1 Lecture 3a: an array of uniformity theorems

recall:

1.1 Theorem (Caporaso-Harris-Mazur)

Assume that Lang's "weak" conjecture holds true. Let K be a number field. Then there is a number $B(K, g)$, such that the number of K-rational points on any curve of genus $g > 1$ over K is bounded by $B(K, g)$.

1. Lang$\Rightarrow$unif. Mordell
2. dependence on K

1.2 Theorem (Pacelli)

Assume that Lang's "weak" conjecture holds true. Then there is a number $B(d, g)$, such that, for any number field K of degree d, the number of K-rational points on any curve of genus $g > 1$ over K is bounded by $B(d, g)$.

1. geometric case
2. how about "strong"

1.3 Theorem (Caporaso-Harris-Mazur)

Assume that Lang's "strong" conjecture holds true. Then there is a number $N(g)$, such that, for any number field extension K, the number of curves of genus $g > 1$ defined over K which have more than $N(g)$ rational points over L is **finite**.

higher dim setup

1.4 Theorem (Abramovich - Voloch)

There is a number $B(d, X \to B)$, such that, for any number field K of degree d, and for any $b \in B(K)$, the number $X_b(K)^{nex}$ is bounded by $B(d, X \to B)$.

1. Lang$\Rightarrow$Unif. Lang
2. integral? stint

1.5 Theorem (Pacelli)

Assume that the Lang-Vojta conjecture holds true. Fix a finite set of primes S in $\mathbb{Z}$, and for any number field K denote by S_K the set of primes of $\mathcal{O}_K$ over S. Then there is a number $B(d, S)$, such that, for any number field K of degree d, the number of stably S_K-integral points on any elliptic curve K is bounded by $B(d, S)$.

S-unit, surfaces

1.6

“Vojta” implies “uniform Vojta”???

2 Lecture 3b: Structure of the proofs

First idea: all curves of a fixed genus fit in a family.

diag Hilbert

Could start if the family is of general type

not prod

Enough if the family **dominates** a variety of general type

still pencil

Second idea: work with fibered products.

diag

Need:

1. arithmetic/geometric argument for showing that using fibered powers is enough.

2. geometric construction of a varity of general type.

Given a family of varieties $X \to B$ and a set of points $\mathcal{P} \subset X(K)$, we say that $\mathcal{P}$ is n-*correlated* in $X \to B$ if $\mathcal{P}^n_B$ is not Zariski dense in X^n_B.

1-correlated
open set
noetherian

2.1 Lemma (CHM)

Suppose $X \to B$ is a family of *curves* and $\mathcal{P} \subset X$ is n-correlated in $X \to B$. Then there is an open Zariski - dense set $U \subset B$ such that the number of points of $\mathcal{P}_b$ on all fibers of $X \to B$ over $b \in U$ is uniformly bounded.

1.give proof
2. higher dim. problem

2.2 Theorem

Let $X \to B$ be a family of varieties of general type over an irredicible base variety B. Then for large enough n, there exixts a positive dimensional variety of general type W_n and a dominant rational map $X_B^n/G \dashrightarrow W$.

1. who
2. moduli
3. Lemma+Th⇒unif

2.3 Theorem (CHM)

Let $g > 1$ be an integer, and fix a subvariety $X_0 \subset M_g$. Let $X_n \subset M_{g,n}$ be the inverse image. Then for large enough n, the variety X_n is of general type.

A family of varieties $X \to B$ is said to be *of maximal variation*, if through a general point $b \in B$ there is no curve $b \in C \subset B$ over which the family is birationally isotrivial. explain

2.4 Theorem

Let $X \to B$ be a family of varieties of general type over an irredicible base variety B. Assume $X \to B$ is of maximal variation. Let $G \subset \mathrm{Aut}(X \to B)$. Then for large enough n, the variety X^n_B/G is a variety of general type. Lem+Theorem$\Rightarrow$ Strong unif

2.5 Exercise

Let X, Y be varieties over K with finite automorphisms, with an isomorphism $X_{\bar{K}} \to Y_{\bar{K}}$. Then for any n there is an isomorphism $X^n/\mathrm{Aut}_K(X) \to Y^n/\mathrm{Aut}_K(Y)$, where the automorphism groups act diagonally on the products.

descent

2.6 Exercise

Consider the projection

$$p_{n,k} : X^n/\mathrm{Aut}_K(X) \to X^k/\mathrm{Aut}_K(X)$$

onto the first k factors. Let $P = (P_1, \ldots, P_k) \in X^k(K)$ be a K-*rational* point, and let $Q \in X^k/\mathrm{Aut}_K(X)(K)$ be its image. Assume P is not fixed by any nontrivial element of $\mathrm{Aut}_K(X)$. Then there is an induced isomorphism

$$X^{n-k} \to p_{n,k}^{-1}\{P\}.$$

Let $\tau : X \to B$ be a projective surjective morphism of reduced schemes of finite type over a field K, with B irreducible. Denote the dimension of the generic fiber of $\tau : X \to B$ by d. Fix a subset $\mathcal{Q} \subset X(K)$, and denote by G_k the Zariski closure of $\mathcal{Q}^k_B$ in the fibered power X^k_B. We say that $\mathcal{Q}$ is *strongly k-correlated* with respect to $\tau : X \to B$, if every irreducible component of G_k which dominates B has relative dimension $< kd$ over B.

strong corr$\Leftarrow$ cor for a related family

Let $X = X_1 \cup \ldots \cup X_c$ be a decomposition into irreducible components. Let $X'_i \to X_i$ be the normalization, and let $X'_i \to B'_i \to B$ be the Stein factorization. There is a dense open set $U_i \subset X_i$ over which $X'_i \to X$ is an isomorphism. Therefore the set $\mathcal{Q}'_i = U_i \cap \mathcal{Q}$ sits naturally in $X'_i(K)$.

