Definition A model $\mathbb{U}$ of a universal theory T of size $\aleph_1 = 2^{\aleph_0}$ is a *universal domain* for T if every countable $M \models T$ embeds into $\mathbb{U}$, uniquely up to an automorphism of $\mathbb{U}$.

Existence Assume (*) T has a unique minimal model. T has a universal domain iff the models of T admit *amalgamation*, i.e. given two maps $f_i : A \to B_i$ there should exist $C \in \mathcal{C}$ with and $g_i : B_i \to C$ with $g_1 f_1 = g_2 f_2$.

Theorem [Model theoretic compactness] Let $\{X_i : i = 1, 2, \ldots\}$ be a fam ily of definable subsets of $\mathbb{U}^{\ltimes}$. If every finite subfamily has non-empty intersection, then $\cap_i X_i \neq \emptyset$

Model completions Assume $\mathbb{U}$ is a universal domain for T. Then if T has a model completion, it is $Th(\mathbb{U})$. This occurs iff the set of consequences of $\phi(x, y)$ in the variables x has a finite basis.

Definition

- A definable set X in the universal domain has Morley rank ≥ 0 iff it is nonempty.
- X has Morley rank $\geq n+1$ if for each m, it contains m disjoint definable subsets of Morley rank $\geq n$.
- If X has Morley rank $\geq n$, but not $\geq n+1$, we say it has Morley rank n. In this case there exists a maximal m, called the *multiplicity*, such that X is the disjoint union of m definable sets of Morley rank n.

Definition An infinite definable subset U of $\mathbb{U}$ is called *minimal* iff every definable subset of U is finite or cofinite.

Examples

1. Sorts = complex algebraic varieties

Basic relations = Zariski closed subsets

Then minimal definable sets = curves. Morley dimension = usual dimension.

Note the entire structure is interpretable inside one minimal set, = the field $\mathbb{C}$.

2. Let G be a finite group, and consider the complex representations of G. The relations are the graph of addition, of the complex scalar multipliers, and of the operations of "multiplication" by any $g \in G$. There exists a universal domain for such representations (a direct sum of infinitely many copies of the regular representation.) The Morley dimension equals the order of the group. The minimal definable sets (up to an equivalence relation to be defined below) correspond to the irreducible representations.

3. Complex tori. Basic closed sets = closed analytic subvarieties.

Minimal sets: curves on Abelian sub-quotients; irreducible non- algebraic tori.

Proposition Within a minimal structure D, algebraic closure induces a matroid, or combinatorial pregeometry. In other words, the exchange law holds: if $a \in acl(B \cup \{c\}) \setminus acl(B)$, then $c \in acl(B \cup a)$. Thus one can define a "transcendence degree" as follows: the Morley transcendence degree of a set X is the cardinality of any maximal subset Y of X, whose elements are algebraically independent.

Definable groups: G a definable set; $\{(x,y,z) : xy = z\}$ a definable subset of G^3.

Lemma (Zilber) Let X be an indecomposable definable subset of a finite - dimensional definable group A, $0 \in X$. Then X generates a subgro up B of A, in finitely many steps; for some n, $B = \{\sum_{i=0}^{n}(-1)^i x_i : x_i \in X\}$.

If $\dim(X) = 1$,$\dim(B) = d$, a generic element of B can be expressed as a sum of d elements of X in k ways, $1 \leq k \leq \infty$.

Proof This generalizes the "indecomposability" theorem of algebraic groups. The proof is the same, once one has learned to replace the set - theoretic stabilizer:

$$Stab_{set}(X) = \{a \in A : a + X = X\}$$

By the more robust dimension-theoretic stabilizer:

$$Stab(X) = \{a \in A : \dim((a + X \setminus X) \cup (X \setminus (a + X))) < \dim(X)\}$$

$\square$

Theorem Assume T is a theory in a countable language, with precisely one isomorphism class of models of cardinality λ, $\lambda > \aleph_0$.

1. T has precisely one isomorphism class of models of any cardinality $\lambda' > \aleph_0$. (Morley ca. 1960)
2. T has either one or $\aleph_0$ models of cardinality $\aleph_0$. (Baldwin-Lachlan 1973)
3. In the first case, T has a finite subset T_0 such that T and T_0 have the same infinite models, and [Zilber 1980] T_0 has infinitely many finite models.

Ideas of proof

- Finite Morley dimension
- Strongly minimal sets
- Coordinatization: internality vs. orthogonality
- Local modularity and classification of strongly minimal sets

Definition Let D_1,D_2 be two minimal definable sets within a struct ure M. D_1, D_2 are *equivalent* if there exists a minimal D and definable, finite-to-one maps $f_1 : D \to D_1$, $f_2 : D \to D_2$, with cofinite ranges.

Lemma[Orthogonality] Let $D_1, \ldots, D_k$ be pairwise inequivalent minimal definable sets in a universal domain $\mathbb{U}$. Then:

- Every definable $R \subset {D_1}^{m_1} \times \ldots \times {D_k}^{m_k}$ is a finite union of "rectangular" sets of the form $R_1 \times \ldots \times R_k$, R_i a definable subset of ${D_i}^{m_i}$.
- If the D_i are defined over $B = acl(B)$,

$$\mathrm{Aut}(D_1 \cup \ldots \cup D_k/B) = \Pi_i \mathrm{Aut}(D_i/B)$$

Theorem $\tilde{T}$ has a unique model of power λ if and only if:

1. Finite Morley dimension
2. Any two minimal sets are equivalent (non-orthogonal).

Internality

Definition G is P*-internal* if there exists a definable isomorphism in $\mathbb{U}$ between G and a group interpretable in P.

Theorem Let G be a definable group, P a definable set.

