5 Hilbert schemes

5.1 Motivation: Graph construction

How do we compactify the space of morphisms $\mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^n, d)$ or $\mathrm{Mor}(\mathbb{P}^1, X, d)$, where X is a projective variety?

Of course, our construction of $\mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^n, d)$ as an algebraic variety includes an implicit compactification

$$\mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^n, d) \subset \mathbb{P}(k[x_0, x_1]^{\oplus n+1}) \simeq \mathbb{P}^{nd+n+d}.$$

However, we would really like a *geometric* interpretation for any compactification we use.

One approach is to use the *graph construction*: Given any morphism of varieties/schemes $\phi : Y \to X$, the graph

$$\Gamma_\phi = \{(y, x) : \phi(y) = x\} \subset Y \times X$$

is a closed subset/subscheme such that

1. $\pi_1 : \Gamma_\phi \xrightarrow{\sim} Y$ is an isomorphism;
2. $\pi_2 : \Gamma_\phi \to X$ is equal to ϕ.

To classifying morphisms $\phi : Y \to X$, we can classify closed

$$Z \subset Y \times X$$

such that

$$\pi_1 : Z \to Y$$

is an isomorphism.

For example, consider $\mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^1, 1)$, whose elements can be represented

$$\phi = [a_{00}x_0 + a_{01}x_1, a_{10}x_0 + a_{11}x_1]$$

with $a_{00}a_{11} - a_{01}a_{01} \neq 0$. If z_0, z_1 are coordinates on the image $\mathbb{P}^1$ then the substitutions

$$z_0 = a_{00}x_0 + a_{01}x_1, \ z_1 = a_{10}x_0 + a_{11}x_1$$

lead to the relation

$$z_0(a_{00}x_0 + a_{01}x_1) - z_1(a_{10}x_0 + a_{11}x_1) = 0.$$

This bihomogeneous polynomial in $k[x_0, x_1; z_0, z_1]_{(1,1)}$ defines $\Gamma_\phi \subset \mathbb{P}^1 \times \mathbb{P}^1$.

Conversely, consider a general bihomogeneous

$$B_{00}x_0z_0 + B_{01}x_0z_1 + B_{10}x_1z_0 + B_{11}x_1z_1 \neq 0 \in k[x_0, x_1; z_0, z_1]_{(1,1)}$$

defining $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$. The projection

$$\pi_1 : Z \to \mathbb{P}^1$$

is an isomorphism if and only if

$$B_{01}B_{10} - B_{00}B_{11} \neq 0. \tag{1}$$

Note that we can extract a rational map $\phi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ given by the substitution

$$z_0 = x_0B_{01} + x_1B_{11}, \quad z_1 = -x_0B_{00} - x_1B_{10}.$$

This is a morphism when the determinant condition (1) is satisfied.

What if (1) is not satisfied? For instance, we might have the locus

$$Z' = \{(x_0, x_1; z_0z_1) : x_0z_0 = 0\}$$

which is a union of fibers of the two projections to $\mathbb{P}^1$.

5.1.1 On equations for graphs

Even for simple maps, it can be challenging to write down equations for Γ_ϕ. For instance, consider the map

$$\phi : \mathbb{P}^1 \to \mathbb{P}^n$$

associated to the substitution $z_j = \phi_j(x_0, x_1)$. We have the 'obvious' equations

$$\phi_j z_i - \phi_i z_j = 0,$$

which we regard as bihomogeneous forms in $k[x_0, x_1; z_0, \ldots, z_n]_{(d,1)}$.

However, these are not the only ones generally. For instance, the morphism

$$\begin{array}{rcl} \phi : \mathbb{P}^1 & \to & \mathbb{P}^2 \\ {[x_0, x_1]} & \mapsto & [x_0^2, x_0x_1, x_1^2] \end{array}$$

has 'obvious' equations

$$z_0x_0x_1 = z_1x_0^2,\ z_0x_1^2 = z_2x_0^2,\ z_1x_1^2 = z_2x_0x_1.$$

However, the ideal of Γ_ϕ also contains

$$z_0 x_1 = z_1 x_0,\ z_1 x_1 = z_2 x_0,\ z_1^2 = z_0 z_2.$$

Exercise 1 Let $\phi : \mathbb{P}^1 \to \mathbb{P}^n$ be a morphism and

$$J = \langle \phi_j z_i - \phi_i z_j \rangle_{i,j=0,\ldots,n}$$

the ideal of obvious equations.

