GLOBAL EXISTENCE FOR QUASILINEAR DISPERSIVE EQUATIONS II

BENOIT PAUSADER

1. Space-time resonant interactions at large scale

We first remark from the defining relation of p:

$$\nabla_\eta \Phi(\xi, p(\xi)) = 0$$

that

$$\frac{dp}{d\xi} = [\nabla^2_{\eta\eta}\Phi(\xi, p(\xi))]^{-1} \circ \nabla^2_{\eta\xi}\Phi(\xi, p(\xi)) \tag{1.1}$$

is invertible.

Here we treat the case of inputs coming from free waves located far away from the origin. In order to simplify the presentation, we will only consider inputs that would have come from our first guess[1] (that is $|x|^{1+\beta} f$ is bounded in L^2).

Hence, we now consider the case when at least one function f or g is supported away from the origin and assume that for all y_1 in the support of f and all y_2 in the support of g,

$$Y_1/2 \le |y_1| \le 2Y_1, \quad Y_2/2 \le |y_2| \le 2Y_2, \quad Y = \max(Y_1, Y_2), \quad T^{\frac{1}{2}} \le Y \le T$$

Note in particular that there are at most $O(\log T)^2$ values of Y_1, Y_2 possible.

We start again by decomposing the Kernel

$$\begin{aligned} K &= K_R + K_N \\ K_N(x, y_1, y_2, s) &= \int_{\mathbb{R}^3\times\mathbb{R}^3} (1 - \varphi(\delta_X^{-1}\nabla_\eta\Phi(\xi,\eta)))\widetilde{\varphi}_{\vec{k}}(\xi,\eta) e^{is\Phi(\xi,\eta)} e^{i\xi[y_1 - x]} e^{i\eta[y_2 - y_1]} d\eta d\xi, \\ \widetilde{\varphi}_{\vec{k}}(\xi,\eta) &= \varphi_k(\xi)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta) \end{aligned}$$

and this time, we see that, upon integrating by parts,

$$\begin{aligned} &\int_{\mathbb{R}^3\times\mathbb{R}^3} (1 - \varphi(\delta_X^{-1}\nabla_\eta\Phi(\xi,\eta)))\widetilde{\varphi}_{\vec{k}}(\xi,\eta) e^{is\Phi(\xi,\eta)} e^{i\xi[y_1-x]} e^{i\eta[y_2-y_1]} d\eta d\xi \\ =&\frac{i}{s}\int_{\mathbb{R}^3\times\mathbb{R}^3} e^{i\xi[y_1-x]} e^{is\Phi(\xi,\eta)} \mathrm{div}_\eta \left\{ \frac{\nabla_\eta\Phi(\xi,\eta)}{|\nabla_\eta\Phi(\xi,\eta)|^2} (1 - \varphi(\delta_X^{-1}\nabla_\eta\Phi(\xi,\eta)))\widetilde{\varphi}_{\vec{k}}(\xi,\eta) e^{i\eta[y_2-y_1]} \right\} d\eta d\xi \end{aligned}$$

and we see that now the worst case happens when the derivative hits the term $e^{i\eta[y_2-y_1]}$. Hence, letting

$$\delta_X = T^{-1} Y T^{\delta}$$

we still get that each integration by parts brings a factor of $T^{-\delta}$ and since we can iterate this as many times as we want, we get that K_N leads to negligible contributions.

It thus suffices to consider the coherent part of the interaction:

$$I_{T,C}(x) = \int_{\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3} \varphi(\delta_X^{-1}\nabla_\eta\Phi(\xi,\eta))\widetilde{\varphi}_{\vec{k}}(\xi,\eta) e^{is\Phi(\xi,\eta)} e^{-ix\xi} \widehat{f}(\xi-\eta)\widehat{g}(\eta) d\eta d\xi,$$

[1] Recall however that the new inputs created by inputs at small scale (i.e. essentially concentrated in a region of size $\ll T^{1/2}$) barely fail to satisfy this integrability condition.

where we consider normalized inputs

$$\||x|^{1+\beta}f\|_{L^2}+\||x|^{1+\beta}g\|_{L^2}\le 2,$$

for some $\beta>0$ such that $\beta\gg\delta$.

