GLOBAL SOLUTIONS FOR THE 3D GRAVITY WATER WAVES SYSTEM

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1. The Dirichlet-to-Neumann operator

In this section, we want to prove

Proposition 1.1. There holds that

$$G(h)\phi = T_\lambda \omega - \text{div}(T_V h) + Q[h, \phi]$$

where

$$\omega = \phi - T_B h,$$

$$\lambda = \sqrt{(1 + |\nabla h|^2)|\zeta|^2 - (\zeta \cdot \nabla h)^2 + \lambda(0)}$$

and $Q[h, \omega]$ denotes terms which are quadratic and higher in $h$ and $\phi$ and one derivative smoother than either $\omega$ or $h$.

The proof of this proposition will take some time since we need to first introduce all the notations.

The following considerations are essentially taken from [1, 3]. We start from the Dirichlet problem

$$\Delta_{x,z} \Phi = 0, \quad \text{on} \quad \Omega = \{z \leq h(x)\}$$

and we consider the change of unknowns $u(x, y) = \Phi(x, z)$ where

$$(x, y) \mapsto (X, Z) = (x, \rho(x, y)),$$

$$\nabla_x = \nabla_X + \nabla_x \rho \partial_z, \quad \partial_y = \partial_y \rho \partial_z, \quad J = \frac{\partial(X, Z)}{\partial(x, y)} = \partial_y \rho.$$}

In order to find the equation satisfied by $u$, we consider the Dirichlet energy

$$E = \int_{\Omega} |\nabla_{x,z} \Phi|^2 dxdz = \int_{\mathbb{R}^+} \left\{ |\nabla_x u - \frac{\partial_y u}{\partial y \rho} \nabla \rho|^2 + \left| \frac{\partial_y u}{\partial y \rho} \right|^2 \right\} |\partial_y \rho| dx dy$$

and looking for an extremal, we obtain the following equation for $u$:

$$0 = \partial_y \left( 1 + \frac{|\nabla_x \rho|^2}{\partial_y \rho} \partial_y u - \nabla_x \rho \cdot \nabla_x u \right) + \text{div} \left( \partial_y \rho \nabla_x u - \nabla_x \rho \partial_y u \right), \quad \text{(1.1)} \quad \{\text{NewEllEqua}\}$$

We also compute the Dirichlet-Neumann operator

$$G(h)\phi = (\partial_z \Phi - \nabla_x \Phi \cdot \nabla_x h)_{|z=h(x)} = \left( \frac{1 + |\nabla_x \rho|^2}{\partial_y \rho} \partial_y u - \nabla_x h \cdot \nabla_x u \right)_{|y=0}. \quad \text{(1.2)} \quad \{\text{DNOp}\}$$

We now introduce

$$V = \nabla_x \Phi = \nabla_x u - \frac{\partial_y u}{\partial y \rho} \nabla \rho,$$

$$B = \partial_z \Phi = \frac{\partial_y u}{\partial y \rho}$$

which are the Eulerian components of the velocity.

Letting $G(h, \phi)(y)$ denote the last quantity in (1.2), we may rewrite (1.1) as

$$\partial_y (G(h, \phi)(y)) + \text{div}_x (\partial_y \rho V) = 0$$
and integrating this, we obtain a new formulation for $G(h)\phi$

$$G(h)\phi = -\text{div}_x \left[ \int_{y=-\infty}^{0} V \partial_y \rho \, dy \right] = -\text{div} \left[ \int_{z=-\infty}^{h(x)} V \, dz \right]$$

which is intuitively consistent with conservation of volume.

We note that a typical choice of $\rho$ would be

$$\rho_1(x,y) = y - e^{y|\nabla|} h(x), \quad \rho(x,y) = y - (1 - y|\nabla|) e^{y|\nabla|} h(x) \quad (1.3)$$

$\rho_1$ is harmonic, while $\rho$ is biharmonic and satisfies $\rho(x,0) = -h(x)$, $\partial_y \rho(x,0) = 0$.

We note that it might be much more advantageous to consider more general mappings of the form

$$(X, Z) = M(x,y)$$

as this would allow to benefit from the whole "gauge" group of diffeomorphisms of the surface. At present we do not know how to do that directly, however.

1.1. Linearization at a flat interface. We would like to obtain some quantitative information about $G(h)\phi$. For this, we will linearize the equation at the free interface $h \equiv 0$. This yield a simple perturbation of an elliptic equation, which we can easily solve. However, the error terms are not smooth enough for our purpose and we will later need to resolve the high frequencies better.

