STABLE POINTED CURVES, ALTERATIONS AND APPLICATIONS.

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Survey of results discussed in my talk.

Introduction

Here a short survey of the results we shall discuss in my talks. The theorems mentioned in this note are the main results I am going to state in my talks. Unfortunately I will not have enough time to discuss all proofs in full detail. In every lecture I shall state at least one clear statement. Some details of its proof will be discussed. Some other parts of the proof we will omit (however enough motivation and references are available).

Terminology and notations as explained in

Dan Abramovich & Frans Oort -
- *Alterations and resolution of singularities,*
(informal notes) cited as [DA.FO].

will be used freely.

0.1. **Motivation:** One of the basic ideas of removing singularities via alterations as started by A. J. de Jong is: fibre a variety Y as a family of curves: $Y \to X$, and change X and Y (via alterations) in such a way that all fibres have only "easy" singularities (X' is nodal over Y').

0.2. **Basic Question:** Here is the central theme of my talk:

- Given a scheme S and a dense open subset $U \subset S$, and
- and a family of curves $\mathcal{C}_U \to U$ (think of a smooth curve over U, or a nodal curve for example),
- does there exist a curve $\mathcal{C} \to S$ extending this ??

By "extending" we mean: the restriction of $\mathcal{C} \to S$ to U is isomorphic with $\mathcal{C}_U \to U$. The question can be asked e.g. in case $\mathcal{C} \to S$ is smooth, or nodal or something alike. [In the proof of (3.1) we need a stronger variant of this question, see [13], 4.18.]

In general the answer, even to the weakest form of this question, is negative! We shall discuss various situations and examples, and see what can be achieved (extending after alterations, extending in certain good situations), we will start the talk by giving various (counter)examples.

By the way, do not worry about "schemes" and so on. If you are accustomed to work with plain varieties over $\mathbb{C}$, then keep doing so, and you can still be perfectly happy with the contents of my talk. But for those of you who

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prefer to think in mixed characteristics, or over arbitrary base fields, I hope the extra data in my theorems will be clear.

Here is a situation which is very instructive and basic for what follows: Suppose moreover that the dimension of S equals one, and that S is non-singular (and, if you wish, U is the complement of one closed point). In this case the extension problem has a positive answer, and even admits a "best" answer:

0.3. **Minimal models, existence and uniqueness**: Let K be a field, and C a complete, regular, geometrically irreducible algebraic curve defined over K; suppose the genus of C is at least 2. Let v be a discrete valuation on K, and $R \subset K$ its ring. Using resolution of singularities and following the ideas of Zariski of studying minimal models of algebraic surface (1956) we arrive at minimal models: using ideas and methods of Abhyankar (1963) we can resolve surface singularities, by Shafarevich (1966) we have the notion of the *minimal model* of C over $S := \mathrm{Spec}(R)$ (see Lichtenbaum, [17], Th. 4.4; also see [6], page 87): *we arrive at a curve $\mathcal{C} \to \mathrm{Spec}(R)$ which is a regular 2-dimensional scheme, and which is relatively minimal.*
Remark: here we use a special case of "resolution of singularities", namely in the case of "arithmetic surfaces".

1. Stable curves.

Given a family of curves $\mathcal{C}' \to U$ over an open set $U \subset S$ we like to extend this to some reasonable family $\mathcal{C} \to S$. (It is a fact of life that this is sometimes not possible, we shall discuss that later.) Even if such an extension is possible, the "best" extension in general is not a nodal family. Moreover, in general the "best extension" *does not commute with base change.* This causes the phenomenon that "taking the limit" of a regular curve is not unique in a geometric sense. If we would try to construct a moduli space of *all* curves (of a given genus, regular or singular), we would arrive at non-Hausdorff moduli spaces! However there is a good notion which singular curves should be admitted, enough to assure that these appear after an extension of the base field, geometrically: after a (ramified) extension. In this way we arrive at compactifications of moduli spaces which are "compact".

Historically, the influential paper [6] by Deligne and Mumford seems to be one of the first printed versions in which the concept of stability, especially in the case of algebraic curves is explained and used. In [20] we see that in 1964 Mumford was trying to find the appropriate notions assuring good compactifications.

