Alterations and resolution of singularities
Lecture 2: towards straightenning up the fibers

0.1. **The situation.** We now have $Z \subset X \xrightarrow{f} P$ (where so far $P = \mathbb{P}^{d-1}$), such that

1. X is normal projective, Z the support of a Cartier divisor
2. f is equidimensional of relative dimension 1
3. The smooth locus of f is dense in all fibers
4. $f|_Z$ is finite and generically étale
5. At least one fiber of f is smooth
6. The fibers are connected.

0.2. **The idea.** It is easy to make an alteration of P, and replace X and Z by their pullbacks, such that Z becomes the union of sections of $X \to P$.

Now we can think of the generic fiber of $X \to P$ as a smooth curve with a number of points marked on it. Say the genus of this curve is g, and the number of points is n. By the defining property of moduli spaces, we get a *rational* map $P \dashrightarrow \overline{M}_{g,n}$.

Suppose for a moment that $\overline{M}_{g,n}$ were a *fine* moduli space. This means that there would be a family of curves $C \to \overline{M}_{g,n}$ of genus g, and n sections $s_i : \overline{M}_{g,n} \to C$, such that any family $Y \to B$ of stable n-pointed curves of genus g is obtained by pulling back along some morphism $B \to \overline{M}_{g,n}$.

Well then, in our case, the generic fiber of $X \to P$ would be the pullback of U along the map

$P \dashrightarrow \overline{M}_{g,n}$. If $P_1 \to P$ resolves indeterminacies of this rational map, then we get a diagram

$$\begin{array}{ccc} X_1 & \dashrightarrow & X \\ \downarrow & & \downarrow \\ P_1 & \to & P \end{array}$$

The new morphism $f_1 : X_1 \to P_1$ is much nicer than f, since at least the fibers are as nice as one can expect: they are nodal curves. Moreover, Z was made much nicer: it is replaced by n sections which are mutually disjoint, and pass through the smooth locus of f_1.

If we can resolve P_1 (say using induction on dimension), then it is easy to resolve X_1 as well.

There are problems though:

First, $\overline{M}_{g,n}$ is not a fine moduli space. This will be one reason for using an alteration in place of the modification $P_1 \to P$.

Second, if we want to repeat this inductively, we cannot allow a rational map $X_1 \dashrightarrow X$ - since we cannot pull back nicely along rational map. So we must find a way to make sure that $X_1 \dashrightarrow X$ is actually a regular map.

We will now make steps to deal with the second problem.

0.3. **Auxiliary divisor.** De Jong's idea is, that if a component of a fiber of $X \to P$ has at least 3 marked points on it, it must survive after stable reduction.

To that end, we have the following lemma (copied from de Jong):

0.4. **Lemma.** Let $X \to P$ be as above. There exists a divisor $H \subset X$ such that

1. $f|_H : H \to P$ is finite and generically étale, and
2. for any irreducible component C of a geometric fiber of f, we have
$$\# sm(X/P) \cap C \cap H \geq 3.$$
Here we count the points without multiplicities.

0.5. **Proof.** Let $n \geq 1$ be an integer. Given a very ample line bundle $\mathcal{L}$ on X, consider the embedding

$$i : X \hookrightarrow \mathbb{P} = \mathbb{P}(\Gamma(X, \mathcal{L}^{\otimes n}))$$

associated to $\mathcal{L}^{\otimes n}$.

0.6. **Claim.** Given any irreducible curve $C \subset X$, the image $i(C) \subset \mathbb{P}$ is not contained in any linear subspace of dimension $n-1$.

0.7. **Proof of claim.** since $\mathcal{L}$ is very ample, the image of $\Gamma(X, \mathcal{L}) \to \Gamma(C, \mathcal{L}_{|C})$ containes a base point free pencil $V \subset \Gamma(C, \mathcal{L}_{|C})$. The map $\mathrm{Sym}^m V \to \Gamma(C, \mathcal{L}_{|C}^{\otimes n})$ has rank $\geq n+1$, therefore $\Gamma(X, \mathcal{L}^{\otimes n}) \to \Gamma(C, \mathcal{L}_{|C}^{\otimes n})$ has rank $\geq n+1$, which is what we claimed.

0.8. Now we consider the collection of "bad" divisors and show that there are "good" ones left. So consider

$$T = \{(H, y) \in \mathbb{P}^\vee \times P | \dim f^{-1} y \cap H = 1\} \subset \mathbb{P}^\vee \times P.$$

It is clear that T is a Zariski closed subset. We can describe the fibers of $pr_2 : T \to P$ using irreducible components of the fibers:

$$pr_2^{-1}(y) = \cup_{C \subset f^{-1} y} \{H | i(C) \subset H\}.$$

But by the fact that $i(C)$ is not contained in any linear subspace of dimension $n - 1$, we have

$$\operatorname{codim}(pr_2^{-1}(y), \mathbb{P}^\vee) \geq n.$$

Therefore $\dim T \leq \dim P + \dim \mathbb{P}^\vee - n$.

Thus if n is large enough, $pr_1(T) \subset \mathbb{P}^\vee$ is of large codimension (at least $n - \dim P$). In particular $pr_1(T) \neq \mathbb{P}^\vee$.

We now fix such large n. Thus there are plenty of H which map finitely to P.

For a fixed closed point $y \in P(k)$ consider the set

$$U(y) = \left\{ H \in \mathbb{P}^\vee(k) \;\middle|\; \begin{array}{l} H \notin pr_1(T) \\ H \cap f^{-1}y \subset sm(X/P) \\ H \cap f^{-1}y \text{ is reduced} \end{array} \right\}.$$

This is clearly a nonempty open set of $\mathbb{P}^\vee$. Moreover, if $H \in U(y)$ then $H \in U(y')$ for all y' in a neighborhood of y.

If moreover $n \geq 3$, then we have that $\# H \cap f^{-1}y \geq 3$. so we are done for all points in a neighborhood V of y.

We deal with points in $P \setminus V$ in the same way. Using Noetherian induction we are done.