2.7 Proposition

Assume that for each X_i of relative dimension d over B, the set $\mathcal{Q}_i'$ is k_i'-correlated with respect to $X_i' \to B_i'$. Then for large k, the set $\mathcal{Q}$ is strongly k-correlated for $X \to B$.

STEP 1. *Let $B' \subset B$ be a nonempty open set, $X' \subset \tau^{-1}B'$ a dense open set, and $\mathcal{Q}' = \mathcal{Q} \cap X'$. Assume $\mathcal{Q}'$ is strongly k-correlated. Then $\mathcal{Q}$ is strongly k-correlated as well.*

STEP 2. *Let $X_1, \ldots, X_{c'}$ be the irreducible components of X of relative dimension d over B. Let $X' = X_1 \cup \ldots \cup X_{c'}$ and let $\mathcal{Q}' = \mathcal{Q} \cap X'$. If $\mathcal{Q}'$ is strongly k-correlated then so is $\mathcal{Q}$.*

STEP 3. *Let* $\mathcal{Q}_i = X_i \cap \mathcal{Q}$. *Assume* $\mathcal{Q}_i$ *is strongly* l*-correlated. Then for* $k = l \cdot c$, $\mathcal{Q}$ *is strongly* k*-correlated.*

STEP 4. *Denote by* c_i *the degree of* $B'_i \to B$. *Assume* $\mathcal{Q}'_i$ *in the proposition is* k'_i*-correlated with respect to* $X'_i \to B'_i$ *. Then for* $k = c_i k'_i$ *we have that* $\mathcal{Q}_i$ *is strongly* k*-correlated with respect to* $X_i \to B$.

Given $X \to B$ a family of varieties of general type, defined over K, consider $\mathcal{P} \subset X(K)$, the set of "nex" rational points. from strong correlation to uniformity

Let $E_n \subset \mathcal{P}_B^n$ be the set of n-tuples of *distinct* points.

Consider the tower of maps

$$\cdots \to E_{n+1} \to E_n \to E_{n-1} \to \cdots \to E_1.$$

Let $E_n^{(m)}$ be the image of E_m in E_n and set $F_n^{(m)} = \overline{E_n^{(m)}}$. We have a descending chain of closed subsets $m > n$ prolongable

$$F_n^{(m)} \supset F_n^{(m+1)} \supset \cdots$$

which must stabilize to a closed set F_n, i.e., there exists m_n such that

$$F_n^{(m_n)} = F_n^{(m_n+1)} = \cdots = F_n.$$

Accordingly, we have a tower of maps

$$\cdots F_{n+1} \overset{\tau_{n+1}}{\to} F_n \to F_{n-1} \overset{\tau_n}{\to} \cdots \to F_1.$$ want $F_n = \emptyset$ $F_0 = .$

Note: $\tau_{n+1} : F_{n+1} \to F_n$ surjective for all n.

Will show: any fiber of $F_{n+1} \to F_n$ cannot have dimension $0, 1, \ldots, \dim X_\eta$ and hence all the F_n are empty.

2.8 Lemma

Suppose no fiber of $\tau_{n+1} : F_{n+1} \to F_n$ has dimension $< d$ for all $n \geq 1$, where $0 \leq d \leq \dim X_\eta$. Assume that the "weak" Lang conjecture holds true. Then no fiber of $\tau_{n+1} : F_{n+1} \to F_n$ has dimension $\leq d$ for all $n \geq 1$.

Step 1. The lemma holds for $d = 0$.

Step 2. It suffices to consider an irreducible component $B_n \subset F_n$ such that $X_n = \tau_{n-1}^{-1} B_n$ has relative dimension d.

Step 3. It suffices to show that $X_n(K)$ is strongly correlated.

Step 4. Proposition $\Rightarrow$ lemma.

"nex" $\Rightarrow$ GT

That shows uniformity in $X \to B, K$.

Suppose $X \to B$ defined over $\mathbb{Q}$.
Let $[K : \mathbb{Q}] = d$

Let $P = (P_1, P_2) \in X^2_B(K)$.
Let $\sigma_1, \ldots, \sigma_d : K \hookrightarrow \bar{\mathbb{Q}}$ distinct.

Then $\{P^{\sigma_1}, \ldots, P^{\sigma_d}\} \in \mathrm{Sym}^d(X^2_B)(\mathbb{Q})$.

1. If P_1 not defined over a subfield of K, then $(P^{\sigma_1}, \dots, P^{\sigma_d})$ not fixed under any nontrivial element of S_d.

2. If so, consider the inverse image Y of $\{P_1^{\sigma_1}, \dots, P_1^{\sigma_d}\} \in \mathrm{Sym}^d(X)$ in $\mathrm{Sym}^d(X_B^2)$.

Then we have an isomorphism
$\prod \left(X^{\sigma_i} \times_{K^{\sigma_i}} \bar{\mathbb{Q}}\right) \to Y \times_{\mathbb{Q}} \bar{\mathbb{Q}}$

Weil restriction

3. If furthermore P_2 is "nex", then $(P_2^{\sigma_1}, \dots, P_2^{\sigma_d})$ is "nex".

4. Apply theorem to the family $\mathrm{Sym}^d(X_B^2) \to \mathrm{Sym}^d(X)$