comes from.

Projective Duality: Each 8-gon P , defined by its vertices, gives rise to an 8-gon P^* in the dual space defined by the successive lines. The successive “vertices” of P^* are the successive lines extending the edges of P . Using our isomorphism, we get a second polygon $(P^*)^\#$ in \mathcal{X} . The operation $P \rightarrow (P^*)^\#$ is an involution given algebraically by the map

$$D(x, y) = \left(-\frac{y(x^2 - x + y^2 - y)}{x(x^2 - 2x + y^2 + 1)}, \frac{y(x + y - 1)}{x(x^2 - 2x + y^2 + 1)} \right) \quad (13)$$

Direct calculations show

$$\Psi \circ D = \Psi, \quad DT_3D^{-1} = T_3^{-1}. \quad (14)$$

One can also deduce these equations from abstract properties of projective duality. I will leave this to the interested reader.

Symmetries and Factorization: Define

$$\sigma_1(x, y) = (y, x), \quad \sigma_2(x, y) = (-x, -y). \quad (15)$$

A direct calculation shows that

$$\sigma_1T_3\sigma_1^{-1} = T_3, \quad \sigma_2T_3\sigma_2^{-1} = T_3^{-1}. \quad (16)$$

Geometrically, the map σ_2 swaps the regular and star-regular 8-gons. Beautifully, a calculation shows that

$$T_3 = (D \circ \sigma_2)^2. \quad (17)$$

In other words T_3 is the square of a simpler map. Readers familiar with the pentagram map will not be surprised by this kind of factorization. The map $D \circ \sigma_2$ satisfies the rule

$$\Psi \circ (D \circ \sigma_2) = -\Psi. \quad (18)$$

This equation would probably be the quickest way for the reader to show, without symbolic manipulation, that Ψ is an invariant for T_3 .

3.2 Special Cases

I first noticed that T_3 behaved nicely on the sets described in this section. I then systematically tested Laurent monomials in the defining functions for these sets and this led me to Ψ .

The Coordinate Axes: First of all

$$T_3(x, 0) = (-x, 0), \quad T_3(0, y) = (0, -y), \quad (19)$$

So, T_3 preserves the coordinate axes and is an involution there. The corresponding octagons look (to me) like the blades of a circular saw.

The Diagonal Line: Let Δ denote the diagonal line $x = y$. The 8-gon $P_0 = P(x, x)$ has 8-fold dihedral symmetry, with the lines of symmetry going through the vertices. To study the $(8, 3)$ -pentagram orbit $\{P_j\}$ we compute

$$T_3(x, x) = (x', x'), \quad x' = \frac{1+x}{1+2x}. \quad (20)$$

The map T_3 is given by a projective transformation of Δ . The fixed points are

$$p_{\pm} = \pm(1/\sqrt{2}, 1/\sqrt{2}). \quad (21)$$

The fixed point p_+ , which corresponds to the regular 8-gon, is attracting. The fixed point p_- , which corresponds to the star-regular 8-gon, is repelling. Thus, every orbit on Δ aside from p_- , is attracted to p_+ . So, if P_0 is not regular then P_j is convex for all $j \geq 0$. The inverse map T_3^{-1} has p_- as an attracting fixed point and p_+ as a repelling fixed point. Hence P_j is non-convex for all j sufficiently negative.

The Unit Circle: Let S^1 denote the unit circle. Here S^1 corresponds to 8-gons with 8-fold dihedral symmetry in which the lines of symmetry bisect the sides. These 8-gons are dual to the ones on Δ and indeed the map D defined above has the property that $D(\Delta) = S^1$. Thus, the action of T_3^{-1} on S^1 is conjugate to the action of T_3 on Δ . In particular, if we start with C_0 convex then C_j is convex for all $j \leq 0$ but C_j is non-convex for all j sufficiently positive.

This is a case that the reader can easily experiment with. Just draw a “stop sign” and see what the 3-diagonal map does.

Other Special Orbits: The material here is not needed for the proof of Theorem 1.2 but it is nice. Let $L_{\pm 2,1}$ denote the diagonal line $y = x \mp 1$ and let $L_{\pm 2,2}$ denote the circle of radius $1/\sqrt{2}$ centered at $(\mp 1/2, \pm 1/2)$. The union $L_{\pm 2} = L_{\pm 2,1} \cup L_{\pm 2,2}$ is the level set $\Psi = \pm 2$. Each of these two level sets is the disjoint union of a diagonal line and a circle. The map T_3^2 preserves each component and acts there with order 3. For example

$$T_3^2(x, x+1) = (x'', x''+1), \quad x'' = \frac{-1-x}{x}. \quad (22)$$

3.3 Nontriviality

Let $\lambda \in \mathbf{R} - \{-2, 0, 2\}$. We know that $E_\lambda \cap \mathbf{R}^2$ contains one bounded loop and one unbounded loop. Since T_3 preserves E_λ and \mathbf{R}^2 and \mathbf{RP}^2 , we see that $f = T_3^2$ preserves both components of $E_\lambda \cap \mathbf{RP}^2$. Let Υ be one of these.

