MODULI OF CONIC SURFACES OVER THE LINE

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Abstract. We study moduli spaces and monodromy groups for surfaces fibered in conics over the projective line, with a view toward explicit presentations and arithmetic applications.

The last decade has seen major advances in our understanding of moduli spaces of Fano varieties through new techniques like K-stability. del Pezzo surfaces are an important guiding example – see [OSS16], for instance – where modern moduli spaces may be related to approaches via Geometric Invariant Theory (GIT). Surfaces with conic bundle structures, beyond the del Pezzo case, have received less attention. Some singular del Pezzo surfaces also come with natural conic fibrations e.g. degree-$2a$ hypersurfaces in $\mathbb{P}(1,1,a,a)$ [LP22]. Threefolds with conic fibrations over the plane have been studied in specific cases [DJKHQ24].

Here we will focus on surfaces fibered in conics over $\mathbb{P}^1$. These have long been studied with a view toward Diophantine problems e.g. [CTS87, §2.6]. Our target application is over finite fields, so we avoid using characteristic-zero or analytic techniques. We focus on explicit constructions of parameter spaces for these varieties, focusing on presentations where sampling can be done quickly and efficiently. A forthcoming paper by the second author [Her26] will analyze the statistical behavior of rationality for these varieties over large finite fields.

Section 1 presents conic bundles and their classification, focusing both on equations and inductive structures under birational maps. We turn to the Picard group in Section 2, presenting Weyl-group symmetries arising from the intersection form. Moduli spaces, and associated birational transformations, are the subject of Section 3: Theorem 28 is our main result; Corollary 29 gives its applications to monodromy. Sections 4 and 5 link our perspective with other moduli frameworks: Parabolic bundles and complete intersections.

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1. Background on conic bundles

Let k be a field of characteristic different from two.

1.1. **Basic notions.**

Definition 1. A *good conic bundle over* $\mathbb{P}^1$ consists of a smooth projective surface X and a dominant morphism $\phi : X \to \mathbb{P}^1$ such that:

(1) the geometric generic fiber of ϕ is $\mathbb{P}^1$:
(2) non-smooth fibers are isomorphic to two copies of $\mathbb{P}^1$ meeting in a node.

Being "good" is an open condition for families of such conic bundles.

Proposition 2. *The enumerated conditions of Definition 1 are equivalent to*

1'. *the geometric generic fiber of ϕ is smooth and connected;*
2'. *ω_ϕ^{-1} is ample relative to ϕ.*

Furthermore, ω_ϕ^{-1} is very ample and globally generated relative to ϕ, inducing an embedding

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}((\phi_*\omega_\phi^{-1})^\vee) \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\varpi} & \\ & \mathbb{P}^1 & \end{array}$$

where π is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$. The discriminant divisor of ϕ is reduced.

Proof. Keep in mind that ϕ is flat and projective.

Assume (1) and (2). Condition (1') is a weakening of (1). Our assumptions imply that, on each fiber $X_p = \phi^{-1}(p)$, global sections of $\omega_{X_p}^{-1}$ imbed X_p as a conic in $\mathbb{P}^2$; further, $\omega_{X_p}^{-1}$ has no higher cohomology. By cohomology-and-base-change, we see that ω_ϕ^{-1} is very ample and globally generated relative to ϕ, inducing an embedding as a conic divisor in a $\mathbb{P}^2$-bundle. Condition (2') follows.

Assume (1') and (2'). A smooth projective geometrically-connected curve with ample anti-canonical classes is necessarily a smooth conic. A Gorenstein projective curve C with $\Gamma(\mathcal{O}_C) = k$ is a plane curve of degree two [Sta25, Lem. 53.10.3, Tag 0C6N]. Of course, a Cartier divisor in a smooth surface is Gorenstein. The case of a double line is precluded, as such a fiber could only occur for singular X; indeed, the smoothness of X guarantees that the discriminant of ϕ is reduced. The cases enumerated in (1) and (2) are the remaining possibilities. □

Definition 3. Consider a good conic bundle $\phi : X \to \mathbb{P}^1$ with

$$(\phi_* \omega_\phi^{-1})^\vee \simeq \mathcal{O}_{\mathbb{P}^1}(a_0) \oplus \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2), \quad a_0 \leq a_1 \leq a_2.$$

The triple $(a_0, a_1, a_2) \in \mathbb{Z}^3$ is the *splitting type* of the bundle; it is *balanced* if $a_2 - a_0 \leq 1$.

Being "balanced" is an open condition on conic bundles, as generic vector bundles on $\mathbb{P}^1$ have this property.

Proposition 4. *Let $\phi : X \to \mathbb{P}^1$ be a good conic bundle of splitting type (a_0, a_1, a_2). Then the discriminant of ϕ has degree $a_0 + a_1 + a_2$ and*

$$\chi(X, T_X) = 6 - 2(a_0 + a_1 + a_2).$$

Proof. Recall that the discriminant divisor is reduced; its degree n is the number of degenerate fibers of ϕ. The topological Euler characteristic $\chi(X) = 4 + n$. By the Noether formula and the Gauss-Bonnet theorem, $n = 8 - c_1(\omega_X)^2$. The Grothendieck-Riemann-Roch formula [Ful98, Th. 15.2] implies

$$\deg(\phi_* \omega_\phi^{-1}) = c_1(\omega_X)^2 - 8.$$

Combining these gives the first statement. Riemann-Roch on X gives the formula for its tangent bundle. □

Corollary 5. *For a balanced good conic bundle $\phi : X \to \mathbb{P}^1$ with n degenerate fibers we have*

$$(\phi_* \omega_\phi^{-1})^\vee = \mathcal{O}_{\mathbb{P}^1}(\lfloor n/3 \rfloor) \oplus \mathcal{O}_{\mathbb{P}^1}(\lfloor (n+1)/3 \rfloor) \oplus \mathcal{O}_{\mathbb{P}^1}(\lfloor (n+2)/3 \rfloor).$$

We analyze conic bundles under birational morphisms:

Proposition 6. *Given a birational morphism of good conic bundles*

$$\begin{array}{ccc} X & \xrightarrow{\beta} & X' \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\varphi} & \\ & \mathbb{P}^1 & \end{array}$$

we have an inclusion

$$\phi_* \omega_\phi^{-1} \hookrightarrow \varphi_* \omega_\varphi^{-1}$$

and $-a_2(\phi) \leq -a_2(\varphi)$. If X has a section of self-intersection $-r$ with $r \geq 0$ then $a_2(\phi) \geq r$.

Proof. The inclusion $\omega_\phi^{-1} \hookrightarrow \beta^*\omega_\varphi^{-1}$ is obtained by iteratively applying the blowup formula for surfaces: If $b : \mathrm{Bl}_s(S) \to S$ is the blowup of a smooth surface with exceptional curve E then

$$\omega_{\mathrm{Bl}_s(S)} = b^*\omega_S(E).$$

Taking direct images gives the inclusion of sheaves on $\mathbb{P}^1$. It follows that the most negative summand of $\phi_*\omega_\phi^{-1}$ is no greater than the most negative summand of the $\varphi_*\omega_\varphi^{-1}$; this gives the first inequality.

