1 Lecture 4: sheaves and singularities on fibered powers

1.1 Theorem

Let $X \to B$ be a family of pos. dim. varieties of gen. type over an irred. base variety B. Then for large enough n, there exixts a positive dimensional variety of general type W_n and a dominant rational map $X^n_B/G \dashrightarrow W$.

1.2 Theorem (CHM)

Let $g > 1$ be an integer, and fix a subvariety $X_0 \subset \mathbf{M}_g$. Let $X_n \subset \mathbf{M}_{g,n}$ be the inverse image. Then for large enough n, the variety X_n is of general type.

1.3 Theorem

Let $X \to B$ be a family of varieties of general type over an irredicible base variety B. Assume $X \to B$ is of maximal variation. Let $G \subset \mathrm{Aut}(X \to B)$. Then for large enough n, the variety X^n_B/G is of general type.

Claim: 1.3 implies 1.2.

1.4 Proposition

Let $X_0 \subset \overline{\mathbf{M}}_g$ be a closed integral subscheme, and let $X_1 \subset \overline{\mathbf{M}}_{g,1}$ be the reduced scheme underlying its inverse image. Then there exists a projective, normal integral scheme B, and a family $C \to B$ of curves of genus g, satisfying the following properties:

1. The natural moduli morphisms $B \to \overline{\mathbf{M}}_g$ and $C \to \overline{\mathbf{M}}_{g,1}$ are finite, with images X_0 and X_1, respectively.
2. Denote $G = \mathrm{Aut}_{\overline{\mathbf{M}}_g}(C \to B)$. Then the two morphisms $B/G \to \overline{\mathbf{M}}_g$ and $C/G \to \overline{\mathbf{M}}_{g,1}$ are birational onto their images.

Moreover, in this situation consider the moduli map $C^n_B \dashrightarrow \overline{\mathbf{M}}_{g,n}$. There is a diagonal action of G on C^n_B, and $C^n_B/G \dashrightarrow \overline{\mathbf{M}}_{g,n}$ is again birational onto the image.

Proof:

1. $Hilb_g \to \overline{\mathbf{M}}_g$
2. take étale slices
3. take Isom., normalize in Galois closure
4. families glue families do not glue)

Situation:

a. the function field extension $K(W_0) \subset K(B)$ isom finite
is Galois.

Denote the Galois group G_0.

Write η for the generic point of B.

b. the geometric points of the finite group $H = \mathrm{Aut}_\eta C_\eta$ are all rational over $K(B)$

c. for any $g \in G_0$ there is a B-isomorphism $(C \to B, \Theta) \xrightarrow{\sim} g^*(C \to B, \Theta)$.

Now consider the set

$$G = \left\{ (g,h) \mid \begin{array}{l} g \in G_0, \\ h : (C \to B) \xrightarrow{\sim} g^*(C \to B) \end{array} \right\}.$$

Note that for any $(g,h) \in G$ and $g' \in G_0$ we can define

$$h^{\{g'\}} : g'^*(C \to B) \xrightarrow{\sim} g'^* g^*(C \to B)\}.$$

We can now define

$$(g,h)(g',h') = (gg', h^{\{g'\}}h').$$

Exercise: check that this is a group!

It is now immediate to verify condition (2)

This "works in higher dimension"

no moduli space...

Need to show that something is of general type so need compactification, desingularization, group actions, differential forms.

1.5 Theorem (Kollár)

Let $f : X \to B$ be a dominant morphism of smooth complex projective varieties. Assume the generic fiber is of general type and the morphism is of maximal variation. Then $\omega_f := \omega_X \otimes f^*\omega_B^{-1}$ is big on X.

Arakelov, Viehweg, Kawamata

1.6 Lemma (CHM)

$f : X \to B$ flat morphism of proj. varieties.
$\mathcal{A}$ big invertible sheaf on X,
$\mathcal{L}$ any invertible sheaf on B.
$p_i : X_B^k \to X$ projections, $p_0 : X_B^k \to B$.