We may rewrite $(1.1)$ as

$$(\partial_y^2 - |\nabla|^2) u = \partial_y Q_a + |\nabla| Q_b,$$

$$Q_a = \left( 1 - \frac{1 + |\nabla x \rho|^2}{\partial_y \rho} \right) \partial_y u + \nabla_x \rho \cdot \nabla_x u,$$

$$Q_b = R_j \left\{ (1 - \partial_y \rho) \partial_j u + \partial_j \rho \partial_y u \right\}$$

which can be readily integrated:

$$u(y) = e^{y|\nabla|} \left( \phi - \frac{1}{2} \int_{-\infty}^{0} e^{|\nabla|} (Q_a(s) - Q_b(s)) \, ds \right) + \frac{1}{2} \int_{-\infty}^{0} e^{-|y-s||\nabla|} \left( \text{sign}(y-s)Q_a(s) - Q_b(s) \right) \, ds \quad (1.4)$$

Letting

$$\|f\|_{X_s} := \|\nabla^{\frac{1}{2}} f\|_{L^\infty_{y} L^2_x} + \|\nabla f\|_{L^2_{x} H^s_x} + \|\partial_y f\|_{L^2_{x} H^s_x}$$

we easily obtain the following result via a fixed-point argument

Lemma 1.2. Let $s > d/2 + 1/2$. Assume that $\phi \in H^s$ and that the mapping $\tilde{\rho} = \rho(x,y) - y$ satisfies $\tilde{\rho} \in X_s$, then there exists a solution $u \in X_s$ of $(1.1)$. In addition, we note that

$$\| (\partial_y - |\nabla|) u \|_{X_{s-1}} \lesssim \| \phi \|_{X_s} \| \tilde{\rho} \|_{X_s}.$$
1.2. Paralinearization. In order to obtain better bounds, we need a better linearization. The paralinearization introduced in [2] works naturally.

Introducing the linear operator

\[ \mathcal{L} f = (\partial_y \rho) \Delta_x f - 2 \nabla_x \rho \nabla_x f + \frac{1 + |\nabla_x \rho|^2}{\partial_y \rho} \partial_y^2 f, \]

we note that (1.1) can be reformulated as

\[ \mathcal{L} u - \frac{\partial_y u}{\partial_y \rho} \mathcal{L} \rho = 0 \]

and this leads naturally to introducing the “good unknown” of Alinhac

\[ \omega = u - T_{\partial_y \rho} \rho. \]

We now want to paralinearize (1.1). We will see that this reduces to

\[ T_{\mathcal{L}} \omega = \mathcal{H} \]

where \( \mathcal{H} \) denote smoothing operators and \( T_{\mathcal{L}} \) is a paralinearized (and symmetrized) version of \( \mathcal{L} \):

\[ T_{\mathcal{L}} = \partial_y \left( T_{1+i|\nabla_x \rho|^2} \partial_y - T_{\nabla_x \rho} \nabla \right) + \text{div}_x \left( T_{\partial_y \rho} \nabla - T_{\nabla_x \rho} \partial_y \right) \]

We may now factorize \( T_{\mathcal{L}} \):

\[ T_{\mathcal{L}} = \partial_y T_{1+i|\nabla_x \rho|^2} \partial_y - i \left[ \partial_y T_{\nabla_x \rho} \zeta + T_{\nabla_x \rho} \zeta \partial_y \right] - T_{(\partial_y \rho)\zeta} + \frac{1}{4} T_{\partial_y \Delta_x \rho} \]

where \( \mathcal{H} \mathcal{H} \) denotes a smoothing operator. Expanding, we find that

\[ \partial_y Q_2 + Q_1 \partial_y + Q_1 T_{1+i|\nabla_x \rho|^2} Q_2 \]

\[ = -i \left[ \partial_y T_{\nabla_x \rho} \zeta + T_{\nabla_x \rho} \zeta \partial_y \right] - T_{(\partial_y \rho)\zeta} + \frac{1}{4} T_{\partial_y \Delta_x \rho} + \mathcal{H} \mathcal{H}. \]

Now, choose

\[ Q_1 = -iT_a + T_b, \quad Q_2 = -iT_a - T_b, \]

\[ a = a^{(1)} + a^{(0)}, \quad b = b^{(1)} + b^{(0)}. \]

Using the symbolic calculus rules, we observe that\(^1\) when \( \sigma_1 \in S^{m_1}, \sigma_2 \in S^{m_2}, m_1 \geq 0 \) and \( |\sigma_1| \geq 1 \),

\[ T_{\sigma_1}^{-1} = T_{\sigma_1} + O(S^{m_1-2}), \]

\[ T_{\sigma_1} T_{\sigma_2} T_{\sigma_1}^{-1} = T_{\sigma_2} + \frac{i}{2} \left( T_{\{\sigma_1, \sigma_2, \sigma_1^{-1}\}} + T_{\{\sigma_1, \sigma_2\}} \sigma_1^{-1} \right) - \frac{1}{4} T_{\{\{\sigma_1, \sigma_2\}, \sigma_1^{-1}\}} + O(S^{2m_1+m_2-2}) \]

\[ = T_{\sigma_2} + \frac{1}{4} T_{\{\sigma_1, \sigma_2\} \sigma_1^{-2}} + O(S^{2m_1+m_2-2}) \]

where we have used the fact that

\[ \{\sigma_1 \sigma_2, \sigma_1^{-1}\} + \{\sigma_1, \sigma_2\} \sigma_1^{-1} = 0. \]