a. Suppose that $n = 1$ and ϕ is nonconstant. Show that $J = J(\Gamma_\phi)$.

b. If $n > 1$, show that $J \subsetneq J(\Gamma_\phi)$.

c. Write down equations for Γ_ϕ for

$$\begin{array}{rcl} \phi : \mathbb{P}^1 & \to & \mathbb{P}^3 \\ [x_0, x_1] & \mapsto & [x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3] \end{array}$$

5.2 Flatness-algebraic facts

Let A be a commutative ring. An A-module M is *flat* if, for each A-linear injection

$$0 \to N_1 \to N_2,$$

tensoring with M yields an injection

$$0 \to M \otimes_A N_1 \to M \otimes_A N_2.$$

In other words, tensoring by M is a *left exact* operation.

Proposition 2 *An A-module is flat if and only if for each ideal $I \subset A$ the tensor product*

$$M \otimes_A I \to M$$

is injective.

Proposition 3 *An A-module M is flat if and only if each of its localizations is flat, i.e., for each prime $\mathfrak{p} \subset A$ $M_\mathfrak{p}$ is flat as an $A_\mathfrak{p}$ module. In fact, it suffices to check the maximal ideals.*

Exercise 4 Let A be an integral domain. Show that each flat module M is torsion-free, i.e., there exists no $m \neq 0 \in M$ and $a \neq 0 \in A$ with $am = 0$. *Hint:* Consider the ideal

$$0 \to \langle a \rangle \to A.$$

Exercise 5 Let $A = k[x_1, x_2]$ where k is a field. Show that $\langle x_1, x_2 \rangle$ is not flat as an A-module.

Exercise 6 Show that a direct sum of flat modules is flat.

Exercise 7 Let A be a principal ideal domain. Show that an A-module is flat if and only if it is torsion-free. *Hint:* You may stick with the case of finitely-generated modules, if necessary.

Exercise 8 Let k be a field and $A = k[\epsilon]/\langle \epsilon^2 \rangle$. Show that an A-module M is flat if and only if the inclusion

$$\{m \in M : m = \epsilon m' \text{ for some } m' \in M\} \subset \{m \in M : \epsilon m = 0\}$$

is an equality.

The following result explains partially why flatness turns out to be a powerful geometric concept:

Theorem 9 *A finitely generated module is flat if and only if it is locally free.*

A finitely generated module over an integral domain is flat if and only if it has constant rank, i.e., the function

$$\begin{array}{rcl} \mathrm{Spec}(A) & \to & \mathbb{Z} \\ \mathfrak{p} & \mapsto & \dim M \otimes_A A_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \end{array}$$

is constant.

Exercise 10 Let $A = k[x_1, \ldots, x_n]$ be a polynomial ring over a field and $I \subset A$ an ideal. Show that I is flat over A if and only if it is principal.

5.3 Flat limits

Here is a geometric reformulation of Exercise 7:

Theorem 11 *Let Δ be a smooth curve, or more generally, a connected regular scheme of dimension one. A morphism $\pi : \mathcal{X} \to \Delta$ is flat if and only if each associated point of $\mathcal{X}$ maps to the generic point of Δ.*

In particular, each irreducible component of $\mathcal{X}$ must dominate Δ.

Corollary 12 *Let $0 \in \Delta$ and $\Delta^* = \Delta \setminus \{0\}$. Each flat family of projective varieties over Δ^*,*

$$\begin{array}{ccc} \mathcal{X}^* & \subset & \mathbb{P}^n \times \Delta^* \\ & \searrow & \downarrow \\ & & \Delta^* \end{array}$$

admits a unique extension to a flat family

$$\begin{array}{ccc} \mathcal{X} & \subset & \mathbb{P}^n \times \Delta \\ & \searrow & \downarrow \\ & & \Delta. \end{array}$$

proof: We take $\mathcal{X}$ to be the closure of $\mathcal{X}^*$ in $\mathbb{P}^n \times \Delta$. Since this is the smallest closed subscheme containing $\mathcal{X}$, it has no associated primes over 0. $\square$

In the special case where $\Delta = \mathrm{Spec} k[[t]]$ and $0 = \{t = 0\}$, we write

$$\mathcal{X}_0 = \lim_{t \to 0} \mathcal{X}_t;$$

this is called the *flat limit* of the scheme $\mathcal{X}_t$.