For the definition of a "stable curve" see [DA.FO], 7.1, 5.3 and 7.7 (we shall use that the two definitions given there are equivalent). A "curve" will be an algebraic curve over a field, complete (proper over that field) and geometrically connected. A "family of curves" will be a flat, proper morphism $\mathcal{C} \to S$ such that all geometric fibres are algebraic curves; sometimes we say "$\mathcal{C}$ is a curve over S". We recall the stable reduction theorem for curves, see [DA.FO], 5.9:

1.1. **Theorem**: *Let S be the spectrum of a discrete valuation ring, $\eta \in S$ the generic point. Let $C_\eta \to \{\eta\}$ be a smooth curve of genus $g > 1$. There* *exists a finite extension of discrete valuation rings $S \hookrightarrow S_1$, with generic point η_1, and an extension*

$$\begin{array}{ccc} C_{\eta_1} & \hookrightarrow & \mathcal{C}_1 \\ \downarrow & & \downarrow \\ \{\eta_1\} & \hookrightarrow & S_1 \end{array}$$

such that $\mathcal{C}_1 \to S_1$ is a family of stable curves.
See [6], page 92, Coroll. 2.7. See [1], Coroll. 2.10.

In slightly different terminology:
(1.1'): *Consider a field K, with a discrete valuation v, and an algebraic curve C defined over K; suppose C is regular (and complete and geometrically irreducible) of genus $g > 1$. Then there exists a finite extension $K \subset K_1$, a valuation v_1 with ring $R_1 \subset K_1$ extending v and an extension of $C \otimes_K K_1$ to a stable curve over R_1.*

Note that the curve C defined over K and the curve C_1 defined over K_1 are indicated by different symbols (in classical literature you will often find them given by the same symbol, confusion is possible!).

1.2. **Remark**: Suppose $C = C_\eta \to \{\eta\}$ extends to $\mathcal{C} \to S$, a minimal model of C, and $\mathcal{C}_1 \to S_1$ is the minimal model of $C_1 := C \otimes K_1$; in general

$$\mathcal{C} \times_S S_1 \to S_1 \quad \text{is not isomorphic with} \quad \mathcal{C}_1 \to S_1.$$

In fact, if $\mathcal{C} \to S$ is not stable, and $\mathcal{C}_1 \to S_1$ is stable (as in the situation of the theorem) this is clear. However if K_1/K is unramified at v, then $\mathcal{C} \times_S S_1 \to S_1$ and $\mathcal{C}_1 \to S_1$ are isomorphic.

There are (at least) two basically different proofs of the stable reduction theorem. There is a proof by M. Artin and G. Winters, see [1], using matrices of intersection numbers of components in special fibres, also see [27]. In the course we shall follow the proof by Grothendieck (based on [SGA]), see [6], §2.

1.3. Here is a (short) description of the proof which is based on ideas by Grothendieck. Let C be a curve smooth over a field K, and $J := \mathbf{Pic}^0_C$ its *Picard variety* (also called its Jacobi variety, "equal" to its Albanese variety if $C(K) \neq \emptyset$). Here is the first of the three basic ideas:

(I) *The curve C has stable reduction at a given discrete valuation v of K iff J has stable reduction at v*
(concepts will be defined in the course), see [6], Th. 2.4.

Choose a prime number ℓ different from the residue characteristic of v, and consider the Tate-ℓ-group $T_\ell(X)$ of an abelian variety X over K (later we choose $J = X$).

(II) *We know that X has stable reduction if and only if the local monodromy*

$$\rho_{X,\ell} = \rho : I_v \to \mathrm{Aut}(T_\ell(X))$$

is unipotent,
see [SGA], page 21, Th. 6.1; see [SGA], IX.3, especially Prop. 3.5 on page 350, and Coroll. 3.8 on page 353, for a proof using [1]; also see [22], pp. 22-26, see [16], Prop. (1.3). We say that an abelian variety has stable reduction

at v, if its Néron minimal model $\mathcal{X}$ at v has a special fibre whose connected component $\mathcal{X}_v^0$ is an extension of an abelian variety by an algebraic torus.

