When are smooth rational surfaces log Fano and/or Mori Dream?

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Introduction

Smooth rational surfaces are Mori Dream if and only if their Cox rings are finitely generated. It's interesting to consider when such a surface is Mori Dream as these can be naturally embedded in some better understood toric variety with related birational geometry, allowing for the combinatorial description of certain fundamental invariants (e.g. the cone of effective divisors).

On the other hand, smooth rational surfaces that are log Fano form a subcategory of Mori Dream spaces, defined as having an anticanonical divisor that can be made ample after subtracting off a suitably constrained effective divisor. While these two categories can be identified under certain conditions in the case of smooth surfaces (e.g. when $-K_X$ is big and movable) this is not true in general.

Results and Future Work

Open questions remain as to when a smooth rational surface is log Fano and/or Mori Dream. For example, it's conjectured that $\mathbb{G}_a^2$-equivariant compactifications have finitely generated Cox rings (hence, are Mori Dream) as such is the case for $\mathbb{G}_a^n$-equivariant compactifications when $n \geq 3$, this is confirmed over the plane but not over the Hirzebruch surfaces. More generally, it's conjectured that all smooth rational surfaces over the plane are Mori Dream spaces.

The existence of smooth rational surfaces over the plane that are Mori Dream but not log Fano is demonstrated by considering particular sequences of blow-ups over a fixed hyperplane, some of these are also $\mathbb{G}_a^2$-equivariant (see the example below).