Note that we are in a situation strictly worse than before since we have localized η in a region larger. In addition, as $Y\to T$, we see that the scale at which we localize goes to 1 and in the end, we consider the whole Fourier support of our functions. To compensate for this, we need norms that penalize the distance to the origin and we need to make sure that the control we gain from this compensates the loss coming from the fact that we get less and less precise description of our output.

Assuming that $Y_1\ge Y_2$, we first remark that a naïve use of Plancherel would already give us that

$$\begin{aligned}\|I_T(x)\|_{L^2}&\lesssim T\sup_t\|e^{is\Lambda_2}f\cdot e^{is\Lambda_3}g\|_{L^2}\lesssim T\|f\|_{L^2}\left[\sup_{s\simeq T}\|e^{is\Lambda_3}g\|_{L^\infty_x}\right]\lesssim T^{-\frac12}\|f\|_{L^2}\|g\|_{L^1}\\&\lesssim T^{-\frac12}Y^{-\frac12-2\beta}.\end{aligned}$$

This is at least $T^{-3/4-\delta}$ and thus we see that $i)$ we already have with little work that $|x|^{\frac34}I\in L^2$ $ii)$ when we lose our localization (i.e. when $\delta_X\simeq 1$), then $Y=T$ and we have already recovered what we wanted. In other words, we are in a transitory regime.

Now, we could try to go the same route as before and perform an integration by parts in time. However, we saw before that the corresponding control needed was the L^∞-norm of the Fourier transform, that we would now need to penalize with respect to distance from the origin (Y_1 or Y_2). Instead, since we have less to gain, we will find a more robust way to get a gain from L^2-orthogonality in the time integral.

We now consider the ξ-derivative of the oscillatory phase:

$$\begin{aligned}\nabla_\xi\left[s\Phi(\xi,\eta)-x\cdot\xi\right]&=s\nabla_\xi\Phi(\xi,\eta)-x=s\nabla\Psi(\xi)-x+O(s\delta_X)\\&=s\nabla\Psi(\xi)-x+O(T^\delta Y).\end{aligned}\tag{1.2}$$

Since by assumption, we have that $|\nabla\Psi|\neq 0$, we see that this phase is large unless the angle in ξ is restricted *and* s is well chosen to be $|x|/|\nabla\Psi|$. This is the extra orthogonality that we want to use.

We start by estimating the uncertainty in (1.2). We remark that a change of s of size $\delta_T=T^\delta Y$ could still be accounted for in the error, and similarly for a change of ξ of size δ_X and a change in x of size δ_T. Thus, we may define the corresponding localizations of the integral:

$$\begin{aligned}&I_T^{z,\zeta,\sigma}(x)=\\&\chi^3(\delta_T^{-1}x-z)\int_{\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3}\theta(\frac{s}{T})\chi^3(\delta_X^{-1}\xi-\zeta)\chi(\delta_T^{-1}s-\sigma)\varphi(\delta_X^{-1}\nabla_\eta\Phi(\xi,\eta))\widetilde{\varphi}_{\vec{k}}(\xi,\eta)e^{i\widetilde{\Phi}(\xi,\eta)}\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta d\xi\\&\widetilde{\Phi}(\xi,\eta)=s\Phi(\xi,\eta)-x\cdot\xi\end{aligned}\tag{1.3}$$

where $z,\zeta\in\mathbb{Z}^3$, $\sigma\in\mathbb{Z}$ and $\chi^3(x,y,z)=\chi(x)\chi(y)\chi(z)$, where

$$\chi\in C^\infty_c(\mathbb{R}),\qquad\sum_{k\in\mathbb{Z}}\chi(x-k)\equiv 1.$$

We claim that all the elements in this family are almost orthogonal (up to a remainder of T^{-100}). This is clear when z or ζ vary. Thus, to show orthogonality in σ, it suffices to show that for each choice of (ζ,σ), there holds that

$$|I_T^{z,\zeta,\sigma}|\lesssim T^{-100},\quad\forall z\notin S_{\zeta,\sigma}\tag{1.4}$$

where $S_{\zeta,\sigma}$ has bounded cardinality.

