Alterations and resolution of singularities
Lecture 3: Toric varieties and toroidal embeddings

3.1. **Definition.** A variety V together with an open dense embedding $T \subset V$ is called a *toric variety* if V is normal, T is a torus (geometrically isomorphic to $\mathbb{G}_m^k$), and the action of T on itself by translations extends to an action on V.

3.2. **Definition.** A variety X together with an open embedding $U \subset X$ is called a *toroidal embedding* if any point $x \in X$ has an étale neighborhood X' such that $X' \times_X U \subset U$ is isomorphic to an étale neighborhood of a point on a toric variety.

Equivalently, given a point $x \in X$, there is a toric variety V, a point $v \in V$, and an isomprphism $\widehat{\mathcal{O}_{X,x}} \to \widehat{\mathcal{O}_{V,v}}$, identifying the ideal of $X \setminus U$ with that of $V \setminus T$.

A toroidal embedding is said to be *strict* (or a *toroidal embedding without self intersections*) if every irreducible component of $X \setminus U$ is normal. For instance, if X is a nonsingular variety and D is a strict normal crossings divisor, and $U = X \setminus D$, then $U \subset X$ is a strict toroidal embedding.

$k = \overline{k}$

3.3. **Some notation.** .

$M = Hom(T, \mathbb{G}_m)$;

$M_{\mathbb{R}} = M \otimes \mathbb{R}$

$N = Hom(M, \mathbb{Z}) = Hom(\mathbb{G}_m, T)$;

$N_{\mathbb{R}} = N \otimes \mathbb{R}$

Note: $M =$ Monomials. $m \in M \mapsto x^m$.

Recall: 1-1 correspondence

$$\{\text{affine toric varieties } T \subset V\}$$
$$\leftrightarrow$$
$$\{\text{str. conv. rat'l polyhedral cones } \sigma \subset N\}$$

$V_\sigma = \operatorname{Spec} k[\sigma^\vee \cap M]$

$\sigma =$ cone spanned by the 1-parameter subgroups $\phi : \mathbb{G}_m \to T$ such that the limit $\lim_{z\to 0} \phi(z)$ exists in V.

3.4. **Divisorial characterization of σ.** Any monomial $m \in M$ defines a Cartier divisor $Div(x^m)$ supported on $V \setminus T$. If $\sigma^\vee$ contains a line through the origin, then for any m on this line the divisor is easily seen to be trivial (both m and $-m$ give regular functions).

Notation:

$\sigma^\perp = \{m \in M_{\mathbb{R}} | \langle m, \sigma \rangle = 0\}$

$M^\sigma =$ Cartier div. supp. on $V \setminus T$.

One can easily see that $M^\sigma = M/\sigma^\perp \cap M$.

$N_\sigma = span(\sigma)$. Clearly $N_\sigma = Hom(M^\sigma, \mathbb{Z})$.

Let $M^\sigma_+ \subset M^\sigma$: the *effective* Cartier divisors.

Easy to see: $M^\sigma_+ = \sigma^\vee \cap M/\sigma^\perp \cap M$.

Then it is not hard to see that $\sigma = (M^\sigma_+)^\vee_{\mathbb{R}}$, the dual cone of the cone spanned by M^σ_+.

In short: σ is the dual cone to the cone of effective Cartier divisors supported on $V \setminus T$.

3.5. **The toroidal picture.** Let $U \subset X$ strict toroidal. $X \setminus U = \cup D_i$, where D_i normal.

$\cap_{i \in I} D_i = \cup X_\alpha$; X_α = STRATA

$Star(X_\alpha) = \cup_{X_\alpha \subset \overline{X_\beta}} X_\beta$.

Note: X_α = unique closed stratum in $Star(X_\alpha)$

M^α = Cartier div. supp. on $Star(X_\alpha) \setminus U$

M^α_+ = Effective Cartier divisors

$N_\alpha = Hom(M_\alpha, \mathbb{Z})$

$\sigma_\alpha = (M^\alpha_+)^\vee_\mathbb{R}$

Thus: to each stratum we associate a cone.

Remark: σ_α has a description analogous to the one using 1-parameter subgroups, in terms of valuations. Let $RS(X)$ be the discrete valuations on X. Let v be a valuation centered in $Star(X_\alpha)$. Let f_j be rational function defining generators of M_α on a small affine open. Then $v(f_j)$ is a vector in σ_α, and in fact σ_α can be described as equivalence classes of discrete valuations centered in $Star(X_\alpha)$.

3.6. **Birational affine morphisms.** Recall: if $\tau \subset \sigma$ two s.c.r.p. cones, then $\tau^{\vee} \supset \sigma^{\vee}$ gives rise to a morphism $V_\tau \to V_\sigma$, which is birational and affine.

Note that $V_\tau \to V_\sigma$ can be described in the following invariant manner:

$$V_\tau = \mathrm{Spec}\ {}_{V_\sigma} \sum_{E \in M_+^\tau} \mathcal{O}_{V_\sigma}(-E),$$

where the sum is taken inside the field of rational functions of V_σ.

This clearly works over $Star(X_\alpha)$ in the toroidal case as well.

3.7. **Principal affine opens.** If $m \in \sigma^{\vee}$ then $\tau = \{n \in \sigma | < n, m >= 0\}$ is a face of σ. We have $\tau^{\vee} = \sigma^{\vee} + \mathbb{R} \cdot m$, and therefore V_{τ} is the principal open set on V_{σ} obtained by inverting the monomial x^{m}.

Again, this can be discribed divisorially in terms of $Div(m)$. Thus the same is true for $Star(X_{\alpha})$: given a face τ of σ_{α}, we get an open set

$$Star(X_{\beta}) \subset Star(X_{\alpha})$$

such that $\tau = \sigma_{\beta}$.

The most important face of a cone is the vertex. It corresponts to the open set $T \subset V$. In the toroidal case you get U.

3.8. **Fans.** Recall: if σ_1 and σ_2 intersect along a common face τ, then V_{σ_1} and V_{σ_2} can be glued together along the common open set V_τ, forming a new toric variety.

In general, whenever you have a *fan* Δ in N, namely a collection of cones σ_i intersecting along faces, you can glue together the V_{σ_i} and get a toric variety V_Δ.

It is not hard to see that EVERY toric variety is obtained in this way in a unique manner. The point is that every toric variety is covered by affine open toric varieties.

In the toroidal case:

X is covered by $\{Star(X_\alpha)\}_\alpha$.

In general

$$Star(X_\alpha) \cap Star(X_\beta) = \cup Star(X_{\gamma_i}),$$

so σ_{γ_i} are faces of both σ_α and σ_β.

Can glue together, as a rational conical polyhedral complex: abstract, not in some N.

3.9. **Modifications and subdivisions.** Let Δ be a fan, and $\Delta' \to \Delta$ a (complete) subdivision. This corresponds to a toric modification $V_{\Delta'} \to V_{\Delta}$.

Since the construction is local (the spec construction and gluing) it works word for word in the toroidal case.

3.10. **Nonsingularity.** Recall: an affine toric V_σ is nonsingular iff σ is simplicial, generated by a basis of N_σ (part of a basis of N). Such a cone is called nonsingular.

In general: a toric variety V_Δ is nonsingular if and only if every cone $\sigma \in \Delta$ is nonsingular.

This is local, so true for toroidal as well.

3.11. **Desingularization.** It is easy to resolve singularities of a toric variety: find a simplicial subdivision such that every cone is nonsingular.

Obviously now, the same works in toroidal case!