Lemma 3.1 *f cannot be the identity on Υ .*

Proof: Suppose f is the identity on Υ . Since f is a orientation preserving isometry E_λ in the flat coordinates, we see that f must be the identity on E_λ . We saw in §2.2 that E_λ intersects the line $\{y = -x\}$ in 3 distinct points. Hence $f(x, -x) = (x, -x)$ for two distinct nonzero points $(x_1, -x_1)$ and $(x_2, -x_2)$ in E_λ . Setting the sum of the coordinates of $f(x, -x)$ equal to 0, we get

$$\frac{(4x^2(-1 - 2x^2 + x^4 - 6x^6 + 32x^{10}))}{((-1 + x^2 + 4x^4)(1 - 2x^2 + x^4 + 24x^6 + 16x^8))} = 0. \quad (23)$$

The only nonzero real roots are

$$x = \pm \frac{1}{2} \sqrt{\frac{1}{2}(1 + \sqrt{17})}.$$

But the corresponding points satisfy $\Psi(x_1, -x_1) = -\Psi(x_2, -x_2) \neq 0$ so these points cannot both lie in E_λ . This is a contradiction. ♠

Remark: The points $(x_1, -x_1)$ and $(x_2, -x_2)$ constructed in the previous proof lie on level sets where f has order 2. In particular, f^2 is the identity on these two level sets.

We call an orientation preserving isometry of a metric circle a *translation*.

Lemma 3.2 *f is a nontrivial translation of Υ in the flat coordinates.*

Proof: Since f is a nontrivial isometry of Υ , we see that f is either a translation or an orientation reversing isometry on Υ . In the latter case, f must have order 2 on Υ . If f is orientation reversing on Υ then f is orientation reversing on nearby level sets. This means that f^2 is the identity on all nearby level sets. But then f^2 is the identity on an open subset of \mathbf{R}^2 . Since f is a birational map, this forces $f^2 = T_3^4$ to be the identity. This is false. ♠

3.4 The End of the Proof

Let $\mathcal{C} \subset \mathcal{X}$ denote the subset of convex 8-gons. Let \mathcal{C}_- and \mathcal{C}_+ respectively denote the subsets of \mathcal{C} corresponding to 8-gons with 8-fold dihedral symmetry with the lines of symmetry respectively going through the vertices and bisecting the edges. Note that $\mathcal{C}_+ \cap \mathcal{C}_-$ is precisely the point representing the regular 8-gon.

The analysis of the special sets in §3.2 reduces us to considering polygons in $\mathcal{C} - \mathcal{C}_+ - \mathcal{C}_-$. To finish the proof it suffices to show that when we have $P_0 \in \mathcal{C} - \mathcal{C}_- - \mathcal{C}_+$ there are indices $i < 0 < j$ such that $P_i, P_j \notin \mathcal{C}$. Our choice of P_0 implies that $\Psi(p) \in \mathbf{R} - \{-2, 0, 2\}$. Here p is the point in \mathcal{C} representing P_0 . But then $p \in E_\lambda \cap \mathbf{R}^2$, where E_λ is the complex torus discussed in §2.2.

Figure 2 shows a picture of the relevant sets. The shaded semidisk is \mathcal{C} . The lightly shaded disk is the unit disk. The sets \mathcal{C}_+ and \mathcal{C}_- are the intersection of \mathcal{C} with the unit circle and with the diagonal line respectively. The blue curve is a rough but topologically accurate sketch of $E_\lambda \cap \mathbf{R}P^2$ when $\lambda \in (0, 2)$. If $\lambda \in (-2, 0)$ the picture would be reflected in the line $\{y = x\}$.

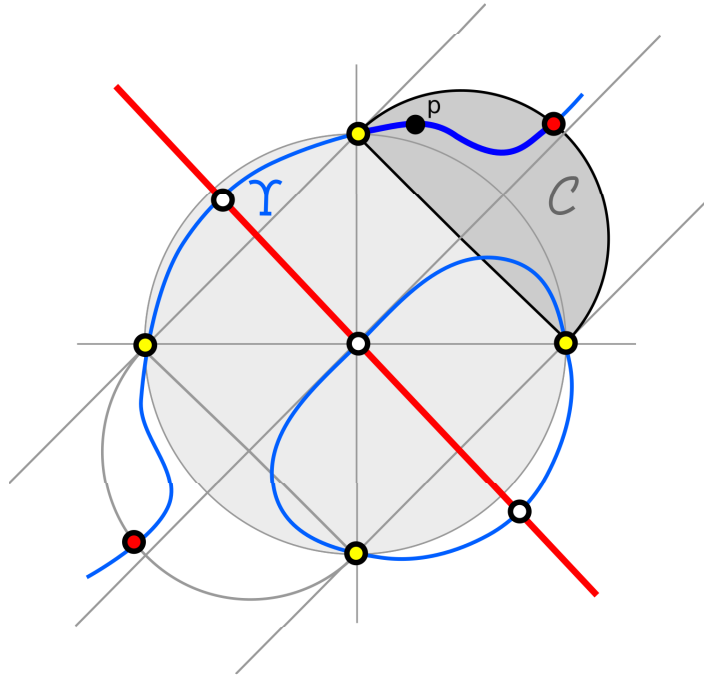


Figure 2: A topologically accurate sketch of $E_\lambda \cap \mathbf{R}^2$ for $\lambda \in (0, 2)$.

Let Υ be the component of $E_\lambda \cap \mathbf{R}P^2$ containing p . We equip Υ with its flat coordinates. Let $f = T_3^2$ as in the previous section. We know that f is a nontrivial translation of Υ . Let ρ be the reflection in the line $\{y = -x\}$. This is the red diagonal in Figure 2. We showed in §2.2 that $\rho(\Upsilon) = \Upsilon$. Being holomorphic, ρ is an isometry of Υ in the flat coordinates. The two sets $\Upsilon \cap \mathcal{C}$ and $\rho(\Upsilon \cap \mathcal{C})$ are disjoint and have equal length. Hence $\Upsilon \cap \mathcal{C}$ is minor in the sense of Lemma 2.1. By Lemma 2.1, we have $f^j(p) \notin \Upsilon \cap \mathcal{C}$ for some $j > 0$. Applying the same argument to f^{-1} gives some $i < 0$ such that $f^i(p) \notin \mathcal{C}$. This completes the proof of Theorem 1.2.

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