Exercise 13 Compute flat limits $\lim_{t\to 0} \mathcal{X}_t$ for the following examples

a. $\mathcal{X}_t = \{x_0^2 + tx_1^2 + t^{-1}x_2^2 = 0\} \subset \mathbb{P}^2$;

b. $\mathcal{X}_t = \{[1, t, t^2], [1, t^2, t], [1, t^2, t^3]\} \subset \mathbb{P}^2$;

c. $\mathcal{X}_t = \{[1, t^a, t^b], [1, t^c, t^d]\} \subset \mathbb{P}^2$ for arbitrary $a, b, c, d \in \mathbb{N}$;

d. $\mathcal{X}_t = \{[1, 0, 0], [1, t, \alpha t]\} \subset \mathbb{P}^2$ for arbitrary $\alpha \in k^*$.

5.4 Flatness and Hilbert polynomials

For a morphism $\pi : \mathcal{X} \to B$, for each $b \in B$ the fiber over b is denoted

$$\mathcal{X}_b = pi^{-1}(b) = \mathcal{X} \times_B \mathrm{Spec} k(b).$$

Theorem 14 *Let B be a variety/integral scheme. Consider a closed subscheme*

$$\begin{array}{ccc} \mathcal{X} & \hookrightarrow & \mathbb{P}^n \times B \\ & \searrow & \downarrow \\ & & B \end{array}$$

and the projection $\pi : \mathcal{X} \to B$. Then π is flat if and only if the function

$$\begin{array}{rcl} \mathrm{HP} : B & \to & \mathbb{Q}(t) \\ b & \mapsto & \mathrm{HP}_{\mathcal{X}_b} \end{array}$$

is constant.

This should be compared with Theorem 9. For a proof, see [1, III.9].

Exercise 15 Suppose that for each $b \in B$, $\mathcal{X}_b$ is finite. (In other words, the fibers of π are zero-dimensional.) Deduce Theorem 14 as a direct consequence of Theorem 9.

5.5 Classifying projective varieties with given Hilbert polynomial

We work over a fixed base field k. Fix $n \in \mathbb{N}$ and $p(t) \in \mathbb{Q}[t]$ polynomial of degree $\leq n$ with $p(\mathbb{Z}) \subset \mathbb{Z}$.

Consider all closed $X \subset \mathbb{P}^n$ with $\mathrm{HP}_X = p(t)$. The set of such subschemes is designated $\mathcal{H}ilb_p$. (Later on we will endow this with additional structure).

Definition 16 Choose d_0 such that $\mathrm{HP}_X(d) = p(d)$ for each $d \geq d_0$. For each $d \geq d_0$, consider the linear subspace

$$J(X)_d \subset k[x_0, \ldots, x_n]$$

which has codimension $p(d)$. The corresponding point in the Grassmannian

$$[X]_d \in \mathrm{Gr}(\binom{n+d}{d} - p(d), k[x_0, \ldots, x_n]_d) \simeq \mathrm{Gr}(\binom{n+d}{d} - p(d), \binom{n+d}{d})$$

is called the *dth Hilbert point* of X.

Proposition 17 *Suppose that $J(X)$ is generated by polynomials of degree $\leq d_0$. Then for each $d \geq d_0$, $[X]_d$ determines X uniquely.*

proof: Each subspace $\Lambda \subset k[x_0, \ldots, x_n]_d$ defines a homogeneous ideal

$$\langle \Lambda \rangle \subset k[x_0, \ldots, x_n]$$

and thus a projective scheme

$$X(\langle \Lambda \rangle) \subset \mathbb{P}^n.$$

Under our assumption, the saturation of $\langle J(X)_d \rangle$ is $J(X)$, so

$$X(\langle J(X)_d \rangle) = X$$

and the X is determined by its dth Hilbert point.□

How do we carry this out uniformly for all schemes with given Hilbert polynomial? The key is the following result of Mumford:

Theorem 18 *[2, ch.14] Fix $p(t) \in \mathbb{Q}[t]$ with $p(\mathbb{Z}) \subset \mathbb{Z}$ and $n > 0$. There exists a uniform integer d_p such that for each $X \subset \mathbb{P}^n$ with $\mathrm{HP}_X(t) = p(t)$, we have*

1. $\mathrm{HF}_X(d) = \mathrm{HP}_X(d)$ *for all $t \geq d_p$;*
2. *$J(X)$ is generated by polynomials of degree $\geq d_p$.*

The proof of this result is an induction on dimension, using techniques from sheaf cohomology.