Suppose X admits a section Σ of self-intersection $-r$. Blowing down fibral components disjoint from Σ induces birational morphsm $\beta : X \to \mathbb{F}_r$ to a Hirzebruch surface. Since $a_2(\mathbb{F}_r/\mathbb{P}^1) = r$ we get the final assertion. □

1.2. **Key constructions.**

Example 7. In each case below ϕ is projection onto the first factor.

(1) [Isk67, §3] Let $X = \{G = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^2$ be smooth of bidegree $(a, 2)$. Then the splitting type is (a, a, a).

(2) Let $X \subset \mathbb{P}^1 \times \mathbb{P}^3$ be a smooth complete intersection of forms F and G of bidegrees $(1, 1)$ and $(a, 2)$. Assume that $Y := \{F = 0\}$ is irreducible, which implies

$$Y \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^2 \oplus \mathcal{O}_{\mathbb{P}^1}(-1)).$$

The sheaf $(\pi_1)_*(\mathcal{O}_Y(0, 1))$ is presented

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}^4 \to (\pi_1)_*(\mathcal{O}_Y(0, 1)) \to 0$$

whence

$$(\pi_1)_*\mathcal{O}_Y(0, 1) \simeq \mathcal{O}_{\mathbb{P}^1}^2 \oplus \mathcal{O}_{\mathbb{P}^1}(1).$$

It follows that

$$\phi_*\omega_\phi^{-1} = \mathcal{O}_{\mathbb{P}^1}(-1 - a)^2 \oplus \mathcal{O}_{\mathbb{P}^1}(-a)$$

and the splitting type is $(a, a + 1, a + 1)$.

(3) Let $X \subset \mathbb{P}^1 \times \mathbb{P}^4$ be a smooth complete intersection cut out by two forms

$$F_1 = L_{11}t_1 + L_{12}t_2, \quad F_2 = L_{21}t_1 + L_{22}t_2, \quad \mathbb{P}^1 = \mathbb{P}^1_{[t_1,t_2]},$$

of bidegree $(1, 1)$ and one form G of bidegree $(a-1, 2)$. Assume that $Y := \{F_1 = F_2 = 0\}$ is generic, i.e. the L_{ij} are linearly independent. It follows that

$$Y \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^2).$$

We find

$$0 \to \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^4}(-2. - 2) \to \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^4}(-1, -1)^2 \to \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^4} \to \mathcal{O}_Y \to 0 \quad (1.1)$$

whence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1)^2 \to \mathcal{O}_{\mathbb{P}^1}^5 \to (\pi_1)_*(\mathcal{O}_Y(0,1)) \to 0$$

and

$$(\pi_1)_*\mathcal{O}_Y(0,1) \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^2.$$

We find that

$$\phi_*\omega_\phi^{-1} = \mathcal{O}_{\mathbb{P}^1}(-1-a) \oplus \mathcal{O}_{\mathbb{P}^1}(-a)^2$$

with splitting type $(a, a, a+1)$.

Remark 8. On first glance, the splitting type $(0, 0, 1)$ is missing. However for $Y = \{F_1 = F_2 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^4$, the restriction homomorphism

$$\Gamma(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^4}(-1,2)) \to \Gamma(\mathcal{O}_Y(-1,2))$$

fails to be surjective; this follows by computing the twist of (1.1) by $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^4}(-1,2)$. Indeed

$$\Gamma(\mathcal{O}_Y(-1,2)) \simeq \Gamma(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^2 + \mathcal{O}_{\mathbb{P}^1}(1)^3),$$

which is eight-dimensional. These divisors all contain the distinguished section of $Y \to \mathbb{P}^1$.

Example 9. Consider the complete intersection

$$Y_0 = \{L_1t_1 - L_2t_2 = L_2t_1 - L_3t_2 = L_1L_3 - L_2^2 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^4,$$

with $L_1, L_2, L_3 \in \Gamma(\mathcal{O}_{\mathbb{P}^4}(1))$ linearly independent. It is also smooth and is the image of

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^2 \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) \hookrightarrow \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^5) \simeq \mathbb{P}^1 \times \mathbb{P}^4$$

induced by global sections. Conic bundles obtained from Y_0 are not balanced.

Example 10. Over algebraically closed fields, representative balanced good conic bundles may be obtained via blowing up $\beta : X \to \mathbb{P}^1 \times \mathbb{P}^1$. Let f_1 and f_2 denote fibers of projections to $\mathbb{P}^1$. We follow the taxonomy of Example 7. In each case, the center $\{s_1, \dots, s_n\}$ of β consists of points in distinct fibers of the first projection; $\phi : X \to \mathbb{P}^1$ is the composed morphism.

(1) Choose points $s_1, \dots, s_{3a}$ imposing independent conditions on the linear series $|2f_2 + af_1|$. Then $\phi : X = \mathrm{Bl}_{s_1,\dots,s_{3a}}(\mathbb{P}^1 \times \mathbb{P}^1) \to \mathbb{P}^1$ yields a good conic fibration of type (a, a, a).
(2) With points $s_1, \dots, s_{3a+2}$ imposing independent conditions on $|2f_2 + af_1|$, $X = \mathrm{Bl}_{s_1,\dots,s_{3a+2}}(\mathbb{P}^1 \times \mathbb{P}^1) \to \mathbb{P}^1$ is a fibration of type $(a, a+1, a+1)$.

(3) With points $s_1, \dots, s_{3a+1}$ imposing independent conditions on $|2f_2 + af_1|$ we get a fibration $X = \mathrm{Bl}_{s_1,\dots,s_{3a+1}}(\mathbb{P}^1 \times \mathbb{P}^1) \to \mathbb{P}^1$ of type $(a, a, a+1)$.

Example 11 (Balance and birationality). Retain the notation of Proposition 6. Then $\phi : X \to \mathbb{P}^1$ may be balanced even when $\varphi : X' \to \mathbb{P}^1$ fails to be so. For example, consider

$$s_1, s_2, s_3, s_4 \in \mathbb{P}^1 \times \mathbb{P}^1, \quad \pi_2(s_1) = \pi_2(s_2)$$

but otherwise generic. Take

$$X := \mathrm{Bl}_{s_1,s_2,s_3,s_4}(\mathbb{P}^1 \times \mathbb{P}^1) \xrightarrow{\beta} \mathrm{Bl}_{s_1,s_2}(\mathbb{P}^1 \times \mathbb{P}^1) =: X',$$

with structure morphisms ϕ and φ projecting onto the first factor. Proposition 6 gives

$$\mathcal{O}_{\mathbb{P}^1}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \simeq \phi_*\omega_\phi^{-1} \hookrightarrow \varphi_*\omega_\varphi^{-1} \simeq \mathcal{O}_{\mathbb{P}^1}^2 \oplus \mathcal{O}_{\mathbb{P}^1}(-2).$$

In particular, $\phi : X \to \mathbb{P}^1$ is balanced despite having a section with self-intersection (-2) i.e. the horizontal ruling containing s_1 and s_2.