1. There exists a unique smallest definable normal subgroup K of G with G/K P-internal.
2. Any definable $R \subset G \times P^m$ is a finite Boolean combination of rectangles $Y \times P^m$, $\dim(Y) < \dim(G)$, and of pullbacks of relations $R' \subset (G/K) \times P^m$.
3. In particular, if every element of G is algebraic over some elements of P, then K is finite.

Local modularity

Theorem The following conditions on a structure of finite Morley rank are equivalent.

1. The lattice of algebraically closed subsets of $\mathbb{U}$ satisfies the modular law. ($(X+Y) \cap Z = X + (Y \cap Z)$ when $X \subset Z$.)
2. Any two algebraically closed subsets are independent over their intersection. (For this, all interpretable sorts must be taken into account.)
3. There is no psuedo-plane $(P, Q; R)$ with P an orbit of $\mathrm{Aut}(\mathbb{U}/\mathbb{B})$ for some finite or countable B, and Q,R B-definable.
4. On a definable set of Morley dimension n, every l-dimensional family of distinct, irreducible, k - dimensional subsets, satisfies: $k + l \leq n$
5. For any definable set Y of Morley dimension 2, there exists a bound $b = b(Y)$ such that there is no $>$ b-dimensional family of distinct irreducible minimal definable subsets of Y.

Example A vector space V over a field or division ring F. Basic definable sets = solution sets to F-linear equations $\sum_{i=1}^{n} \alpha_i x_i = 0$.

Theorem [Zilber, Cherlin-Harrington-Lachlan,Buechler Local-Global principle] A structure of finite Morley rank is locally modular iff every minimal structure interpretable therein is locally modular.

Theorem If D is minimal, locally modular and not disintegrated, then D is equivalent to a minimal D' admitting an Abelian group structure. (In fact there exists an explicitly describable representative of the non-orthogonality equivalence class.)

Theorem [H.-Pillay] Let G be an Abelian group interpretable in a locally modular stable theory, and let X be a definable subset. Then X is a finite Boolean combination of cosets of definable subgroups.

Classification of strongly minimal sets: Zilber's Trichotomy

Three types of minimal structures D.

(1) D is called *disintegrated* if the corresponding combinatorial geometry is trivial: $\mathrm{acl}(X \cup Y) = \mathrm{acl}(X) \cup \mathrm{acl}(Y)$ for $X, Y \subset D$. Equivalently, every *family* of definable subsets of D^2 is a combination of degenerate ones, in one of the following senses: it is constant (consists of a single definable subset of D^2); it is a family of points; or it is one of the familes $(D \times \{b\} : b \in D)$ or $(\{a\} \times D : a \in D)$. There is no *moving* geometry of interest on D^n, for such a D.

(2) D non-disintegrated, but locally modular.

(3) An algebraically closed field, or a curve over such a field; the basic closed sets are the Zariski closed ones.

Zilber's Conjecture for a strongly minimal D:

D is equivalent to one of one of the three types above.

- Strongly minimal sets failing ZC can be constructed.

- True in locally finite context (Zilber), or under Galois-theoretic assumptions. In this case (1) or (2) hold.
- True with additional assumption of topological nature that "holds in practice" so far.
- Analog for real algebraic geometry (O-minimal structures) true (Peterzil-Starchenko).

Definition A *Zariski geometry* is a minimal structure D, together with a distinguished subcollection $\mathcal{C} \subset \mathcal{D}$ (called "closed sets"), and satisfying:

1. C contains the diagonals, and is closed under intersections, finite unions, and inverse images of projections.
2. $\mathcal{D}$ is generated by $\mathcal{C}$ as a Boolean algebra (on each D^n).
3. Every closed set is the union of finitely many closed, irreducible sets.
4. If X is a proper subset of Y, both closed subsets of D^n, and Y is irreducible, then $\dim(X) < \dim(Y)$
5. If X is a closed, irreducible subset of D^n, $\dim(X) = m$, and Δ is a diagonal $x_i = x_j$, then $X \cap \Delta$ is the union of closed irreducible sets of dimension at least $m - 1$.

Theorem Zilber's conjecture holds for minimal structures admitting a Zariski geometry.

Remark We will only use part of this theorem: a Zariski geometry is either locally modular, or interprets a field of Morley dimension 1.

Lemma

- A minimal subset of a group A cannot be disintegrated.
- Suppose A is generated by finitely many minimal subsets X_i, with each X_i equivalent to a minimal set k. Then there exists a k-internal group A' and a definable map $A \to A'$ with finite kernel.

Proof By Zilber's indecomposability lemma, a minimal subset X of A may be taken to generate a definable subgroup B. Let (b_1, b_2) be a generic element of B^2, $b_3 = b_1 + b_2$. By the lemma, each b_i has the same algebraic closure as a d-tuple from X. Putting together the three d-tuples contradicts the triviality of algebraic dependence on X.

Similarly, if A is generated by the minimal sets X_i, then any point of A is algebraic over some points of k. $\square$

Definition The *socle* of A is the sum of all connected definable subgroups, generated by minimal subsets of A.

By the orthogonality lemma, the socle is a direct sum of subgroups, each generated by a finite union of minimal sets in the same non-orthogonality class. The structure of the socle reduces to the structure of groups interpretable over various minimal sets.

Call a definable group B *rigid* if every connected definable subgroup of B is defined over $\mathrm{acl}(B)$. This sometimes follows by consideration of torsion points; it is automatically true for locally modular groups.

Proposition[Reduction to the socle] Let G be an Abelian group of finite Morley dimension, X an definable subset of G. Assume :
i) The socle A of G is rigid.
ii) $Stab(X)$ is finite.
Then X is contained in a the union of finitely many cosets of A, and a definable set of smaller dimension.