To show this, we just examine (1.2) and we see that for s, ξ in the support of $I^{z,\zeta,\sigma}$, there holds that

$$\nabla_\xi\widetilde{\Phi}(\xi,\eta)=\nabla_\xi\left[s\Phi(\xi,\eta)-x\cdot\xi\right]=\sigma\delta_T\nabla\Psi(\delta_X\zeta)-x+O(\delta_T).$$

This is essentially the distance of x to a fixed point, up to an uncertainty of δ_T. Thus, we see that, outside of a ball of radius $O(\delta_T)$, there holds that

$$|\nabla_\xi \widetilde{\Phi}| \gtrsim \delta_T.$$

On the other hand, we may integrate by parts in ξ in the definition of K to get

$$\int_{\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3} \chi^3(\delta_X^{-1}\xi-\zeta)\chi(\delta_T^{-1}s-\sigma)\varphi(\delta_X^{-1}\nabla_\eta\Phi(\xi,\eta))\widetilde{\varphi}_{\vec{k}}(\xi,\eta)e^{is\widetilde{\Phi}(\xi,\eta)}\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta d\xi$$
$$= \int_{\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3} \chi(\delta_T^{-1}s-\sigma)e^{is\widetilde{\Phi}(\xi,\eta)}\widehat{g}(\eta)\mathrm{div}_\xi\left\{\frac{\nabla\widetilde{\Phi}}{|\nabla_\xi\widetilde{\Phi}(\xi,\eta)|^2}\chi^3(\delta_X^{-1}\xi-\zeta)\varphi(\delta_X^{-1}\nabla_\eta\Phi(\xi,\eta))\widetilde{\varphi}_{\vec{k}}(\xi,\eta)\widehat{f}(\xi-\eta)\right\}d\eta d\xi,$$

and we see that at each integration by parts, we have a gain of $Y\delta_T^{-1}+\delta_X^{-1}\delta_T^{-1}\simeq T^{-\delta}$. This gives (1.4). Now that we have transfered the obvious orthogonality in x to an orthogonality in s, we deduce that

$$\|I_T\|_{L^2}^2 = \sum_{z,\sigma,\zeta}\|I_T^{z,\zeta,\sigma}\|_{L^2}^2. \tag{1.5}$$

Besides, we have also seen that each choice of (σ,ζ) determines at most a bounded number of z and therefore, we can ignore the summation in z in the sum above.

Now, consider an elementary interaction in (1.3), and remark that both η and ξ are restricted to balls of size δ_X. But this also restricts the support of the relevant functions f, g, so that, introducing

$$\begin{aligned}\widehat{f_\zeta}(\xi) &:= \widehat{f}(\xi)\chi(\delta_X^{-1}\xi-\zeta+p(\zeta))\\ \widehat{g_\zeta}(\xi) &:= \widehat{g}(\xi)\chi(\delta_X^{-1}\xi-p(\zeta)),\end{aligned}$$

we see, if p is a nice diffeomorphism (which we may always assume from (1.1)) that $I_T^{z,\zeta,\sigma}$ only depends on f_ζ and g_ζ and that

$$\sum_\zeta\|f_\zeta\|_{L^2}^2\lesssim\|f\|_{L^2}^2,\quad \sum_\zeta\|g_\zeta\|_{L^2}^2\lesssim\|g\|_{L^2}^2,$$

which essentially take care of the sum in ζ in (1.5).