\(^1\)Note that in general, we would expect \( \sigma_1^{-1} \) to not be of order \( m_1 \) and it is probably better to look at the bounds on a case by case basis. In any case, when \( m_1 = 0 \) and \( \sigma_1 \geq 1 \), the bounds below are correct and sufficient for our purpose.
We also observe that
\[
T_{b^{-ia}} T_g T_{b^{ia}} = T_g(b^2 - a^2) - T_\pi + O(\mathcal{S}^0), \quad \pi = a\{b, g\} + b\{g, a\} - g\{a, b\}
\]

Plugging back into our factorization, we find that
\[
-i (\partial_y T_a + T_a \partial_y) - T_{\partial_y b} - T \frac{\partial_y \rho}{1 + |\nabla_x \rho|^2} (a^2 + \rho) + T_\pi
\]
\[
= -i [\partial_y T_{\nabla_x \rho} \cdot \nabla + T_{\nabla_x \rho} \cdot \partial_y] - T_0 (\partial_y \rho) |\nabla_x \rho|^2 + \frac{1}{4} T_0 \partial_y \Delta_x \rho + \mathcal{H} \mathcal{H}.
\]
We may now solve for the highest order terms:
\[
a^{(1)} = \nabla_x \rho \cdot \zeta,
\]
\[
\frac{\partial_y \rho}{1 + |\nabla_x \rho|^2} ((a^{(1)})^2 + (b^{(1)})^2) = \partial_y \rho |\zeta|^2
\]
so that we may choose
\[
b^{(1)} = \sqrt{(1 + |\nabla_x \rho|^2)} |\zeta|^2 - (\zeta \cdot \nabla_x \rho)^2.
\]
To find the subprincipal terms, we go to next order:
\[
-i (\partial_y T_{a \leq 0} + T_{a \leq 0} \partial_y) - T_{\partial_y b} - T \frac{\partial_y \rho}{1 + |\nabla_x \rho|^2} (2(a^{(1)} a^{(0)} + 2b^{(1)} b^{(0)}) + T_\pi = \frac{1}{4} T_0 \partial_y \Delta_x \rho + O(\mathcal{S}^0).
\]
So we may choose
\[
a^{(0)} = 0, \quad b^{(0)} = \frac{1}{2b^{(1)}} \left( a^{(1)} \{b^{(1)}, \ln \frac{\partial_y \rho}{1 + |\nabla_x \rho|^2} \} + b^{(1)} \{\ln \frac{\partial_y \rho}{1 + |\nabla_x \rho|^2}, a^{(1)} \} - \{a^{(1)}, b^{(1)}\} \right).
\]
Note that proceeding by induction, we may find \(a^{(-i)} \equiv 0\) and \(b^{(-i)}\) so as to make the error terms are smoothing as we want.

Finally, we arrive at
\[
\left( \partial_y + T_{b^{-ia}} T_{1 + |\nabla_x \rho|^2} \right) \left( T_{\frac{1}{1 + |\nabla_x \rho|^2} \partial_y - T_{a^{-ib}}} \omega = Q^0(\omega, h)
\right)
\]
where \(Q^0(\omega, h)\) denotes a quadratic or higher expression of similar smoothness as either \(\omega\) or \(\tilde{\rho}\), and using the Lemma about parabolic regularity, we find that
\[
\left( T_{\frac{1}{1 + |\nabla_x \rho|^2} \partial_y - T_{a^{-ib}}} \right) \omega = Q^{-1}(\omega, h)
\]
where \(Q^{-1}(\omega, h)\) denotes a quadratic or higher expression smoother by one derivative as either \(\omega\) or \(\tilde{\rho}\).

Finally, we may obtain an expression for the Dirichlet-to-Neumann operator:
\[
\frac{1}{1 + |\nabla_x \rho|^2} \partial_y u - \nabla_x h \cdot \nabla_x u = \frac{T_{\frac{1}{1 + |\nabla_x \rho|^2} \partial_y - T_{\nabla_x \rho} + Q^{-1}(\omega, h)}
\]
\[
+ \left( T_{\frac{1}{1 + |\nabla_x \rho|^2} \partial_y - T_{\nabla_x \rho} \partial_y \nabla \rho + \left( T_{\frac{1}{1 + |\nabla_x \rho|^2} \partial_y (\frac{\partial_u u}{\partial_y \rho}) - T_{\nabla_x \rho} \nabla (\frac{\partial_u u}{\partial_y \rho}) \right) \rho
\]
\[
= T_0 \omega + \frac{1}{2} T_0 \Delta_x \omega - \text{div}(T_{\nabla_x u} \frac{\partial_u u}{\partial_y \rho} \nabla \rho) + Q^{-1}(\omega, h)
\]
\[
+ \left( T_{\Delta u} - T_{\text{div}(\frac{\partial_u u}{\partial_y \rho} \nabla \rho)} - T_{\nabla_x \rho} \nabla (\frac{\partial_u u}{\partial_y \rho}) + T_{\frac{1}{1 + |\nabla_x \rho|^2} \partial_y \frac{\partial_u u}{\partial_y \rho}} \right) \rho.
\]
Note that the term in the last line equals
\[
T_{\frac{1}{\partial_y \rho}} (\mathcal{L} u - \frac{\partial_u u}{\partial_y \rho} \mathcal{L} \rho) \rho = 0.
\]
REFERENCES


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