Alterations and resolution of singularities
Lecture 4: Weak resolution in characteristic 0

The main result today is:

0.1. **Theorem.** Let X be a variety over an algenraically closed field of characteristic 0, $Z \subset X$ a proper closed subset. Then there exists a modification $f : X' \to X$ such that X' is regular and $f^{-1}Z = Z'$ is a simple normal crossings divisor.

We present a proof by $\aleph$ - de Jong.

0.2. **simple reductions.** As in de Jong's theorem, we may assume:

1. X is projective
2. Z is a divisor
3. X is normal

using this, we can project and reduce to the case:

4. There exists a projection $X \to P$, equidimensional of relative dimension 1, with generically reduced connected fibers, and generic fiber smooth, such that $Z \to P$ is finite
5. We may enlarge Z as in de Jong's theorem so it intersects each component of a fiber in at least 3 smooth points

We can now repeat the stable reduction argument and get

$$\begin{array}{ccc} X_1 & \to & X \\ \downarrow & & \downarrow \\ P_1 & \to & P \end{array}$$

such that

0.3. **making $P_1 \to P$ Galois.** Note that $P_1 \to P$ is generically étale. We can actually replace it by a Galois morphism:

Let $K(P) \subset L$ be the Galois closure of $K(P) \subset K(P_1)$, with Galois group G. Let Q be the normalization of P in L. we have a natural rational map $h : Q \dashrightarrow P_1$, so for each $g \in G$ we have $h \circ g : Q \dashrightarrow P_1$. The closure P_2 of the graph of the product map

$$Q \to \prod_{g \in G} P_1$$

is G equivariant, and admits a morphism to P_1. Thus we can pull backX_1 to P_2, and therefore we may as well assume that G acts on P_1 so that $P_1/G \to P$ is birational.

We want G to respect X_1 as well. We use the following:

0.4. **Fact.** (Separatedness of the moduli problem) Let B be normal, $U \subset B$ be a dense open, and $C_i \to B$ two families of stable pointed curves. If $(C_1)|_U \simeq (C_2)|_U$ over U then $C_1 \simeq C_2$ over B.

0.5. In our situation: let $U \subset P$ be the open subset over which X is smooth and Z is étale. Let $U_1 \subset P_1$ the inverse image. Over U_1 we have $X_1|_{U_1} \simeq X \times_P U_1$, and $\cup s_i(U_1) = Z \times_P U_1$. The group G acts on $X \times_P U_1$ and $Z \times_P U_1$, thus permuting $s_i(U_1)$. Thus we have $X_1|_{U_1} \subset g^* X_1$ for every $g \in G$. By the fact above these are all isomorphic (up to permuting the sections). Thus we have:

0.6. The action of G on P_1 lifts to X_1, permuting the sections s_i.

We may now replace P by P_1/G, X by X_1/G and Z by Z_1/G. So the alterations are finite quotient maps.

By induction on dimension we may resolve (P, Δ). We pull back X etc.. so we may assume

P is nonsingular, and Δ, which includes the degeneracy locus of X, Z and P_1, is a strict normal crossings divisor.

In particular $(P \setminus \Delta) \subset P$ is a toroidal embedding!

0.7. **Abhyankar's lemma.** If B nonsingular, $D \subset B$ a d.n.c., and $B_1 \to B$ finite, (tamely) ramified in D only, then B_1 is toroidal.

If Galois then the Galois group acts *toroidally*.

0.8. **Proof.** The formal, 1-dimensional case is obvious by the Galois theory of a power series ring.

The general case follows by the Nagata-Zariski purity theorem.

0.9. **Family of nodal curves.** Let $U \subset B$ be a toroidal embedding. Let $\pi : C \to B, s_i : B \to C$ be a family of pointed nodal curves, nondegenerate over U. Then $U_C \subset C$ is toroidal where $U_C = \pi^{-1}(U) \setminus \cup s_i(B)$.

0.10. **Proof.** Local equation can be written in the form $uv = m$, where m is a monomial.

Conclusion: P_1 is toroidal, with G acting toroidally. X_1 is toroidal, with G acting.

If G acted toroidally we would be done!

0.11. **node case.** Local equations for X_1 are $uv = m$. now m is an eigenvector. if G does not switch the branches $u = 0$ and $v = 0$ we could choose u, v eigenvectors and be happy.

To correct this, blowup $Sing f_1 =$ zero scheme of (u, v), and normalize

0.12. **Smooth (or double) case.** X_1 is locally Spec$R[u]$, where R is the ring of P_1. Can choose u as an eigenvector for G_x. but $u = 0$ might not be uniquely determined, and may not be part of the toroidal divisor!!

0.13. **Theorem ($\aleph$ - de Jong).** There exists a canonical G equivariant blowup $\widetilde{X_1} \to X_1$ such that G_x acts toroidally on $\widetilde{X_1}$.

Thus $\widetilde{X_1}/G$ is toroidal, and we can resolve.

0.14. **Proof.** Let I_G be the ideal generated by u and all the monomials in R on which G acts via the same character. Then the normalized blowup $\widetilde{B_{I_G}}(X_1)$ does the trick!