Now, we may estimate $I_T^{z,\zeta,\sigma}$ rather crudely using a variant of the Plancherel theorem as above (in fact (2.1) below) to get (assuming that $Y_1\geq Y_2$)

$$\|I_T^{z,\zeta,\sigma}\|_{L^2}\lesssim\delta_T\|f_\zeta\|_{L^2}\sup_{s\geq T/4}\|e^{is\Lambda_3}g_\zeta\|_{L^\infty}\lesssim\delta_T T^{-3/2}\|f_\zeta\|_{L^2}\|g\|_{L^1}.$$

Remarking that there are $O(T\delta_T^{-1})$ values of σ for which

$$\theta(\frac{s}{T})\chi(\delta_T^{-1}s-\sigma)\neq 0$$

and that each choice of (ζ,σ) defines a bounded number of z, we finally obtain that

$$\begin{aligned}\|I_T\|_{L^2}^2 &\lesssim\sum_{\sigma,\zeta}\|I_T^{z,\zeta,\sigma,\sigma}\|_{L^2}^2\lesssim T\delta_T^{-1}\sum_\zeta(\delta_T T^{-\frac{3}{2}}\|f_\zeta\|_{L^2}\|g\|_{L^1})^2\\ &\lesssim\delta_T T^{-2}\|f\|_{L^2}^2\|g\|_{L^1}^2.\end{aligned}$$

Hence, assuming that $|x|^{1+\beta}f$, $|x|^{1+\beta}g\in L^2$, we see that

$$T^{1+\beta}\|I_T\|_{L^2}\lesssim T^{1+\beta}\cdot(YT^{\delta-2})^{\frac{1}{2}}\cdot Y^{-1-\beta}\cdot Y_2^{\frac{3}{2}-1-\beta}\lesssim T^{\beta+\frac{\delta}{2}}Y^{-\frac{1}{2}-\beta}Y_2^{\frac{1}{2}-\beta},$$

and we see that we have a good chance of making this summable.

2. Some simple results from Fourier analysis

The elementary lemma to control bilinear term is the following simple lemma:

Lemma 2.1.

$$\|\mathcal{F}^{-1}\int_{\mathbb{R}^3} m(\xi,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta\|_{L^p}\lesssim\|\mathcal{F}m\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)}\|f\|_{L^q(\mathbb{R}^3)}\|g\|_{L^r(\mathbb{R}^3)},\quad \frac{1}{p}=\frac{1}{q}+\frac{1}{r}. \tag{2.1}$$

This follows directly from the formula

$$\mathcal{F}^{-1}\int_{\mathbb{R}^3} m(\xi,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta=\int_{\mathbb{R}^3}(\mathcal{F}m)(z,t)f(x+z)g(t+z+x)dzdt.$$

We also need an efficient stationary phase lemma:

Lemma 2.2. *Assume that $0<\epsilon\leq 1/\epsilon\leq K$, $N\geq 1$ is an integer, and $f,g\in C^N(\mathbb{R}^n)$. Then*

$$\Big|\int_{\mathbb{R}^n} e^{iKf}g\,dx\Big|\lesssim_N (K\epsilon)^{-N}\big[\sum_{|\rho|\leq N}\epsilon^{|\rho|}\|D^\rho_x g\|_{L^1}\big], \tag{2.2}$$

provided that f is real-valued,

$$|\nabla_x f|\geq \mathbf{1}_{\mathrm{supp}\,g},\qquad \textit{and}\qquad \|D^\rho_x f\cdot\mathbf{1}_{\mathrm{supp}\,g}\|_{L^\infty}\lesssim_N \epsilon^{1-|\rho|},\ 2\leq|\rho|\leq N. \tag{2.3}$$

Proof of Lemma 2.2. We localize first to balls of size $\approx\epsilon$. Using the assumptions in (2.3) we may assume that inside each small ball, one of the directional derivatives of f is bounded away from 0, say $|\partial_1 f|\gtrsim_N 1$. Then we integrate by parts N times in x_1, and the desired bound (2.2) follows. □