Example 12 (Balance and Singularity). Now consider X obtained by blowing up four points on the diagonal

$$s_1, s_2, s_3, s_4 \in \Delta_{\mathbb{P}^1} \subset \mathbb{P}^1 \times \mathbb{P}^1$$

but otherwise generic. Let $D \subset X$ denote the proper transform of the diagonal, with $D^2 = -2$. We obtain an embedding

$$X \hookrightarrow \mathbb{P}((\phi_*\omega_\phi^{-1})^\vee) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)^2 \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^1}) \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^4.$$

However, on projection to $\mathbb{P}^4$, D is contracted to a point x_0; this morphism comes from the linear series of bidegree $(2,2)$ forms on $\mathbb{P}^1 \times \mathbb{P}^1$ vanishing at s_1, s_2, s_3, s_4. The image of X in $\mathbb{P}^4$ has an ordinary double point at x_0.

2. Lattices and Weyl groups

Let $\phi : X \to \mathbb{P}^1$ be a good conic bundle over an algebraically closed field with $n \geq 1$ degenerate fibers. Write f for the class of a fiber of ϕ and take Σ to be a section of ϕ, which exists by Tsen's Theorem. Write $E_1', E_1'', \dots, E_n', E_n''$ for the irreducible components of the degenerate fibers so that

$$f \equiv E_1' + E_1'' \equiv \cdots \equiv E_n' + E_n'';$$

we label so that

$$\Sigma \cdot E_i' = 1, \Sigma \cdot E_i'' = 0, \quad i = 1, \dots, n.$$

We have

$$(E_i')^2 = (E_i'')^2 = -1, \quad E_i' \cdot E_i'' = 1,$$

with the other intersection numbers among $\{E'_1, \dots, E''_n\}$ equal to zero. Note that $\{f, \Sigma, E''_1, \dots, E''_n\}$ freely generates $\mathrm{Pic}(X)$ and

$$K_X = -(r+2)f - 2\Sigma + E''_1 + \cdots + E''_n, \quad \Sigma^2 = -r.$$

Remark 13. Example 10 gives special cases with

$$f = f_1, \Sigma = f_2, E''_i = E_i, E'_i = f_1 - E_i$$

so that $r = 0$ and

$$K_X = -2f_1 - 2f_2 + E_1 + \cdots + E_n.$$

The basis $\{f_1, f_2, E_1, \dots, E_n\}$ may be used for all good conic fibrations with a section of even self-intersection, not just those arising as blowups of $\mathbb{P}^1 \times \mathbb{P}^1$. Indeed, we may take

$$f = f_1, \Sigma = f_2 - \frac{r}{2} f_1, E''_i = E_i, E'_i = f_1 - E_i.$$

Definition 14. Let $\phi : X \to \mathbb{P}^1$ be a good conic bundle with $n \geq 1$ degenerate fibers

$$E'_1 \sqcup E''_1, \dots, E'_n \sqcup E''_n.$$

The collection of disjoint fibral curves

$$\{E'_1, \dots, E'_n\}$$

is *even* (resp. *odd*) if the rational ruled surface obtained by blowing it down

$$\beta : X \to \mathbb{F}_r$$

has r even (resp. odd).

When r is even (resp. odd) then sections of $\mathbb{F}_r \to \mathbb{P}^1$ all have even (resp. odd) self-intersection. Note that if $\{E'_1, E'_2 \dots, E'_n\}$ is even then $\{E''_1, E'_2, \dots, E'_n\}$ is odd, as the blow-down associated with the second collection is isomorphic to $\mathbb{F}_{r-1}$ or $\mathbb{F}_{r+1}$.

We record a classical result (see, for instance, [Sko86, Has09]) on intersections:

Proposition 15. *Consider* $\mathrm{Pic}(X)$ *as a unimodular lattice under the intersection form. For* $n \geq 2$*,*

$$R_n := \langle K_X, f \rangle^{\perp} \subset \mathrm{Pic}(X)$$

is isomorphic to the root lattice for D_n *and its extension*

$$W_n := \mathrm{Pic}(X) / \langle K_X, f \rangle$$

is isomorphic to the weight lattice. The group

$$\{g \in \mathrm{Aut}(\mathrm{Pic}(X)) : g(f) = f, g(K_X) = K_X\}$$

equals the Weyl group $W(D_n)$.

Proof. The intersection pairing induces

$$R_n \subset W_n \subset R_n \otimes \mathbb{Q}.$$

We use the basis of Remark 13, so that

$$R_n = \langle f_1 - E_1 - E_2, E_1 - E_2, E_2 - E_3, \dots, E_{n-1} - E_n \rangle ,$$

which has the desired intersections. Take the images of

$$E_1, \dots, E_n, f_2$$

as generators of W_n; setting

$$L_i := E_i - \frac{1}{2} f_1 \in R_n \otimes \mathbb{Q}$$

we have

$$\frac{L_1 + \cdots + L_n}{2} \equiv f_2 \pmod{\mathbb{Q}K_X + \mathbb{Q}f_1}.$$

This corresponds to the weights of D_n, in the notation of [FH91, §18.1,19.2]; the roots are $\pm L_i \pm L_j, i \neq j$, i.e. differences of components in distinct singular fibers. The Weyl group $W(D_n)$ in the basis $\{L_i\}$ consists of signed permutations with an even number of -1's; this coincides with permutations of irreducible components of degenerate fibers, compatible with intersections and preserving parities of collections (see Definition 14) .

We turn to the last assertion of the Proposition. Observe that the discriminant groups

$$R_n^\vee / R_n \simeq \langle K_X, f \rangle^\vee / \langle K_X, f \rangle$$

are isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ (for n even) or $\mathbb{Z}/4\mathbb{Z}$ (for n odd). Elements of the Weyl group act trivially on discriminant group. Every automorphism of R_n acting as the identity on the discriminant group $R_n^\vee/R_n$ is in the Weyl group [Ser01, V.11]. □

Remark 16. For $n \geq 3$, the full group $\mathrm{Aut}(R_n)$ is a semidirect product of $W(D_n)$ with the automorphisms of the Dynkin diagram D_n [Ser01, V.11]. The latter factor encodes outer automorphisms of the associated complex Lie group [Ser01, VI.3]: These are the symmetric group $\mathfrak{S}_3$ for D_4 and the cyclic group C_2 for the other n.