As Grothendieck showed:

(III) *eigenvalues of algebraic monodromy are roots of unity*

(see [30], appendix, page 515; see [22], (1.3); see [23], 7.4).

Conclusion: there exists a finite extension $K \subset L$, and a discrete valuation w of L over v such that:

$$\text{the eigenvalues of} \quad \rho_{X,\ell} \quad \text{are roots of unity} \Longrightarrow$$

$$\text{the eigenvalues of} \quad \rho_{X_L,\ell} \quad \text{are equal to } 1 \Longrightarrow$$

$$X_L \quad \text{has stable reduction at} \quad w \quad \Longrightarrow$$

(take $X = J := \mathbf{Pic}_C^0$)

$$C_L \quad \text{has stable reduction at} \quad w.$$

1.4. **Warning**: We see that

$$J \quad \text{has stable reduction at} \quad v \quad \Longrightarrow \quad C \quad \text{has stable reduction at} \quad v,$$

however

$$J \quad \text{has good reduction at} \quad v \quad \not\Longrightarrow \quad C \quad \text{has good reduction at} \quad v.$$

1.5. **Remark,** not used in the course: In order to test properties of good reduction of an algebraic curve over a discrete valuation ring using properties of monodromy, one can consider monodromy actions on the fundamental group of an algebraic curve, see [2].
We have:

- An abelian variety has stable reduction iff the monodromy representation (on the Tate-ℓ-group, or on homology) is unipotent, it has good reduction iff this monodromy representation is trivial.
- An algebraic curve has stable reduction iff the monodromy representation (on homology, or on the Tate-ℓ-group of its Jacobian) is unipotent,
- it has "compact type of reduction" (i.e. its Jacobian has good reduction, i.e. we can choose a reduction to a nodal curve which is a tree of nonsingular components) iff this representation is trivial.
- An algebraic curve has good reduction iff the monodromy representation by outer automorphisms on the fundamental group is trivial.

2. Moduli spaces.

2.1. **Motivation.** In order to be able to use stable curves, and compactified moduli space in order to extend families of curves we need existence of families of curves over a compactified moduli space in such a way that the moduli point of a fibre is that point in the moduli space. Warning: we use the terminology "universal family" only in case we have a representable functor; in what we need we shall often speak of a "tautological family", or a "tautological curve". We recall theorems proved in [GIT], [6], and in [15]; see also [9].

2.2. Fix:

- an integer $g \in \mathbb{Z}_{\geq 0}$ (the genus),
- an integer $n \in \mathbb{Z}_{\geq 0}$ (the number of marked points),
- such that $2g - 2 + n > 0$,
- and an integer $m \in \mathbb{Z}_{\geq 1}$ (the level).

2.3. **Theorem**: *Suppose $m \in \mathbb{Z}_{\geq 3}$. There is a fine moduli scheme*

$$M_{g,n}^{(m)} \to \mathrm{Spec}\ \mathbb{Z}[1/m],$$

and a universal family

$$\pi^{(m)} : C^{(m)} \to M_{g,n}^{(m)}, \quad \textit{with sections} \quad s_i : M_{g,n}^{(m)} \to C^{(m)}, \quad 1 \leq i \leq n,$$

representing the functor of stable n-pointed smooth curves of genus g, with a level m-structure over schemes on which m is invertible.

2.4. **Theorem**: *Suppose $m \geq 3$. There exists a family of stable n-pointed curves*

$$\pi^{(m)} : C^{(m)} \to \overline{M_{g,n}^{(m)}}, \quad \textit{with sections} \quad s_i : \overline{M_{g,n}^{(m)}} \to C^{(m)}, \quad 1 \leq i \leq n,$$

such that $\pi^{(m)}$ coincides with the moduli map of this family.

