1 Lecture 1: The conjectures of Lang and Vojta

- `http://math.bu.edu/people/abrmovic/IHP`

WORK IN CHARACTERISTIC 0

ARITHMETIC $\overset{?}{\leftrightarrow}$ GEOMETRY

1.1 Theorem (Caporaso-Harris-Mazur)

Assume that Lang's "weak" conjecture holds true. Let K be a number field. Then there is a number $B(K, g)$, such that the number of K-rational points on any curve of genus $g > 1$ over K is bounded by $B(K, g)$.

Exerc. on many points
What's known on Lang

Varieties:

1.2 Definition

A *variety* defined over k is a geometrically integral scheme of finite type over k. If $k \subset k_1$ we write X_{k_1} for $X \underset{\mathrm{Spec}\ k}{\times} \mathrm{Spec} k_1$.

1.3 Definition

Let X be a variety. A subscheme $Z \subset X$ is called a *strict normal crossings divisor* if for each point $x \in Z$, there is a regular system of parameters $y_1, \ldots, y_{\dim X}$ for x in X (in particular the point $x \in X$ is supposed to be a regular point on X), such that Z is given on a Zariski neighborhood of x by the equation $y_1 \cdots y_l = 0$.

Kodaira dimensions

Let X be a variety, and L an invertible sheaf. Denote

$$\mathcal{R}(X, L) = \oplus_{n \geq 0} H^0(X, L^n),$$

the algebra of sections of L.

1.4 Definition

The *Iitaka dimension* $\kappa(X, L)$ is defined as follows:

- $\kappa(X, L) = \mathrm{tr.deg.}\mathcal{R}(X, L) - 1$,
  if this is non-negative;

- $\kappa(X, L) = -\infty$, otherwise.

why $-\infty$
properties
characterize

If $\kappa(X, L) = \dim X$ we say that L is *big* on X.

Given a smooth variety, denote by ω_X its dualizing sheaf. It is common to denote the associated divisor class by K_X, and then write $\omega_X = \mathcal{O}_X(K_X)$.

abuse

1.5 Definition

Let X be a **smooth** proper variety. Then the *Kodaira dimension* $\kappa(X)$ of X is defined to be $\kappa(X, \omega_X)$.

properties

1.6 Definition

Let $\bar{X}$ be a smooth proper variety, and let $D \subset \bar{X}$ be a reduced normal crossings divisor. Denote $X = \bar{X} \smallsetminus D$. The *Logarithmic Kodaira dimension* $\bar{\kappa}(X)$ is defined to be $\kappa(X, \omega_X(D))$.

sense
properties

1.7 Definition

Let X be a variety. Let X' be a nonsingular proper variety birational to X. Then the *Kodaira dimension* $\kappa(X)$ of X is defined to be $\kappa(X)'$. sense

1.8 Definition

Let X be a variety. Let $X' \to X$ be a desingularization of X. Then the *logarithmic Kodaira dimension* $\bar{\kappa}(X)$ of X is defined to be $\bar{\kappa}(X')$. sense

1.9 Definition

A variety X is said to be *of general type* if $\kappa(X) = \dim X$.

ω big

1.10 Definition

A variety X is said to be *of logarithmic general type* if $\bar{\kappa}(X) = \dim X$

$\omega(D)$ big

Curves:

$\kappa(X) = -\infty$.

Geometry: $\kappa(X) = -\infty \iff X_{\bar{K}} = \mathbb{P}^1$

Arithmetic: after a number field extension we have infinitely many points.

twists

0.

Geometry: $\kappa(X) = 0 \iff X_{\bar{K}}$ is elliptic

Arithmetic: after a number field extension we have infinitely many points (but fewer).

twists

1.

Geometry: $\kappa(X) = 1 \iff g(X) > 1$

Arithmetic: at most finitely many points (Faltings)

Lang-Vojta?

Curves with divisors:

$\mathbb{P}^1 \smallsetminus \{p_1, \ldots, p_n\}$:

Siegel

$E \smallsetminus \{p_1, \ldots, p_n\}$:

$K =$ a number field
$S =$ a finite set of finite places
$\mathcal{O}_{K,S} =$ the ring of S-integers in K

1.11 Lang's "weak" conjecture

Let X be a variety of general type over a number field K. Then the set of K-rational points $X(K)$ is not Zariski-dense in X.

gen. Mordell

1.12 Lang-Vojta conjecture

Let $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_{K,S}$ be a scheme of finite type such that $X := \mathcal{X}_K$ is a variety of logarithmic general type. Then the set of points $\mathcal{X}(\mathcal{O}_{K,S})$ is not Zariski dense in $\mathcal{X}$.

1. in X
2. compatible
3. gen. Siegel

1.13 Lang's "geometric" conjecture

Let X be a variety of general type over an algebraically closed field k. Then the union of all subvarieties of X which are not of general type is not Zariski dense in X.

will come back

1.14 Lang's "strong" conjecture

Let X be a variety of general type over a number field K. Then for any number field extension $L \supset K$, the set $X(L) \smallsetminus Z(X)$ of rational points on X *outside* $Z(X)$ is finite.

gen. Mordell

1.15 Theorem (Caporaso-Harris-Mazur)

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Lang-Vojta?

Curves with divisors:

$\mathbb{P}^1 \smallsetminus \{p_1, \ldots, p_n\}$:

Siegel

$E \smallsetminus \{p_1, \ldots, p_n\}$:

Surfaces:

MINIMAL
embarrassing

$-\infty$.

Geometry: $X_{\bar{K}}$ ruled.

Arithmetic:

0.

Geometry: $X_{\bar{K}}$ Abelian / finite group,
or K3 / finite group

cases
enriques

Arithmetic:

1.

Geometry: $X_{\bar{K}}$ "elliptic" surface

cases

Arithmetic:

B-S-D
CT-S-S-D

---

2. "general type".

Lang geom.
Miyaoka

1.16 Theorem (Iitaka)

Suppose $\kappa(X) \geq 0$. There is a dominant map $X \dashrightarrow I(X)$, where rational

1. $\dim I(X) = \kappa(X)$
2. X_η a variety, where $\eta \in I(X)$ the generic point.
3. $\kappa(X_\eta) = 0$.

1.17 Conjecture

Suppose $\kappa(X) = -\infty$. Then X is covered by rational curves.

1.18 Conjecture

The Kodaira dimension is constant in a smooth projective family. sing.

1.19 "Additivity"

$X \to Y$
$\eta \in Y$ generic point.
Assume X_η a variety

gen. finite case

1.20 Theorem (Iitaka)

$$\kappa(X) \leq \kappa(X_\eta) + \dim Y.$$

1.21 Conjecture (Iitaka)

$$\kappa(X) \geq \kappa(X_\eta) + \kappa(Y).$$

what's known
Kollár, Kawamata MM

1.22 Log case?