Leaving the conic fibration unspecified yields more complicated indefinite root systems:

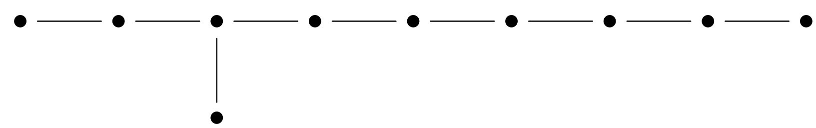

FIGURE 1. $T_{3,7,2}$ diagram for $n = 9$

Example 17. The blowup $X' \to \mathbb{P}^1 \times \mathbb{P}^1$ at n sufficiently generic points is a good conic bundle with n degenerate fibers, with respect to either of the projections to $\mathbb{P}^1$. Now $K_{X'}^{\perp}$ has basis

$$f_2 - f_1, f_1 - E_1 - E_2, E_1 - E_2, E_2 - E_3, \ldots, E_{n-2} - E_{n-1}, E_{n-1} - E_n,$$

where the f_i are the classes associated with fibrations over $\mathbb{P}^1$ and $E_1, \ldots, E_n$ are the exceptional classes. This lattice is associated with the root system $T_{3,n-2,2}$:

The diagram $T_{3,n-2,2}$ is a tree of (-2)-curves with chains of lengths $3, n-2$, and 2 attached to the unique trivalent vertex (Figure 1). See [Muk04, p. 127] but note the typos in the formulas. The subgroup fixing f_1 is isomorphic to $W(D_n)$.

Let $G = \mathrm{Spin}_{2n}$ denote the simply-connected semisimple linear algebraic group associated with the Dynkin diagram D_n. Consider the half-spin representations [FH91, §20.1]

$$\rho_{\pm} : \mathrm{Spin}_{2n} \to \mathrm{GL}_{2^{n-1}}.$$

They both have weights taken from

$$\left\{ \frac{\pm L_1 \pm L_2 \cdots \pm L_n}{2} \right\}; \qquad (2.1)$$

the weight sets are those consisting of elements with even and odd numbers of -1's; these sets are conjugate under an outer automorphism.

Proposition 18. *The half-spin representations have weights corresponding to*

$$\{E \in \mathrm{Pic}(X) : E^2 = -1, K_X E = -1, f \cdot E = 1\}$$

and

$$\{C \in \mathrm{Pic}(X) : C^2 = 0, K_X C = -2, f \cdot C = 1\},$$

i.e. numerical exceptional curves and conics of degree one with respect to ϕ.

Proof. The weights of each half-spin representation form a single orbit under the $W(D_n)$-action. On the other hand, the geometric descriptions of the exceptional curves and conics are also compatible with the

$W(D_n)$-action. Thus it suffices to compute the image in the weight lattice for a single class of each type.

This is easy to check using the basis of Remark 13 and the identification of Proposition 15. The first collection corresponds to

$$f_2 - E_{i_i}, f_2 + f_1 - E_{i_1} - E_{i_2} - E_{i_3}, \ldots, f_2 + kf_1 - E_{i_1} - \cdots - E_{i_{2k+1}}, \ldots,$$

where $1 \leq i_1 < \cdots < i_{2k+1} \leq n$ and $2k+1 \leq n$. In other words, these are expressions (2.1) whose number of positive signs has parity different from n. The second collection is

$$f_2, f_2 + f_1 - E_{i_1} - E_{i_2}, \ldots, f_2 + kf_1 - E_{i_1} - \cdots - E_{i_{2k}}, \cdots, 2k \leq n,$$

which are expressions (2.1) whose number of positive signs has the same parity as n. □

Remark 19. The notation of Proposition 18 suggests that the sets are represented by (-1)-curves and conic fibrations. Degenerate cases of Example 10 make clear what can go wrong: The points $\{s_i\}$ might not have distinct images under the second projection or may fail to impose independent conditions on the linear series $f_2 + kf_1$.

3. Moduli, Cremona transformations, and monodromy

We continue to assume that $\phi : X \to \mathbb{P}^1$ is a good conic bundle over an algebraically closed field with $n \geq 1$ degenerate fibers.

3.1. Infinitesimal automorphisms.

Proposition 20. *If $\phi : X \to \mathbb{P}^1$ is a good balanced conic bundle, with $n \geq 3$ degenerate fibers, then $\Gamma(X, T_X) = 0$.*

Proof. Let $\mathrm{Aut}^\circ(X)$ denote the identity component of the autmorphism group; its tangent space, at the identity, is $\Gamma(X, T_X)$. Since ϕ has at least three degenerate fibers, automorphisms in $\mathrm{Aut}^\circ(X)$ must commute with ϕ i.e. $\mathrm{Aut}^\circ(X) = \mathrm{Aut}^\circ(X/\mathbb{P}^1)$. Thus vector fields in $\Gamma(X, T_X)$ lie in the fibers of ϕ.

Since $\phi : X \to \mathbb{P}^1$ is a good conic bundle, it admits a birational morphism

$$\beta : X \to \mathbb{F}_r,$$

blowing down (-1)-curves in distinct fibers over $\mathbb{P}^1$ (see Proposition 4). Let $s_1, \ldots, s_n \in \mathbb{F}_r$ denote the center of β. We may assume that none of these points are contained in a negative section of $\mathbb{F}_r \to \mathbb{P}^1$ (if there is such a section). Indeed, if ϕ admits a section of negative self-intersection, β blows down curves disjoint from that section. Finally, Proposition 6 and the balanced condition implies that $r \leq \lfloor (n+2)/3 \rfloor$.

For $n = 4$, classification shows that the following conditions are equivalent:

- X is balanced;
- ϕ has at most one section of self-intersection (-2);
- $X \simeq \mathrm{Bl}_{s_1,s_2,s_3,s_4}(\mathbb{P}^1 \times \mathbb{P}^1)$ where the $\pi_1(s_i)$ are distinct and at most two points have the same image under π_2;
- $\Gamma(X, T_X) = 0$.

Below, we assume $n \neq 4$.

Suppose that $r = 0$. Then there are non-zero vector fields on $\mathbb{F}_0$ vanishing at $s_1, \ldots, s_n$ only if these points are contained in two horizontal rulings. Now at least half the points are on one of the rulings. The proper transform Σ of this ruling in X satisfies $-\Sigma^2 \geq n/2$; Proposition 6 implies $-\Sigma^2 \leq \lfloor (n+2)/3 \rfloor$. These are inconsistent unless $n = 4$.

For $r > 0$, any vector fields necessarily vanish along the negative section, of self-intersection $-r$. Suppose $\mathbb{F}_r$ has non-zero vector fields vanishing at $s_1, \ldots, s_n$; this can happen only if they are all contained in a section with self-intersection r. Its proper transform $\Sigma \subset X$ would have self intersection

$$\Sigma^2 = r - n \leq \lfloor (n+2)/3 \rfloor - n,$$

again contradicting Proposition 6. □

3.2. **Parametrizing conic bundles.** Choose an even collection of n disjoint irreducible components of degenerate fibers, as in Definition 14. Blowing down yields a birational morphism over $\mathbb{P}^1$

$$\beta : X \to \mathbb{F}_r, \quad r \text{ even}, \tag{3.1}$$

with center $s_1, \ldots, s_n \in \mathbb{F}_r$, points in distinct fibers over $\mathbb{P}^1$.

Proposition 21. *Good balanced conic bundles $\phi : X \to \mathbb{P}^1$ with n degenerate fibers may be parametrized by a smooth irreducible scheme of finite type. The generic such X is isomorphic to a blowup of $\mathbb{F}_0$.*

The balance assumption is necessary for finite type: Otherwise, we would have blow-ups of $\mathbb{F}_r$ for any even r!