3. Choice of norms

3.1. **Definition of the norms.** We naturally introduce a partition of unity consistent with the information that we want to quantify (momentum and frequency):

$$\begin{aligned}(Q_{j,k}f)(x)&=\varphi^{(k)}_j(x)\cdot P_kf(x)\\ \widehat{P_kf}(\xi)&=\varphi(2^{-k}\xi)\widehat{f}(\xi)\qquad \varphi^{(k)}_j(x)=\begin{cases}\varphi(2^{-j}x) & \text{if } j+k>0\\ \phi(2^kx) & \text{if } j+k=0\end{cases}\end{aligned}$$

where $\varphi(x)=\phi(x)-\phi(2x)$ and $\phi(x)=1$ when $|x|\leq 1$ and $\phi(x)=0$ when $|x|\geq 2$. Thus we see that $Q_{j,k}$, defined for all $(k,j)\in\mathbb{Z}\times\mathbb{N}$ such that $j+k\geq 0$ essentially localizes to distance about 2^j from the origin and to frequency about 2^k *provided* that we respect the uncertainty principle $2^k\cdot 2^j\geq 1$.

Indeed, in our situation, we do not expect small spatial scales (corresponding to $j\leq 0$) to play a particular role (e.g. the initial data does not constrain these at all); therefore, it makes little sense to allow for $j\leq 0$. In addition, there is no point localizing below the uncertainty principle: since

$$P_kf=2^{3k}\varphi(2^k\cdot)*f,$$

we see that $|\nabla P_kf|\simeq 2^k|P_kf|$ and therefore, P_kf is locally constant at scales smaller than 2^{-k}, and the norms

$$\|P_kf\|_{L^2(|x|\simeq 2^j)},\quad j\leq -k$$

are monotonically increasing (and summable) in j until $j=-k$. Thus in this situation, it suffices to consider $\varphi^{(k)}_jP_kf$.

Now, we are equipped with a prototype for the norms we want to consider[2], at least for frequencies about 1:

$$\|f\|_{B^1_{j,0}} = \sup_j \left\{ 2^{(1+\beta)j}\|Q_{j,0}f\|_{L^2} + \|\widehat{Q_{j,0}f}\|_{L^\infty} \right\},$$
$$\|f\|_{B^2_{j,0}} = \sup_j \left\{ 2^{(1-\beta)j}\|Q_{j,0}f\|_{L^2} + \|\widehat{Q_{j,0}f}\|_{L^\infty} + 2^{\gamma j}\|\widehat{Q_{j,0}f}\|_{L^1} \right\}.$$

Now, a way to normalize these norms in their dependence in the frequency parameter k in a consistent way is to ask that all the terms give out the same number for a typical function at the uncertainty principle level and centered at 0:

$$f(x) = 2^{3k}\varphi(2^k x), \quad \widehat{f}(\xi) = \widehat{\varphi}(2^{-k}\xi) \qquad 2^{(1+\beta)j}\|Q_{j,k}f\|_{L^2} \simeq 2^{(\frac{1}{2}-\beta)k} \simeq 2^{(\frac{1}{2}-\beta)k}\|\widehat{Q_{j,k}f}\|_{L^\infty}, \quad j=-k,$$

and similarly, for the same function,

$$2^{(\frac{1}{2}-\beta)k} \simeq 2^{2\beta k}2^{(1-\beta)j}\|Q_{j,k}f\|_{L^2} \simeq 2^{(\frac{1}{2}-\beta)k}\|\widehat{Q_{j,k}f}\|_{L^\infty} \simeq 2^{(\gamma-\frac{5}{2}-\beta)k}2^{\gamma j}\|\widehat{Q_{j,k}f}\|_{L^1}.$$

Finally, since we want to be able to sum these norms and allow them to control a large number of derivative (say 10), we obtain the norm

$$\|f\|_Z = \sup_{j+k\geq 0} \left[2^{\alpha k} + 2^{10k}\right]\|Q_{j,k}f\|_{B_{j,k}},$$
$$\|f\|_{B_{j,k}} = \sup_{f=g+h} \left\{ \|g\|_{B^1_{j,k}} + \|h\|_{B^2_{j,k}} \right\},$$
$$\|f\|_{B^1_{j,k}} = 2^{(1+\beta)j}\|f\|_{L^2} + 2^{(1/2-\beta)k}\|\widehat{f}\|_{L^\infty},$$
$$\|f\|_{B^2_{j,k}} = 2^{(1-\beta)j}2^{2\beta k}\|f\|_{L^2} + 2^{(1/2-\beta)k}\|\widehat{f}\|_{L^\infty} + 2^{\gamma j}2^{(\gamma-5/2-\beta)k}\|\widehat{f}\|_{L^1}.$$

In fact, we will have to precise a little the last component of the B^2 norm, but then the corresponding modification will be handled similarly.

