

Consider 2×2 matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

and the vector fields

$$X = (a_{11}x_1 + a_{12}x_2)\frac{\partial}{\partial x_1} + (a_{21}x_1 + a_{22}x_2)\frac{\partial}{\partial x_2}$$

and

$$Y = (b_{11}x_1 + b_{12}x_2)\frac{\partial}{\partial x_1} + (b_{21}x_1 + b_{22}x_2)\frac{\partial}{\partial x_2}$$

on $\mathbb{R}^2$.

a. Assuming

$$AB - BA = 0,$$

compute the Lie bracket $[X, Y]$ and explain your answer.

b. Discuss $[X, Y]$ in the case where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$