Exercise 19 Prove this result in the special case where $p(t) = D$, a constant polynomial. Note that any scheme X with $\mathrm{HP}_X(t) = D$ has dimension zero and is supported in at most D points.

Exercise 20 Find the smallest possible values for d_p in the following cases:

a. $p(t) = 1$ and n arbitrary;

b. $p(t) = 2$ and n arbitrary;

c. $p(t) = t + 1$ and n arbitrary;

d. $p(t) = 3$ and $n = 2$;

Combining Theorem 18 and Proposition 17, we obtain:

Corollary 21 *Retain the notation of Theorem 18. For each $d \geq d_p$ the map*

$$\begin{array}{rcl} \mathcal{H}ilb_p & \hookrightarrow & \mathrm{Gr}(\binom{n+d}{d} - p(d), \binom{n+d}{d}) \\ X & \mapsto & [X]_d \end{array}$$

is a bijection onto its image.

Exercise 22 Consider subschemes in $\mathbb{P}^1$ with $p(t) = 2$ and take $d = 3$. Describe the image of the map

$$\begin{array}{rcl} \mathcal{H}ilb_p & \hookrightarrow & \mathrm{Gr}(2,4) \\ X & \mapsto & [X]_3 \end{array}$$

by writing down its defining equations.

5.6 Why the Hilbert scheme is a scheme

Recall the inclusion

$$\mathcal{H}ilb_p \subset \mathrm{Gr}(\binom{n+d}{d} - p(d), \binom{n+d}{d})$$

introduced in Corollary 21. We endow this set with a natural scheme structure, following [2, ch.15].

First, Corollary 12 (combined the valuative criterion for properness) suffices to show that $\mathcal{H}ilb_p$ is closed in the Grassmannian. **MORE DETAILS TO FOLLOW**

For each

$$\Lambda \in \mathrm{Gr}(\binom{n+d}{d} - p(d), \binom{n+d}{d})$$

let $\langle \Lambda \rangle$ denote the corresponding ideal. If $\Lambda \in \mathcal{H}ilb_p$ then we know (Theorem 18) that

$$\dim \langle \Lambda \rangle_e = \binom{n+e}{e} - p(e)$$

for each $e \geq d$.

For each $e \geq d$ and $R \in \mathbb{N}$, consider the locus

$$H_p(e,R) = \{\Lambda : \dim \langle \Lambda \rangle_e \leq R\} \subset \mathrm{Gr}(\binom{n+d}{d} - p(d), \binom{n+d}{d}).$$

We have

$$\mathcal{H}ilb_p \subset \cap_{e\geq d} H_p(e, \binom{n+e}{e} - p(e)).$$

Lemma 23 *For each $e \geq d$ and $R \in \mathbb{N}$, $H_p(e,R)$ is a closed subscheme of the Grassmannian, with an explicit set of defining polynomial equations.*

proof: We give a quick sketch, followed by a more detailed argument. Consider the linear transformation

$$\begin{array}{rcl} k[x_0,\ldots,x_n]_{e-d} \otimes \Lambda & \rightarrow & k[x_0,\ldots,x_n]_d \\ (F,G) & \mapsto & FG \end{array}$$

with image $\langle\Lambda\rangle_e$. We are interested in the Λ at which this has rank $\leq R$; this is a determinantal locus in the Grassmannian.

Suppose that Λ is expressed

$$\operatorname{span}(x^{\mu(i)} + \sum_{j=1}^{p(d)} a_{ij} x^{\mu(j)}, i = p(d)+1, \ldots, \binom{n+d}{d})$$

where $\{x^{\mu(i)}, i = 1, \ldots, \binom{n+d}{d}$ is a suitable ordering of the monomials of degrees d in $x_0, \ldots, x_n$. Then $\langle\Lambda\rangle_e$ is equal to

$$\operatorname{span}(x^{\nu} x^{\mu(i)} + \sum_{j=1}^{p(d)} a_{ij} x^{\mu(j)}, i = p(d)+1, \ldots, \binom{n+d}{d}, \deg(\nu) = e-d),$$