3.2. **Requirements.** At this point, we are almost ready to use this norm to prove the boundedness of the quadratic interactions. We only need to check first

(1) that this norm is invariant by Calderón-Zygmund operators in the sense that
$$\|Qf\|_Z \lesssim \|f\|_Z$$
whenever $\widehat{Qf} = q(\xi)\widehat{f}(\xi)$ and
$$\sup_{|\alpha|\leq 100} |\partial^\alpha q(\xi)| \leq |\xi|^{-|\alpha|}.$$
(2) that boundedness in this norm guarantees that the linear flow is integrable:
$$\|e^{it\Lambda}f\|_{W^{5,\infty}} \lesssim (1+|t|)^{-1-\beta}\|f\|_Z.$$
(3) that the evolution leads to linear profiles continuous in the Z norm, i.e.
$$\sup_{[0,T]} \|f\|_Z < \infty, \quad \|f(t+s)-f(t)\|_Z \to 0, \quad s\to 0$$
for all t in the domain of definition of f.

The last bullet is obtained by showing that

$$\|\partial_t f\|_Z < +\infty$$

Whenever $f(0)$ is bounded in Z and $f\in C([0,T]:H^N)$. In fact, we even show boundedness of $\partial_t f$ only in the B^1-norm.

[2]Here and in the following, we let $\gamma = 9/8$.

3.3. **Decay of the free flow.** That boundedness in the B^1-norm implies integrable decay for free solutions follows from the decay estimates. For the B^2-norm, it is just a bit more complicated. For notational simplicity, we only consider the case $k = 0$. In this case, we have, on the one hand

$$\|e^{it\Lambda}Q_jf\|_{L^\infty} \lesssim t^{-\frac{3}{2}}\|Q_jf\|_{L^1} \lesssim t^{-\frac{3}{2}}2^{\frac{3}{2}j}\|Q_jf\|_{L^2} \lesssim t^{-1-\beta}\left[t^{-\frac{1}{2}+\beta}2^{(\frac{1}{2}+\beta)j}\right]\|f\|_{B^2_{j,k}}.$$

This is sufficient so long as $j \le 9/10\log_2 t$ (summable in j). On the other hand, we also have that

$$\|e^{it\Lambda}Q_jf\|_{L^\infty} \lesssim \|\widehat{Q_jf}\|_{L^1} \lesssim 2^{-\gamma j}\|f\|_{B^2_{j,k}} \lesssim t^{-1-\beta}\left[t^{1+\beta}2^{-\gamma j}\right]\|f\|_{B^2_{j,k}}$$

which is good when $j \ge 9/10\log_2 t$ (summable in j).

4. Overview of the proof

As explained above, we proceed in two main steps:

- An energy estimate step which reduces the proof to the *a priori* control of $t^{1+\beta}\|u(t)\|_{W^{5,\infty}}$ assuming a global bound on some H^N-norm, $N \gg 1$,
- A decay estimate to enforce this control,

and we focus here on this last step. After a suitable choice of norms and making sure that these norms are invariant by Calderón-Zygmund operators, we need to prove an estimate like

$$\sup_t\left\{\|T^{\sigma;\mu,\nu}[f,g](t)\|_Z + \||\nabla|T^{\sigma;\mu,\nu}[f,g](t)\|_Z\right\} \lesssim \sup_{t\in\mathbb{R}}\left\{\|f(t)\|_{Z\cap H^N}\|g(t)\|_{Z\cap H^N}\right\},$$

where

$$\mathcal{F}T^{\sigma;\mu,\nu}[f,g](\xi,t) = \int_{s=0}^t\int_{\mathbb{R}^d} e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)d\eta ds.$$

