Alterations and resolution of singularities
Lecture 1b: first reduction steps

Our goal in the next few lectures is:

0.1. **Theorem (de Jong).** Let X be a variety over an algebraically closed field, $Z \subset X$ a proper closed subset. There is an alteration $f : Y \to X$, and an open immersion $j : Y \subset \bar{Y}$, such that $\bar{Y}$ is projective and regular, and the subset $j(f^{-1}Z) \cup \left(\bar{Y} \setminus Y\right)$ is a strict normal crossings divisor.

This will be proven in a number of steps.

The proof is by induction on dimension. The case of curves is well known (however, the proof also relies on the case of surfaces).

1. ELEMENTARY REDUCTION STEPS

1.1. **Situation.** The field k is algebraically closed. We have a variety X/k, $Z \subset X$ a subvariety.

1.2. **Remark.** We may always replace X by a modification, and Z by its inverse image.

1.3. **Making Z into a divisor.** Reacall that the blowup of X along Z is a projective morphism, such that the inverse image of Z is a Cartier divisor.

1.4. **Remark.** In this situation, we may always enlarge Z into a bigger divisor Z' - if we prove the theorem for Z' it works for Z.

1.5. **Chow's Lemma.** Given a variety X, there is a modification $Y \to X$ such that Y is quasi-projective.

1.6. **The idea.** X is covered by *finitely many* affine patches, each having a projective completion. Take the closure of the graph of the isomorphisms between these opens...

1.7. We replace X by such a Y. We replace Z by its inverse image.

now let $X \subset \overline{X}$ be a projective completion

1.8. We replace X by such $\overline{X}$. We replace Z by $Z \cup \left(\overline{X} \setminus X\right)$.

Thus we may assume X projective.

1.9. Moreover, we may replace X by its normalization.

2. Projection

2.1. **Plan.** Let $\dim(X) = d$. After blowing up X, We will produce a projection $X \to \mathbb{P}^{d-1}$, which is a family of curves with some nice properties. We will use the theory of moduli of curves to deal with the fibers of the morphism. Then we will use induction on dimension to deal with the base. After we put these together, a few elementary blowups will bring us to the conclusion of the theorem.

2.2. **General projections.** Let $Y \subset \mathbb{P}^N$ be a projective variety over an algebraically closed field (separably closed would suffice). For any closed point $p \in \mathbb{P}^N \setminus Y$ we have a projection $pr_p : Y \to \mathbb{P}^{N-1}$.

2.3. **Claim.** Suppose $\dim Y < N-1$. Then there is a nonempty open set $U \subset \mathbb{P}^N$, such that if $p \in U$ then pr_p sends Y birationally to its image.

2.4. **Proof.** Let $q \in Y$ be a regular point. The cone $C_{Y,q}$ over Y with vertex q has dimension $\dim Y + 1 < N$. The projective tangent space $\mathbb{T}_{Y,q}$ at q has dimension $< N$ too. Therefore if $p \in \mathbb{P}^N \setminus (\mathbb{T}_{Y,q} \cup C_{Y,q}$ then the line through p and q meets Y transversally, at q only. This property holds as well for the line through p and q', for any $q' \in Y$ in a neighborhood of q. Hence the claim.

2.5. **Claim.** Suppose $\dim Y = N-1$. Then there is a nonempty open set $U \subset \mathbb{P}^N$, such that if $p \in U$ then pr_p maps Y generically étale to $\mathbb{P}^{N-1}$.

2.6. **Proof.** Same as before, without the cone.

We now go back to our X and Z.

2.7. **Lemma.** There exists a modification $\phi : X' \to X$ and a morphism $f : X \to \mathbb{P}^{d-1}$ such that

1. There exists a finite set of nonsingular closed points $S \subset X_{ns}$ disjoint from Z, such that X' is the blowup of X at the points of S.
2. f is equidimensional of relative dimension 1
3. The smooth locus of f is dense in all fibers
4. Let $Z' = \phi^{-1}Z$. Then $f|_{Z'}$ is finite and generically étale
5. At least one fiber of f is smooth.

2.8. **proof.** First project $\pi : X \to \mathbb{P}^d$ using the above.

Let $B \subset \mathbb{P}^d$ be the locus over which π is not étale.

If we choose a general $p \in \mathbb{P}^d$, then $pr_p : \pi(Z) \to \mathbb{P}^{d-1}$ is generically étale - simply use the above for all components of $\pi(Z)$.

We choose such a p away from B. By the local description of blowing up, we can identify the variety

$$X' = \{(x, \ell) \in X \times \mathbb{P}^{d-1} | \pi(x) \in \ell\}$$

with the blowing up of X at the points in $\pi^{-1}(p)$.

We define $f : X' \to \mathbb{P}^{d-1}$ to be the second projection.

What are the fibers? The fiber over a point ℓ is the scheme theoretic inverse image $\pi^{-1}(L)$ where L is the line corresponding to ℓ.

It follows immediately that f is equidimensional: all fibers have dimension at most 1, and are defined by $d-1$ equations (the equations of L).

Since no line through p is contained in B, every fiber has a dense smooth locus.

The last assertion follows by Bertini's theorem, since the fibers are obtained by intersecting X with linear subspaces.

2.9. **Claim.** The morphism f has connected fibers.

2.10. **Proof.** Since the smooth locus is dense in every fiber, the Stein factorization is étale. Since projective space has no nontrivial finite étale covers, the Stein factorization is trivial.