Proof. The splitting type of $\phi_* \omega_\phi^{-1}$ is determined by n, the degree of the discriminant (see Proposition 4). By Proposition 6, sections Σ of ϕ have self-intersections bounded from below. Specifically, $\beta : X \to \mathbb{F}_r$ for $r \leq \lceil n/3 \rceil + 1$. Thus our conic bundles are parametrized by a scheme of finite type. The deformation space of $\mathbb{F}_r$ is smooth of dimension $r/2$; the space of n-tuples of distinct points on these surfaces in also smooth. Thus the parameter space is smooth and irreducible. The description of the generic member follows because a generic deformation of $\mathbb{F}_r$ (for even r) is isomorphic to $\mathbb{F}_0$. □

Definition 22. A good conic bundle $\phi : X \to \mathbb{P}^1$ with $n \geq 1$ degenerate fibers is *typical* if every even collection of curves blows down to $\mathbb{F}_0$.

A typical conic bundle $\phi : X \to \mathbb{P}^1$ with $n \geq 3$ degenerate fibers is *uniform* if, for each $\beta : X \to \mathbb{F}_0$ arising from an even collection and indices $1 \leq i_1 < i_2 < i_3 \leq n$, the points $\beta(E'_{i_1}), \beta(E'_{i_2}), \beta(E'_{i_3}) \in \mathbb{F}_0$ are in general position.

A uniform conic bundle is *uniformly balanced* if each blow down $\beta : X \to X'$ along a collection of $< n$ fibral curves is balanced.

These conditions are all stable under blow downs of fibral curves

$$\begin{array}{ccc} X & \xrightarrow{\beta} & X' \\ & \searrow \quad \swarrow & \\ & \mathbb{P}^1 & \end{array}$$

even though being balanced is *not* respected under blow down (Example 11).

Proposition 23. *Consider good conic bundles with $n \geq 1$ degenerate fibers.*

- *Balanced, typical, uniform, and uniformly balanced are dense Zariski-open conditions on good conic bundles with the requisite number of degenerate fibers.*
- *A uniform conic bundle $X \to \mathbb{P}^1$ with n degenerate fibers, equipped with an ordered collection $\{E'_1, \ldots, E'_n\}$, has no automorphisms.*

Proof. The first assertion follows from Proposition 21. Any automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ fixing three points in general position is the identity, which gives the second assertion. □

There may be automorphisms acting non-trivially on the fibral curves (see Proposition 32).

3.3. **Introduction of moduli stacks.**

Definition 24. The moduli stack $\mathcal{C}_n$ (resp. $\mathcal{C}_n^b, \mathcal{C}_n^t, \mathcal{C}_n^u$ or $\mathcal{C}_n^{ub}$) parametrizes

$$(\phi : X \to \mathbb{P}^1; E'_1, \ldots, E'_n)$$

where ϕ is a good (resp. balanced, typical, uniform, or uniformly balanced) conic bundle with n degenerate fibers and $E'_1, \ldots, E'_n$ is an even tuple of components of degenerate fibers. Isomorphisms are commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\sim} & Y \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \mathbb{P}^1 & \xrightarrow{\sim} & \mathbb{P}^1 \end{array}$$

where the horizontal arrows are isomorphisms respecting the fibral curves and their order.

Except for $\mathcal{C}_n$, these stacks are of finite type. The group C_2^{n-1} acts on all these spaces via relabeling the components of the degenerate fibers i.e. for an even number of indices we replace E_i' with E_i''. The Weyl group $W(D_n)$ acts transitively on even collections of fibral curves; see the proof of Proposition 15 for an explanation why. Being good, balanced, typical, uniform, and uniformly balanced are compatible with this action, thus $W(D_n)$ acts naturally on $\mathcal{C}_n, \mathcal{C}_n^b$, $\mathcal{C}_n^t$, $\mathcal{C}_n^u$, and $\mathcal{C}_n^{ub}$.

Proposition 25. *Consider the diagonal action of* $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ *on* $n \geq 3$ *copies of* $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. *The mapping*

$$\begin{aligned} \iota_n : \mathcal{C}_n^t &\hookrightarrow [\mathrm{PGL}_2 \times \mathrm{PGL}_2 \backslash \mathbb{F}_0^n] \\ (\phi : X \to \mathbb{P}^1; E_1', \dots, E_n') &\mapsto (\beta(E_1'), \dots, \beta(E_n')) \end{aligned}$$

is an open embedding.

For generic $s_1, s_2, s_3 \in \mathbb{F}_0^3$ there is a unique $\alpha \in \mathrm{PGL}_2 \times \mathrm{PGL}_2$ with

$$s_1 \overset{\alpha}{\mapsto} (0,0), \;\; s_2 \overset{\alpha}{\mapsto} (1,1), \;\; s_3 \overset{\alpha}{\mapsto} (\infty, \infty).$$

Thus we have another open embedding

$$\begin{aligned} \mathbb{F}_0^{n-3} &\hookrightarrow [\mathrm{PGL}_2 \times \mathrm{PGL}_2 \backslash \mathbb{F}_0^n] \\ (s_4, \dots, s_n) &\mapsto [(0,0), (1,1), (\infty, \infty), s_4, \dots, s_n]. \end{aligned}$$

Proposition 26. *The embedding* ι_n *restricted to the uniform locus is also an open embedding*

$$\begin{aligned} j_n : \mathcal{C}_n^u &\hookrightarrow \mathbb{F}_0^{n-3} \\ (\phi : X \to \mathbb{P}^1; E_1', \dots, E_n') &\mapsto (\alpha(\beta(E_4')), \dots, \alpha(\beta(E_n'))). \end{aligned}$$

The spaces $\mathcal{C}_n^u$, and $\mathcal{C}_n^{ub}$ are open subsets of $\mathbb{F}_0^{n-3}$, which inherits a birational action of the Weyl group.

Remark 27. Constructions of birational actions of generalized Weyl groups on products of projective spaces are discussed in [DO88, ch. VI]; the formalism there does not directly address our case. Mukai [Muk04] extended this framework; the case relevant for us is Example 17. Section 2 of [CT06] (see Lemmas 2.2 and 2.3) explains how $W(D_n)$ arises via small birational modifications

$$\mathrm{Bl}_{q_1,\ldots,q_n}(\mathbb{P}^{n-3}) \overset{\sim}{\dashrightarrow} \mathrm{Bl}_{q_1,\ldots,q_n}(\mathbb{P}^{n-3}),$$

where $q_1, \ldots, q_n \in \mathbb{P}^{n-3}$ are in general position.

Quotients of quasi-projective varieties under finite groups (like $W(D_n)$) are separated stacks with quasi-projective coarse moduli spaces. We refer the reader to [Mus, App. 1] for discussion, including behavior of quotients under field extensions.