In order to fully use the atomic structure of our spaces, we need to decompose the functions f and g. We also decompose the time into slices of dyadic length $s \in [T, 2T]$, $T \le t$. This gives the sum

$$\begin{aligned}&\sup_{t,k,j,m}\sum_{\substack{k_1+j_1\ge 0\\ k_2+j_2\ge 0}} 2^{\beta^{10}(|k_1|+j_1+|k_2|+j_2+m)}(1+2^k)\|\varphi_j^{(k)}(x)T^{\sigma;\mu,\nu}_{m,k,k_1,k_2}[f_{k_1,j_1},g_{k_2,j_2}]\|_Z\\ &\lesssim \sup_{t\in\mathbb{R}}\left\{\|f(t)\|_{Z\cap H^N}\|g(t)\|_{Z\cap H^N}\right\},\end{aligned} \tag{4.1}$$

where

$$\mathcal{F}T^{\sigma;\mu,\nu}_{m,k,k_1,k_2}[f,g](\xi) = \int_{\mathbb{R}\times\mathbb{R}^d} q_m(s)\varphi_k(\xi)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta)e^{is\Phi^{\sigma;\mu,\nu}(\xi,\eta)}\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)d\eta ds,$$

where

$$f_{k_1,j_1} = \varphi^{(k_1)}_{j_1}(x)\cdot P_{k_1}f,\quad g_{k_2,j_2} = \varphi^{(k_2)}_{j_2}(x)\cdot P_{k_2}g$$

and q_m is a positive function supported into $[2^m, 2^{m+1}]$ satisfying

$$\int_{\mathbb{R}} q_m'(s)ds = 1.$$

At this point, the only fact that we need to remember from the fact that f, g satisfy specific equations is the following bound, which is a somewhat easy consequence of simple bounds on the nonlinearity[3]

$$\|\partial_t P_kf\|_{L^2} \lesssim t^{-1-\beta}\min(2^k, 2^{-30k}),\qquad \|\partial_t\hat{f}\|_{L^\infty} \lesssim t^{-1+1/10},$$

and similarly for g.

Once we have simplified matters this much, we need to start estimating the sum above as follows:

[3]Recall that $\partial_t f$ precisely equals the nonlinearity we are facing.

(1) First, we use simple estimates (describing the inputs using the energy estimate norm) to remove the most simple cases (essentially large k, k_1, k_2 and when one parameter is too unbalanced). This allows us to reduce to the case of only a logarithmic (in $j+m$) number of cases, so that it suffices to bound each summand in (4.1) uniformly.

(2) Second, we use the weighted norm description in order to i) by finite speed of propagation reduce to the case when $j, j_1, j_2 \leq m$ (i.e. with our notations $j \leq m$), ii) remove the non coherent-resonant cases.

(3) Third, we use the sum-space decomposition to finish the analysis by i) removing the "joint point" $(1-\beta/10)m \leq j_1 \leq j_2 \leq m$ and finally ii) treating the coherent-resonant case.

Informally, the idea in steps 2 and 3 is that we can rewrite the evolution as

$$\partial_t f = \partial_t \{Q(f,f)\} + \mathcal{T}(f,f) + R(f,f)$$

where $Q(f,f)$ is a quadratic change of unknown such that the mapping $f \mapsto f + Q(f,f)$ is bounded $H^N \cap Z \to Z$, $\mathcal{T}(f,f)$ represents only *transverse* interactions carried by waves which move in a transverse way and finally $R(f,f)$ is the left-over part which is, hopefully more localized and hence has a simpler structure.

Informally speaking, step 2 essentially treats the effect of Q and $\mathcal{T}$ while step 3 treats the effect of R once we have understood how to allow for the outputs it produces.

We refer to [1] for more details.

References

[1] A. D. Ionescu and B. Pausader, Global solutions of quasilinear systems of Klein–Gordon equations in 3D, J. Eur. Math. Soc., to appear.

LAGA, Université Paris 13 (UMR 7539)
E-mail address: pausader@math.u-paris13.fr