

Math 465 Lecture Notes

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9 Lecture 9: Elimination Theory

Remark 9.1 (Fields of definition) We continue to work over an infinite field, e.g., a field of characteristic zero or an algebraically closed field of characteristic p .

First, we tie up some loose ends on morphisms between affine varieties. Suppose we have affine varieties $X \subset k^n$ and $Y \subset k^m$ and a morphism $\phi : X \rightarrow Y$. Then there is an induced k -linear homomorphism

$$\phi^* : k[Y] \rightarrow k[X].$$

Moreover, since ϕ is induced by a morphism $\phi' : k^n \rightarrow k^m$, we have a diagram

$$\begin{array}{ccc} k[y_1, \dots, y_m] & \xrightarrow{\phi'^*} & k[x_1, \dots, x_n] \\ \downarrow & & \downarrow \\ k[Y] & \xrightarrow{\phi^*} & k[X]. \end{array}$$

Proposition 9.2 *Suppose that X and Y are affine varieties and $\psi : k[Y] \rightarrow k[X]$ is a k -linear homomorphism. Then there exists a morphism $\phi : X \rightarrow Y$ with $\phi^* = \psi$.*

proof Realize $X \subset k^n$ and $Y \subset k^m$, with coordinates x_1, \dots, x_n and y_1, \dots, y_m respectively. We have the following diagram

$$\begin{array}{ccc} k[y_1, \dots, y_m] & & k[x_1, \dots, x_n] \\ \downarrow & \searrow^{\bar{\psi}} & \downarrow \\ k[Y] & \xrightarrow{\psi} & k[X] \end{array}$$

Next, we turn to elimination theory. This topic arises from one of our original guiding problems:

Problem 9.8 (Implicitization Problem) Given a polynomial map $\phi : k^n \rightarrow k^m$, describe equations for the image.

We will actually generalize and reformulate this a bit:

Problem 9.9 (Implicitization Problem II) Given a morphism $\phi : X \rightarrow k^m$, describe generators for the ideal $I(\phi(X))$.

Warning: The image of a polynomial morphism is not necessarily closed. For example, consider the varieties

$$X = \{(x_1, x_2) : x_1 x_2 = 1\} \quad Y = k$$

and the map

$$\begin{aligned} \phi : X &\rightarrow Y \\ (x_1, x_2) &\rightarrow x_1 \end{aligned}$$

The image $\phi(X) = \{x_1 : x_1 \neq 0\}$, which is not closed. This shows that it is only reasonable to seek equations for the *closure* of the image $\overline{\phi(X)}$.

Definition 9.10 The *graph* Γ_ϕ of a morphism $\phi : X \rightarrow k^m$ is the locus of pairs

$$\{(x, \phi(x)) : x \in X\} \subset X \times k^m \subset k^n \times k^m.$$

Proposition 9.11 *The graph of a morphism is an affine variety.*

proof Assume that ϕ is given in coordinates by $\phi_1, \dots, \phi_m \in k[x_1, \dots, x_n]$. Then the equations of Γ_ϕ in $X \times k^m$ are given by

$$y_1 - \phi_1 = y_2 - \phi_2 = \dots = y_m - \phi_m = 0. \quad \square$$

Note that Γ_ϕ admits maps

$$\begin{array}{ccccc} & & \Gamma_\phi & & (x_1, \dots, x_n, y_1, \dots, y_m) \\ & \swarrow \pi_1 & & \searrow \pi_2 & \\ X & & & & k^m \quad (x_1, \dots, x_n) \quad (y_1, \dots, y_m) \end{array}$$

where π_1 is invertible $\pi_2(\Gamma_\phi) = \phi(X)$.

The following observation translates the Implicitization Problem into a purely algebraic problem:

Let $J \subset k[x_1, \dots, x_n, y_1, \dots, y_m]$ be the ideal $I(\Gamma_\phi)$, including the equations in x_1, \dots, x_n for the affine variety X . The equations of the image $\phi(X)$ are precisely the polynomials in the intersection $J \cap k[y_1, \dots, y_m]$.

Indeed, a polynomial $g \in k[y_1, \dots, y_m]$ vanishing on $\phi(X)$ can be regarded as a polynomial in $k[x_1, \dots, x_n, y_1, \dots, y_m]$ vanishing on all the points with y -coordinates in $\phi(X)$, in particular, on all the points of Γ_ϕ . Conversely, if $g \in k[y_1, \dots, y_m]$ vanishes on Γ_ϕ then it also vanishes on each point in $\pi_2(\Gamma_\phi) = \phi(X)$.

Definition 9.12 A monomial order on $k[x_1, \dots, x_n, y_1, \dots, y_m]$ is an *elimination order* for x_1, \dots, x_n if, each polynomial with leading monomial in $k[y_1, \dots, y_m]$ is actually contained in $k[y_1, \dots, y_m]$, i.e.,

$$\text{LM}(g) \in k[y_1, \dots, y_m] \Rightarrow g \in k[y_1, \dots, y_m].$$

Example 9.13 1. Lexicographic order with $x_i > y_j$ for each i, j is an elimination order for x_1, \dots, x_n . However, this is very inefficient computationally.

2. Fix monomial orders $>_x$ and $>_y$ on x_1, \dots, x_n and y_1, \dots, y_m respectively. (Graded reverse lexicographic is usually efficient.) Then take the *product order*

$$x^\alpha y^\beta > x^\gamma y^\delta$$

if

$$\begin{cases} x^\alpha >_x x^\gamma \text{ or} \\ x^\alpha =_x x^\gamma \text{ and } y^\beta >_y y^\delta. \end{cases}$$

Theorem 9.14 (Elimination Theorem) Let $J \subset k[x_1, \dots, x_n, y_1, \dots, y_m]$ be an ideal and $>$ an elimination order for x_1, \dots, x_n . Let $\{f_1, \dots, f_r\}$ be a Gröbner basis for J . Then $J \cap k[y_1, \dots, y_m]$ is generated by the elements of the Gröbner basis contained in $k[y_1, \dots, y_m]$, i.e.,

$$J \cap k[y_1, \dots, y_m] = \langle f_j : f_j \in k[y_1, \dots, y_m] \rangle \subset k[y_1, \dots, y_m].$$

proof: It suffices to show that each element $g \in J \cap k[y_1, \dots, y_m]$ is generated by the $f_j \in k[y_1, \dots, y_m]$. Choose

$$g \in (J \cap k[y_1, \dots, y_m]) \setminus \langle f_j : f_j \in k[y_1, \dots, y_m] \rangle$$

with $\text{LM}(g)$ minimal. Apply the division algorithm to g and let f_j be a Gröbner basis element with $\text{LM}(f_j) \mid \text{LM}(g)$. Hence $\text{LM}(f_j) \in k[y_1, \dots, y_m]$ so the definition of the elimination order implies $f_j \in k[y_1, \dots, y_m]$. Thus

$$\tilde{g} := g - f_j \text{LT}(g) / \text{LT}(f_j)$$

is an element of $(J \cap k[y_1, \dots, y_m]) \setminus \langle f_j : f_j \in k[y_1, \dots, y_m] \rangle$ with $\text{LM}(\tilde{g}) < \text{LM}(g)$, a contradiction. \square

References

[Shaf] I. R. Shafarevich, *Basic algebraic geometry. 1. Varieties in projective space, Basic algebraic geometry. 2. Schemes and complex manifolds*, Second edition. Translated from the 1988 Russian edition and with notes by Miles Reid. Springer-Verlag, Berlin, 1994.