1 Lecture 2: Vojta's conjectures

1.1 Vojta's setup

Notation:

K = a number field

deffers from Soulé

$\mathbb{M}_K$ = the places of K

(abs val or emb?
norm or norm^2?)

$k(v)$ = residue field at $v \in \mathbb{M}_K$

$(\mathbb{M}_K)_{na}$ = nonarchimedean

maxspec

X = variety over K

$\mathcal{X}$ = *model* of X over K.

arithmetic variety
nice properties
$\mathbb{M}_K$ itself is

D = a Cartier divisor on $\mathcal{X}$, endowed with Green functions $g_{D,v}$ on X for every archimedean place $v \in \mathbb{M}_K$:

$g(f) = -\log ||f||^2$
factor 2

$$dd^c g + \delta_D = [\omega]$$

$dd^c = -(2\pi i)^{-1}\partial\bar{\partial}$

Assume D a cartier divisor on $\mathbb{M}_K$, assume $v \in (\mathbb{M}_K)_{na}$,
the multiplicity of D at v is denoted $n_v(D)$.
We denote by $\deg_v(D) = n_v \log \#k(v)$.

If, on the other hand v is archimedean, denote $\deg_v(D) = g_{D,v}$ (this is a real number!)

For $S \subset \mathbb{M}_K$ a subset (not necessarily finite yet), denote

$$\deg_S(D) = \sum_{v \in S} \deg_v(D)$$

and $\deg D = \deg_{\mathbb{M}_K} D$.

Product formula implies: $\deg D$ is invariant under linear equivalences (where the Green function of (f) at an archimedean v is $-\log ||f||_v^{2?}$).

inv of metr. lb

Let $\mathcal{L}$ be a metrized line bundle on $\mathcal{X}$, and let P be an algebraic point on $\mathcal{X}$, defined over E. there is an associated morphism
$\sigma : \mathbb{M}_E \to \mathcal{X}$.
Define

$$h_{\mathcal{L},K(P)} = \frac{1}{[E:K]} \deg \sigma^* \mathcal{L}.$$

If $P \notin \operatorname{Supp} D$, then

$$h_{\mathcal{O}(D),K(P)} = \tfrac{1}{[E:K]} \deg \sigma^* D.$$

Now S will be a finite set of places of K, containing all infinite places. For $E \supset K$ a finite extension, we denote S_E the set of places of E above S.

We break the height into S-part and non-S-part.

$$m_{K,S}(D,P) = \frac{1}{[E:K]} \deg_{S_E} \sigma^* D$$
= "proximity function".

$$N_{K,S}(D,P) = \frac{1}{[E:K]} \deg_{\mathbb{M}_K \smallsetminus S_E} \sigma^* D$$
= "counting function".

A closely related quantity is the "truncated counting function"

$$N^{(1)}_{K,S}(D,P)$$
$$= \frac{1}{[E:K]} \sum_{w \notin S_E} \min(1, n_{D,w}(P)) \log \#k(w)$$

FACT: All these quantities depend on the choice of the model $\mathcal{X}$ of X only up to adding a function which is bounded over $P \in X(\bar{K})$.

Relative discriminant:
if $E \supset K$ be a finite extension. Denote

$$d_K(E) = \tfrac{1}{[E:K]} \log |D_E| - \log |D_K|.$$

It can be identified as

$$d_K(E) = \frac{1}{[E:K]} \deg \omega_{\mathcal{O}_E/\mathcal{O}_K},$$

the latter having zero Green functions

$\mathbb{M}_E \to \mathbb{M}_K$ decl. unram. at ∞.

We can analogously define

$$d_{K,S}(E) = \tfrac{1}{[E:K]} \deg_{\mathbb{M}_E \smallsetminus S_E} \omega_{\operatorname{Spec} \mathcal{O}_E / \operatorname{Spec} \mathcal{O}_K}.$$

If $P \in X$ and $K(P) = E$ we denote $d_K(P) = d_K(E)$.

1.2 Vojta's conjecture

$K =$ a number field;
$S =$ a finite set of places, containing archimedean
$X =$ smooth complete variety over K;
$D \subset X$ a normal crossings divisor;
$\mathcal{A} =$ a big line bundle on X;
$r =$ positive integer; $\quad \epsilon =$ positive real

Then there exists a proper Zariski closed subset Z (depending on all the data) such that

$$h_{\omega_X}(P)+m(D,P) \leq d_K(P)+\epsilon h_{\mathcal{A}}(P)+O(1)$$

nevanlinna

for all $P \subset X(\bar{K}) \smallsetminus Z$ s.t. $[K(P):K] \leq r$.

Equivalently:
$$N(D,P)+d_K(P) \geq$$
$$h_{\omega_{X(D)}}(P)-\epsilon h_{\mathcal{A}}(P)-O(1).$$

1.3 Proposition: Vojta implies Lang-Vojta

1.4 Vojta's abc conjecture

$$N^{(1)}(D,P) + d_K(P) \geq$$
$$h_{\omega_X(D)}(P) - \epsilon h_{\mathcal{A}}(P) - O(1).$$

abc

1.5 Proposition

Vojta's abc implies abc.

of M-Oé

1.6 Theorem

Vojta's conjecture implies Vojta's abc conjecture.

1.7 Corollary

Vojta's conjecture implies abc