We summarize our results:

Theorem 28. *The moduli stack of uniform (resp. uniformly balanced) conic fibrations $\phi : X \to \mathbb{P}^1$ with $n \geq 3$ degenerate fibers is isomorphic to $[W(D_n)\backslash\mathcal{C}_n^u]$ (resp. $[W(D_n)\backslash\mathcal{C}_n^{ub}]$). These are separated with quasi-projective coarse moduli spaces.*

Corollary 29. *The monodromy representation on Picard groups of good conic bundles with $n \geq 3$ degenerate fibers is the Weyl group $W(D_n)$.*

Proof. Theorem 28 reflects that each even n-tuple $\{E'_1, \ldots, E'_n\}$ canonically determines a blowup representation $\beta : X \to \mathbb{F}_0$. Proposition 15 guarantees the monodromy is at most $W(D_n)$. A dense Zariski-open subset of moduli has a geometrically-connected $W(D_n)$ cover, i.e. $\mathcal{C}_n^u$; since the monodromy over $\mathcal{C}_n^u$ is trivial by construction, so the corollary follows. □

Remark 30. Over infinite fields, unirational varieties have Zariski-dense rational points. Thus we can always find points $s_1, \ldots, s_n \in \mathbb{F}_0$ such that the resulting conic bundle is uniformly balanced. Further, the classes of Proposition 18 may be realized by irreducible (-1)-curves and conic fibration. The configurations where these fail are Zariski-closed proper subsets of $\mathbb{F}_0^n$.

The recursive structure of our definitions yields the following:

Proposition 31. *There exist natural dominant forgetting morphisms*

$$\Phi : \mathcal{C}_n \longrightarrow M_{0,n}$$
$$(\phi : X \to \mathbb{P}^1, E'_1, \ldots, E'_n) \mapsto (\mathbb{P}^1, \phi(E'_1), \ldots, \phi(E'_n))$$

and

$$\Psi : \mathcal{C}_n^{ub} \longrightarrow \mathcal{C}_{n-r}^{ub}, \quad r = 1, \ldots, n-3$$
$$(\phi : X \to \mathbb{P}^1, E'_1, \ldots, E'_n) \mapsto (\varphi : X' \to \mathbb{P}^1, E'_1, \ldots, E'_{n-r})$$

where $\beta : X \to X'$ blows down $E'_{n-r+1}, \ldots, E'_n$. There are $\binom{n}{r}2^r$ choices for Ψ depending on which fibral curves are blown down.

3.4. **Small discriminant cases.** The cases $n = 1, 2$ are not representative as the underlying surfaces X have no moduli and positive-dimensional automorphisms.

When $n = 3$ the moduli space consists of a single point, albeit with non-trivial twists: Let $\phi : X \to \mathbb{P}^1$ be a uniform conic bundle with three degenerate fibers. Then $X \simeq \mathrm{Bl}_{s_1,s_2,s_3}(\mathbb{P}^1 \times \mathbb{P}^1)$ where the three points are in distinct fibers of both projections. The fibration is isomorphic to the morphism of moduli spaces of pointed stable curves

$$\varphi_5 : \overline{M}_{0,5} \to \overline{M}_{0,4}$$

forgetting the last point. Its automorphism group is permutations of the first four points

$$\mathrm{Aut}(X) = \mathfrak{S}_4 \simeq W(D_3).$$

For $n = 4$, see the proof of Proposition 20 for classification details. Here the moduli space is two-dimensional, with non-trivial generic stabilizer:

Proposition 32. *A generic conic bundle $X \to \mathbb{P}^1$ with four degenerate fibers over an algebraically closed field k has automorphism group C_2^3.*

Proof. Recall that a quartic del Pezzo surface

$$X = \{Q_0 = Q_\infty = 0\} \subset \mathbb{P}^4, \quad Q_0, Q_\infty \in k[x_0, x_1, x_2, x_3, x_4]_2$$

admits a C_2^4 action; indeed, diagonalizing

$$Q_0 = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2, \quad Q_\infty = a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$$

we may let ± 1 act on each coordinate. The pencil $\langle Q_0, Q_\infty \rangle$ contains five rank-four quadrics $Q_1, \ldots, Q_5$; each gives a double cover

$$g_i : X \to \mathbb{P}^1 \times \mathbb{P}^1$$

branched over a $(2, 2)$ curve E_i. Projecting to the factors gives conic fibrations

$$\phi_{\pm i} : X \to \mathbb{P}^1$$

with four degenerate fibers.

Which elements of C_2^4 are equivariant for a given fibrations? The group acts on

$$\{\phi_{\pm 1}, \ldots, \phi_{\pm 5}\}$$

by switching an even number of signs; the stabilizer of any element is isomorphic to C_2^3. The element

$$\phi_{+i} \mapsto \phi_{-i}, \quad i = 2, 3, 4, 5$$

corresponds to the covering involution of g_1. The remaining elements, in pairs indexed by partitions

$$\{2,3,4,5\} = \{2,3\} \sqcup \{4,5\}, \{2,4\} \sqcup \{3,5\}, \{2,5\} \sqcup \{3,4\},$$

correspond to two-torsion elements of the Jacobian $J(E_1)$ acting on E_1. This interpretation remains valid over non-closed fields. □

4. Relations with parabolic bundles

4.1. **Background.** Parabolic bundles on curves were introduced in [MS80]; here we focus on the special case of rank-two bundles on $\mathbb{P}^1$ with weight 1/2 [BHK10, Cas15].

Fix a collection of distinct points $p_1, \ldots, p_n \in \mathbb{P}^1$. We consider rank-two bundles with parabolic structure

$$\mathbf{E} = (E, L_1, \ldots, L_n),$$

consisting of a rank-two vector bundle $E \to \mathbb{P}^1$ along with one-dimensional subspaces $L_i \subset E|_{p_i}$ for $i = 1, \ldots, n$. Its parabolic degree is

$$\mathrm{pardeg}(\mathbf{E}) = \deg(E) + n/2.$$

We say that $\mathbf{E}$ is even or odd based on the parity of $\deg(E)$, i.e. of the Hirzebruch surface $\mathbb{F}_r := \mathbb{P}(E)$.

Given a rank-one subbundle $F \subset E$, the parabolic degree of the associated $\mathbf{F} \subset \mathbf{E}$ is

$$\mathrm{pardeg}(\mathbf{F}) = \deg(F) + \frac{1}{2}|\{i : L_i = F|_{p_i} \subset E_{p_i}\}|.$$

We say $\mathbf{E}$ is *semistable* (resp. *stable*) if

$$\mathrm{pardeg}(\mathbf{F}) \leq \text{(resp. } <\text{)} \ \mathrm{pardeg}(\mathbf{E})/2$$

for every rank-one subbundle. It is *unstable* if it is not semistable. These are projective notions: Tensoring $\mathbf{E}$ by a line bundle does not change its stability. For odd n, stability and semistability coincide.