the span of a collection of

$$\binom{n+e-d}{e-d}(\binom{n+d}{d} - p(d))$$

vectors in $k[x_0,\ldots,x_n]_e$. Write each of these vectors in terms of the monomial basis for $k[x_0,\ldots,x_n]_e$, and arrange the coordinates in a

$$\binom{n+e-d}{e-d}(\binom{n+d}{d} - p(d)) \times \binom{n+e}{e}$$

matrix $M(\Lambda)$. The entries of $M(\Lambda)$ are linear polynomials in the a_{ij}. Observe that

$$\Lambda \in H_p(e,R) \Leftrightarrow \operatorname{rank}(M(\Lambda)) \leq R,$$

which is defined by simultaneous vanishing of the $(R+1) \times (R+1)$ minors of $M(\Lambda)$. These are polynomials in the a_{ij} of degree $\leq R+1$. $\square$

Definition 24 Fix $n \in \mathbb{N}$ and $p(t) \in \mathbb{Q}[t]$ polynomial of degree $\leq n$ with $p(\mathbb{Z}) \subset \mathbb{Z}$. Choose d_p as specified in Theorem 18 and $d \geq d_p$. The scheme structure on

$$\mathcal{H}ilb_p \subset \mathrm{Gr}(\binom{n+d}{d} - p(d), \binom{n+d}{d})$$

is defined

$$\cap_{e \geq d} H_p(e, \binom{n+e}{e} - p(e)).$$

Exercise 25 Consider the $n = 2$ and $p(t) = 2$. Write out explicit equations for

$$\mathcal{H}ilb_{p(t)} \subset \mathrm{Gr}(4, k[x_0, x_1, x_2]_2) = \mathrm{Gr}(4, 6).$$

5.7 Statement of main theorem

Fix $p(t) \in \mathbb{Q}[t]$ with $p(\mathbb{Z}) \subset \mathbb{Z}$ and a positive integer n. We would like to classify closed $X \subset \mathbb{P}^n$ with Hilbert polynomial $\mathrm{HP}_X(t) = p(t)$.

Theorem 26 *There exists a projective scheme $\mathcal{H}ilb_p$ representing flat families of subschemes of $\mathbb{P}^n$ with Hilbert polynomial $p(t)$.*

Thus to give a flat family $\pi : \mathcal{X} \to B$ of closed subschemes in $\mathbb{P}^n$ with Hilbert polynomial $p(t)$ is equivalent to specifying a morphism $\mu : B \to \mathcal{H}ilb_p$. There is a *universal flat family*

$$\mathcal{Z} \to \mathcal{H}ilb_p$$

such that $\mathcal{X} = \mathcal{Z} \times_{\mathcal{H}ilb_p} B$.

Exercise 27 Prove this by hand for $\mathbb{P}^1$. For the constant polynomial $p(t) = D$, show that

$$\mathcal{H}ilb_D \simeq \mathbb{P}(k[x_0, x_1]_D)$$

with universal family

$$\mathcal{Z} = \{c_0 x_0^D + c_1 x_0^{D-1} x_1 + \ldots + c_D x_1^D = 0\} \subset \mathbb{P}^1 \times \mathbb{P}(k[x_0, x_1]_D).$$

Exercise 28 Consider $X \subset \mathbb{P}^2$ with $\mathrm{HP}_X(t) = 2$. Show that the Hilbert function $\mathrm{HF}_X(t) = 2$ for each $t \geq 2$ and $J(X)$ is generated in degrees ≤ 2. Prove that

$$\begin{array}{rcl} \mathcal{H}ilb_2 & \hookrightarrow & \mathrm{Gr}(4,6) \\ [X] & \mapsto & \{J(X)_2 \subset k[x_0, x_1, x_2]_2\} \end{array}$$

is an embedding.

5.8 Schemes contained in subschemes

Theorem 29 *Fix a closed $W \subset \mathbb{P}^n$. There exists a projective scheme $\mathcal{H}ilb_{p,W}$ representing flat families of subschemes of W with Hilbert polynomail p.*

References

[1] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[2] David Mumford. *Lectures on curves on an algebraic surface*. With a section by G. M. Bergman. Annals of Mathematics Studies, No. 59. Princeton University Press, Princeton, N.J., 1966.