Then for large k,

$$p_0^*\mathcal{L} \otimes (\bigotimes_{i=1}^{k} p_i^*\mathcal{A})$$

big on X_B^k.

proof

Immediate deduction of FPT

is false
1. invariants singular
2. powers singular

2 Rational Gorenstein Singularities

Q: is there a nice class of singularities, s.t. canonical
1. ω_X makes sense, and
2. ω^n pulls back to desingularization ??

2.1 Dualizing complex and Gorenstein singularities

$\omega^\bullet_{\mathbb{P}^N}$ is represented by the complex of sheaves with all stages $= 0$ apart from $\omega^{(-N)} = \omega_{\mathbb{P}^N}$

If $X \subset \mathbb{P}^N$ take $\mathbf{R}\mathcal{H}om(\mathcal{O}_X, \omega^\bullet_{\mathbb{P}^N})$

well defined locally as.
exercise divisor

X is Cohen-Macaulay $\Leftrightarrow$

$$\omega_X^{(i)} = 0 \text{ for } i \neq -\dim(X).$$

Denote $\omega_X^{(-\dim(X))}$ by ω_X.

If X normal, denote $\iota : X_{\mathrm{sm}} \to X$, then $\omega_X = \iota_* \omega_{X_{\mathrm{sm}}}$

X **Gorenstein** if Cohen-Macaulay and ω_X invertible.

examples

2.2 Rational singularities

Let X be a variety and $f : X' \to X$ a desingularization. We say that X has **rational singularities** if $\mathbf{R}f_*\mathcal{O}_{X'} = \mathcal{O}_X$.

expand smooth, toric,...

By duality, this is the same as saying that $\mathbf{R}f_*\omega^\bullet_{X'} = \omega^\bullet_X$.

In particular: X is normal, Cohen-Macaulay.

Using G-R vanishing, we have:

∃ morphism anyway

2.3 Proposition

Assume X Gorenstein, $X' \to X$ a desingularization, and $f_*\omega_{X'} = \omega_X$. Then X has rational singularities.

Note that it suffices if X' has rational Gorenstein singularities.

2.4 Observation

If X has rational Gorenstein singularities, and $X' \to X$ a desingularization, then $f_*\omega^n_{X'} = \omega^n_X$ for all $n \geq 0$.

2.5 Lemma (CHM)

V with rational Gorenstein singularities,
$G \subset \operatorname{Aut} V$ a finite group,
$\Sigma \subset V$ fixed points of nontrivial elements of G,
$q : V \to W = V/G$ the quotient morphism,
$r : W' \to W$ a desingularization.

Assume $s \in H^0(V, \omega_V^k)^G$
vanishes along Σ to order at least $k \cdot e(G)$.

Then there is an element $t \in H^0(W, \omega_W)$ such that $q^* r_* t = s$.

étale case
normalization
Abhyankar

2.6 Mildness

A flat morphism $X \to B$ is said to be **mild** if whenever B' has rational-Gorenstein singularities and $B' \to B$ dominant, then $X \times_B B'$ has rational-Gorenstein singularities.

2.7 Observation

If $X \to B$ mild then $X^n_B \to B$ is mild as well.

2.8 Proposition

Assume $X \to B$ is a Gorenstein flat morphism, both B and B' have rational-Gorenstein singularities and $B' \to B$ dominant. Assume $X \times_B B'$ has rational-Gorenstein singularities. Then X has rational-Gorenstein singularities.

2.9 Theorem (Hochster)

a toric singularity is rational.

2.10 Proposition

a family $X \to B$ of nodal curves is mild.

2.11 Proposition

Let $X \to B$ be a flat generically smooth projective morphism. Then there is an alteration $B' \to B$ and a modification $Y \to X \times_B B'$ such that $Y \to B$ is mild.

Galois