4.2. **Extracting conic bundles.**

Definition 33. Given $\mathbf{E} \to \mathbb{P}^1$ a parabolic bundle as above, the associated good conic bundle is defined

$$X := \mathrm{Bl}_{\mathbb{P}(L_1),\ldots,\mathbb{P}(L_n)} \mathbb{P}(E) \to \mathbb{P}^1.$$

The group C_2^{n-1} acts transitively on the even parabolic projective bundles associated with $\phi : X \to \mathbb{P}^1$, by elementary (or Hecke) transformations over an even subset of $\{s_1, \ldots, s_n\}$. Changing the ordering on these n points, we obtain an action of $W(D_n)$ on these even parabolic projective bundles.

Remark 34. We made the convention in (3.1) to realize conic bundles as blowups of *even* Hirzebruch surfaces. This is a good choice because $\mathbb{F}_0$ has simpler automorphisms than $\mathbb{F}_1$. However, we could have used odd Hirzebruch surfaces instead.

Proposition 35. *Consider the moduli space of*

$$(\mathbb{P}(E) \to \mathbb{P}^1, p_1, \dots, p_n)$$

of projective bundles with parabolic structure on $\{p_1, \dots, p_n\}$*, distinct points on* $\mathbb{P}^1$*. The quotient under our* $W(D_n)$*-action gives a moduli space of good conic bundles over* $\mathbb{P}^1$ *with* n *degenerate fibers.*

Definition 36. A good conic bundle $\phi : X \to \mathbb{P}^1$ is *semistable* (resp. *stable*) if all the associated even parabolic bundles $\mathbf{E} \to \mathbb{P}^1$ are semistable (resp. stable).

The general theory of parabolic bundles pays dividends in our situation. Compactifications of universal moduli spaces of parabolic bundles, over the moduli space of pointed curves, have been constructed via GIT [Sch11, Pin23]. These show that the moduli space of stable parabolic bundles over the moduli space of pointed curves of genus zero has quasi-projective coarse moduli space.

Proposition 37. *The moduli space of stable good conic bundles with* n *degenerate fibers is separated with quasi-projective coarse moduli space.*

Proof. Let $\psi : \mathcal{P}_n \to \mathcal{C}_n$ be the morphism of moduli stacks, from even stable parabolic projective bundles to good conic bundles with labeled degenerate fibers, arising from Definition 33. This is equivariant under the actions of C_2^{n-1} via elementary transformations on $\mathcal{P}_n$ and relabelings on $\mathcal{C}_n$. Let $\mathcal{P}_n^s \subset \mathcal{P}_n$ denote the stable locus and and $\mathcal{C}_n^s$ the image

$$\psi : \cap_{\tau \in C_2^{n-1}} \tau(\mathcal{P}_n^s) \to \mathcal{C}_n.$$

This is also quasi-projective, and its quotients under finite groups (like $W(D_n)$) remain quasi-projective[Mus, App. 1]. □

Proposition 38. *Fix distinct points* $p_1, \dots, p_n \in \mathbb{P}^1$*. Let* $\mathbf{E} \to \mathbb{P}^1$ *be parabolic of rank two and even degree with respect to these points; write* $\phi : X \to \mathbb{P}^1$ *for the associated conic bundle. If* $n \geq 3$ *and* $\mathbf{E}$ *is unstable (resp.* $n \geq 5$ *and* $\mathbf{E}$ *is strictly semistable) then* $\phi : X \to \mathbb{P}^1$ *is unbalanced.*

The edge cases for $n = 4$ are discussed in Examples 11 and 12.

Proof. In the balanced case, Corollary 5 gives

$$(\phi_* \omega_\phi^{-1})^\vee = \mathcal{O}_{\mathbb{P}^1}(\lfloor n/3 \rfloor) \oplus \mathcal{O}_{\mathbb{P}^1}(\lfloor (n+1)/3 \rfloor) \oplus \mathcal{O}_{\mathbb{P}^1}(\lfloor (n+2)/3 \rfloor).$$

By Proposition 6, sections $\Sigma \subset X$ of ϕ satisfy

$$\Sigma^2 \geq -\lfloor (n+2)/3 \rfloor.$$

After tensoring by a line bundle, we may assume $\deg(E) = 0$. Assume that $\mathbf{E}$ is unstable with destabilizing subsheaf $\mathbf{F} \subset \mathbf{E}$. Set

$$k = |\{i : L_i = F|p_i\}|$$

so that being unstable translates to

$$k > \frac{n}{2} - 2\deg(F).$$

Let $\Sigma \subset X$ denote the proper transform of $\mathbb{P}(F) \subset \mathbb{P}(E)$. Now

$$(\mathbb{P}(F))^2_{\mathbb{P}(E)} = -2\deg(F)$$

and

$$\Sigma^2 = -2\deg(F) - k < -\frac{n}{2}.$$

Combining our inequalities involving Σ^2 yields

$$\lfloor (n+2)/3 \rfloor > \frac{n}{2},$$

a contradiction for $n \geq 3$. In the strictly semistable case we obtain

$$\lfloor (n+2)/3 \rfloor \geq \frac{n}{2},$$

a contradiction for $n \geq 5$. □

Example 39. Stable parabolic bundles can yield unbalanced conic bundles. Let $n = 7$ and take points

$$s_1, \ldots, s_7 \in \mathbb{P}^1 \times \mathbb{P}^1, \quad \pi_2(s_1) = \pi_2(s_2) = \pi_2(s_3), \pi_2(s_5) = \pi_2(s_6) = \pi_2(s_7)$$

but otherwise generic. Consider the conic bundle

$$\phi : X = \mathrm{Bl}_{s_1,\ldots,s_7}(\mathbb{P}^1 \times \mathbb{P}^1) \xrightarrow{\pi_1} \mathbb{P}^1$$

arising from the parabolic bundle

$$\mathbf{E} = (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}, L_1, \ldots, L_7) \quad \mathbb{P}(L_i) = s_i,$$

which is stable. However, we have

$$(\phi_* \omega_\phi^{-1})^\vee = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3)^2$$

as $s_1, \ldots, s_7$ sit on a reducible curve of class $f_1 + 2f_2$, where f_i is the fiber of π_i.

Since being balanced is a Zariski-open condition, Propositions 37 and 38 yield:

Corollary 40. *For $n \geq 4$, the moduli stack of balanced conic bundles $[W(D_n) \backslash \mathcal{C}_n^b]$ is separated with quasi-projective coarse moduli space.*

The $n = 4$ case follows from the proof of Proposition 20.

Remark 41. Moduli spaces of conic bundles over general curves have also been studied in [GS00]. They develop formulates of stability grounded in slope-stability of vector bundles. Our approach gives very special instances of these spaces – even restricting to genus zero.

4.3. **Linear subspaces on complete intersections of quadrics.** Here we follow [Cas15]. Let $n = 2g + 1$ and consider smooth complete intersections of quadrics

$$Z = \{Q_1 = Q_2 = 0\} \subset \mathbb{P}^{2g}$$

with pencil

$$\mathcal{Q} = \{t_1 Q_1 + t_2 Q_2 = 0\} \to \mathbb{P}^1_{[t_1,t_2]}.$$

Write $p_1, \ldots, p_{2g+1} \in \mathbb{P}^1$ for the singular members of the pencil.

