Alterations and resolution of singularities
Lecture 1a: overview

1. Objects

1.1. **Definition.** A *variety* defined over k is a geometrically integral scheme of finite type over k. If $k \subset k_1$ we write X_{k_1} for $X \times_{\mathrm{Spec}\ k} \mathrm{Spec}\ k_1$.

1.2. **Definition.** Let X be a variety. A subscheme $Z \subset X$ is called a *strict normal crossings divisor* if for each point $x \in Z$, there is a regular system of parameters $y_1, \ldots, y_k$ for x in X (in particular the point $x \in X$ is supposed to be a regular point on X), such that Z is given on a Zariski neighborhood of x by the equation $y_1 \cdots y_l = 0$.

2. Operations

2.1. **Definition.** A morphism of varieties $Y \to X$ is called a *modification* if it is proper and birational.

2.2. **Definition (de Jong).** A morphism of varieties $Y \to X$ is called an *alteration* if it is proper, surjective and generically finite. This notion of alteration will also be used for integral schemes.

2.3. **Definition.** Let X be a variety. A *resolution of singularities* in the weak sense is a modification $Y \to X$ such that Y is nonsingular.

2.4. **Definition.** Let X be a variety. A *resolution of singularities* in the strong sense is a modification $Y \to X$, which is an isomorphism over the nonsingular locus X_{reg}, such that Y is nonsingular.

3. Results

3.1. **Theorem (Hironaka).** Suppose X is a variety over a fied k of characteristic 0. There is a sequence of blowings up $X_n \to \cdots \to X_0 = X$, each along a regular variety lying over $\mathrm{Sing} X$, such that X_n is regular. Thus $X_n \to X$ is a resolution of singularities in the strong sense.

3.2. **Theorem.** Let X be a variety over an algebraically closed field. There is a separable alteration $Y \to X$ such that Y is quasi projective and regular.

3.3. **Corollary.** Let X/k be a variety. There is a finite extension $k \subset k_1$ and a separable alteration $Y \to X_{k_1}$ such that Y is quasi projective and regular.

3.4. **Theorem (de Jong).** Let X be a variety over an algebraically closed field, $Z \subset X$ a proper closed subset. There is a separable alteration $f : Y \to X$, and an open immersion $j : Y \subset \bar{Y}$, such that $\bar{Y}$ is projective and regular, and the subset $j(f^{-1}Z) \cup \left(\bar{Y} \setminus Y\right)$ is a strict normal crossings divisor.

3.5. **Theorem of semistable reduction up to alteration (de Jong).** Let R be a discrete valuation ring, with fraction field K and residue field k. Let $X \to \operatorname{Spec} R$ be an integral scheme of finite type such that X_K is a variety. There exists a finite extension $R \subset R_1$, where R_1 is a discrete valuation ring with residue field k_1, and an alteration $Y \to X_{R_1}$, such that Y is nonsingular, the special fiber Y_{k_1} is a reduced, strict normal crossings divisor.

3.6. **Theorem of equivariant resolution up to alteration (de Jong).** Let X be a variety over an algebraically closed field, $Z \subset X$ a proper closed subset, and a finite group $G \subset \operatorname{Aut}(Z \subset X)$. There is an alteration $f : Y \to X$, and a finite subgroup $G_1 \subset \operatorname{Aut} Y$, satisfying:

1. there is a surjection $G_1 \to G$ such that f is G_1 equivariant, and the field extension $K(X)^G \subset K(Y)^{G_1}$ is purely inseparable;
2. Y is quasi projective and nonsingular; and
3. $f^{-1}Z$ is a G-strict normal crossings divisor.

3.7. **Corollary.** Let X be a variety over an algebraically closed field. There is a purely inseparable alteration $Y \to X$ such that Y is a quotient of a nonsingular variety by the action of a finite group.

3.8. **Weak Resolution Theorem.** Let X be a variety over an algebraically closed field of characteristic 0. Then there is a projective resolution of singularities (weak sense) $Y \to X$.

3.9. **Question.** Can we improve the methods and obtain a weak resolution of singularities in all characteristics? At least weak resolution up to purely inseparable alterations?