1 Addendum to lecture 3: strong uniformity

1.1 Theorem (Caporaso-Harris-Mazur)

Assume that Lang's "strong" conjecture holds true. Then there is a number $N(g)$, such that, for any number field extension K, the number of curves of genus $g > 1$ defined over K which have more than $N(g)$ rational points over L is **finite**.

Start with family over Hilbert scheme.
Apply:

1.2 Theorem (CHM)

Let $g > 1$ be an integer, and fix a subvariety $X_0 \subset M_g$. Let $X_n \subset M_{g,n}$ be the inverse image. Then for large enough n, the variety X_n is of general type.

and proof of:

1.3 Lemma (CHM)

Suppose $X \to B$ is a family of *curves* and $\mathcal{P} \subset X$ is n-correlated in $X \to B$. Then there is an open Zariski - dense set $U \subset B$ such that the number of points of $\mathcal{P}_b$ on all fibers of $X \to B$ over $b \in U$ is uniformly bounded.

this time using the Lang exceptional set Z. Get

1. a bound $N_0(g)$ which is valid outside a finite set of *geometric* isomorphism classes.
2. a finite number of K-rational points P_i on M_{g,n_i} giving rise to the exceptions. May assume $n_i > 1$.

Let $N_1(g) =$ maximum number of fixed points of automorphisms on a curve of genus g.
Claim: $N(g) = N_0(g) + N_1(g)$ works.

$P = P_i$ lies over $[C] \in M_g$, the fiber is an open set in $C^n/\mathrm{Aut}C$. Assume $D_{\bar{K}} = C_{\bar{K}}$ and there is a rational point $Q \in D^n$ whose image is P. Also assume Q not fixed by $\mathrm{Aut}D$.
Denote $\pi : D^n \to C^n/\mathrm{Aut}C$.
$Q = (x_1, \ldots x_n)$. At least one x_i not fixed by automorphisms, say x_1.
Therefore $(x_1, \ldots x_{n-1})$ not fixed.
By the exercise, the inverse image of $\pi(x_1, \ldots x_{n-1})$ is isomorphic to D.
Thus D is determined by P.