Recall (see, for example, [HKT22, §2]) that the primitive cohomology

$$H^{2g-2}(Z, \mathbb{Z})_{\mathrm{prim}}$$

has $W(D_{2g+1})$ symmetry. Furthermore, the finite variety $F_{g-1}(Z)$, parametrizing $(g-1)$-dimensional projective subspaces isotropic for the pencil, also has this symmetry.

Let $F_{g-2}(Z)$ denote the variety of $(g-2)$-dimensional subspaces isotropic for $\mathcal{Q}$. We have

$$\dim(F_{g-2}(Z)) = 2g - 2 = n - 3.$$

Casagrande shows it is isomorphic to the moduli space of stable bundles of rank two and degree zero, with parabolic structure at $p_1, \ldots, p_{2g+1}$ of weight $1/2$.

Given $\mathbb{P}(\Lambda) \in F_{g-2}(Z)$, we construct a conic bundle $X \to \mathbb{P}^1_{[t_1,t_2]}$ with degenerate fibers over $p_1, \ldots, p_{2g+1}$. Let q be a non-degenerate quadratic form in n variables over a field K and Λ an isotropic K-subspace of dimension m. Then $q|\Lambda^\perp$ is degenerate along Λ, inducing a non-degenerate quadratic form q' on $\Lambda^\perp/\Lambda$, which has dimension $n - 2m$. We say that q' is the *Witt reduction* of q for Λ.

We apply this for $K = k(\mathbb{P}^1)$, q the generic fiber of $\mathcal{Q} \to \mathbb{P}^1$, Λ the constant isotropic subspace. Since $n = 2g + 1$ and $m = g - 1$, the Witt reduction is a ternary quadratic form. We take $X \to \mathbb{P}^1$ to be the associated conic bundle.

5. Moduli of complete intersections

We follow the trichotomy presented in Example 7, giving complete-intersection descriptions of balanced good conic bundles.

5.1. **Type (a,a,a):** Consider $X = \{G = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^2$ smooth of bidegree $(a, 2)$, of splitting type (a, a, a) with $a \geq 1$. The GIT quotient

$$(\mathrm{SL}_2 \times \mathrm{SL}_3)\backslash\backslash\mathbb{P}(\Gamma(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(a, 2)))$$

is a birational model for balanced conic bundles with $3a$ degenerate fibers.

Proposition 42. *Smooth surfaces of this type have finite reduced automorphism groups and are GIT stable. The resulting moduli stack* $[W(D_{3a})\backslash\mathcal{C}^b_{3a}]$ *for* $a \geq 1$ *is Deligne-Mumford with quasi-projective coarse moduli space.*

Its proof follows the strategy of [MFK94, ch. 4, §2] and [Ben13]. The discriminant divisor

$$\Delta \subset \mathbb{P}(\Gamma(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(a, 2)))$$

parametrizes singular surfaces. It is irreducible, in fact, birational to a projective bundle over $\mathbb{P}^1 \times \mathbb{P}^2$; indeed, for $p \in \mathbb{P}^1 \times \mathbb{P}^2$, the surfaces singular at p are a codimension-four linear subspace in the parameter space. Since Δ is invariant under the group action, the smooth surfaces are at least GIT semistable. Smooth $X \subset \mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(a, 2)$ $(a > 0)$ are balanced. By Proposition 20, it has no global vector fields.

5.2. **Type (a,a+1,a+1):** Consider

$$X = \{F = G = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^3$$

be a smooth complete intersection of forms of bidegree $(1, 1)$ and $(a, 2)$ for $a \geq 1$. Now F may be put in standard form: There are independent linear forms L_1 and L_2 on $\mathbb{P}^3$ so that

$$F = L_1t_1 + L_2t_2$$

and

$$Y = \{F = 0\} \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}^2_{\mathbb{P}^1}).$$

In particular, X is necessarily balanced and $\Gamma(X, T_X) = 0$ by Proposition 20.

Following [Ben14], these may be compactified by

$$\mathbb{P}(\mathcal{F}) \to \mathbb{P}^7,$$

where $\mathcal{F} \to \mathbb{P}^7$ is locally-free of rank $6a + 10$ over the open locus of standard F. We interpret

$$[F] \in \mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(1, 1))) = \mathbb{P}^7$$

and

$$[G] \in \mathbb{P}\left(\Gamma(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(a, 2))/[F] \cdot \Gamma(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(a - 1, 1))\right).$$

However, unlike in the previous case there is no canonical way to linearize a GIT problem whose stable locus contains all smooth X. See [Ben14, §2] and [Ben12] for discussion of the birational geometry.

5.3. **Type (a,a,a+1):** Consider

$$X = \{F_1 = F_2 = G = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^4$$

a smooth complete intersection of two forms of bidegree $(1,1)$ and one of bidegree $(a-1,2)$. This is more complicated than the previous cases.

Remark 43. The formalism of complete intersections is poorly adapted to this case. Assuming the forms F_1 and F_2 are generic, we may choose coordinates so that

$$F_1 = L_{11}t_1 + L_{12}t_2, \quad F_2 = L_{21}t_1 + L_{22}t_2 \tag{5.1}$$

so that the image of

$$Y = \{F_1 = F_2 = 0\}$$

in $\mathbb{P}^4$ satisfies

$$\pi_2(Y) = \{L_{11}L_{22} = L_{12}L_{21}\}.$$

This relation is not in the ideal $\langle F_1, F_2\rangle$ but is in its saturation with respect to $\langle t_1, t_2\rangle$. We obtain an alternate resolution to (1.1):

$$0 \to \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^4}(-1,-2)^2 \to \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^4}(-1,-1)^2 \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^4}(-2,-2) \to \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^4} \to \mathcal{O}_Y \to 0,$$

which is not a Koszul complex.

Remark 44. We cannot hope for vanishing of vector fields, as in Proposition 20, for all such X. Recall Example 9, where

$$Y_0 = \{L_1t_1 - L_2t_2 = L_2t_1 - L_3t_2 = 0\}$$

has extra automorphisms. When $a = 1$ and $X \subset Y_0$, we have $\Gamma(T_X) \neq 0$ even when X is smooth. Here the image in $\mathbb{P}^4$

$$\pi_2(X) = \{L_1L_3 - L_2^2 = G(L_1, \dots, L_5) = 0\}$$

generally has two ordinary singularities on the line $L_1 = L_2 = L_3 = 0$. The morphism $X \to \pi_2(X)$ resolves these singularities; explicitly

$$X = \mathrm{Bl}_{s_1,s_2,s_3,s_4}(\mathbb{P}^1 \times \mathbb{P}^1), \quad \pi_2(s_1) = \pi_2(s_2), \quad \pi_2(s_3) = \pi_2(s_4).$$

The group $\mathbb{G}_m$ acts on X, i.e., automorphisms of the second $\mathbb{P}^1$ fixing the two distinguished points.

Question 45. Do $[W(D_{3a+1})\backslash\mathcal{C}^b_{3a+1}]$ and $[W(D_{3a+2})\backslash\mathcal{C}^b_{3a+2}]$, for $a \geq 1$, have explicit constructions via complete intersections?

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