2.5. **Remark, exercise**: Using the valuative criterion of properness, see [HAG], Th. II.4.7, and using the stable reduction theorem for curves, see (1.1), and by methods from [15], also see [12], Section 4, we conclude that

$$\overline{M_{g,n}^{(m)}} \to \mathrm{Spec}\ \mathbb{Z}[\frac{1}{m}]$$

is a proper morphism for every $m \in \mathbb{Z}_{\geq 1}$, and every $n \in \mathbb{Z}_{\geq 0}$. Remark: one can even show (Mumford, Knudsen), that this morphism is projective.

2.6. **Exercise**: Let us consider a Zariski-dense open set $U \subset M_{g,n}^{(m)}$, and take a family of curves $\mathcal{C}_U \to U$ (possibly with extra structures like level and marked points). We say that $\mathcal{C}_U \to U$ is a *tautological curve* if for every geometric $u \in U$ the moduli point $u' \in M_g$ (image of u) is the moduli point of the fibre $\mathcal{C}_U \times_U \{u\}$. (Sometimes $\mathcal{C}_U \to U$ with these properties is called a "universal family", a terminology which I find confusing.) [In this exercise you can either work over $\mathbb{Z}$, or over a field, as you like!]
a) Show that there exists a dense open set $U \in M_{1,1}$ (here $m = 1$) which carries a tautological curve; show that this tautological curve is not unique (up to isomorphism over $M_{1,1}$).
b) Show that no dense open set $U \in M_2$ (here $n = 0$, $m = 1$) carries a tautological curve (see [32], page 177, Theorem 3).
c) For every $g \in \mathbb{Z}_{\geq 2}$ show the existence of a dense open subset $U \in M_g$ (here $n = 0$, $m = 1$) carrying a tautological curve. Show that this is unique. Define the moduli functor for all smooth curves of genus g, where $g \in \mathbb{Z}_{\geq 2}$

is fixed, considering only curves which have no non-trivial automorphisms (even after extending base fields); show that this functor is representable.

2.7. **Remark, exercise**: Let S be a Noetherian scheme, and $U \subset S$ a dense open subset. Let $\mathcal{C}_U \to U$ be a stable curve. There exists an alteration $T \to S$, with $U' := U \times_S T$ and $\mathcal{D}_{U'} := \mathcal{C}_U \times_U U'$ and a stable curve $\mathcal{D} \to T$ extending $\mathcal{D}_{U'} \to U'$.
[This will be used in the proof of (3.1), but there we shall need a refinement of this exercise.]

3. Alterations can remove singularities.

We recall the central theorem by A. J. de Jong which will be proved in the course:

3.1. **Theorem** (A J. de Jong): *Let X be a variety over an algebraically closed field, $Z \subset X$ a proper closed subset. There is an alteration $f : Y \to X$, and an open immersion $j : Y \subset \overline{Y}$, such that $\overline{Y}$ is projective and regular, and the subset $j(f^{-1}Z) \cup (\overline{Y} - Y)$ is a strict normal crossings divisor.*

$$\begin{array}{ccccc} f^{-1}Z & \hookrightarrow & Y & \stackrel{j}{\hookrightarrow} & \overline{Y} \\ \downarrow & & \downarrow f & & \\ Z & \hookrightarrow & X. & & \end{array}$$

See [13], 4.1.

4. Intersection multiplicities (a conjecture by Serre).

Intersection theory has a long history (and certainly we are not going to say much about it here). One aspect: it is not so easy to have a good definition for intersection multiplicities.

4.1. **Remark, exercise**: let $C \subset \mathbb{P}^2_k$ be a plane algebraic curves, $P \in C$ a closed point at which C is regular, and $D = \mathcal{Z}(F) \subset \mathbb{P}^2$ a plane curve given by a homogeneuous polynomial F; suppose F is not identical zero on a neighborhood of P in C (i.e. no component of D contains the component of C containing P. Show that the following two definitions of *the intersection multiplicity $i(C, D; P)$ of C and D at P* are equivalent:

- the dimension of the k-vector space
$$\mathcal{O}_{C,P} \quad \otimes_{\mathcal{O}_{\mathbb{P}^2,P}} \quad \mathcal{O}_{D,P},$$
- the value of the valuation $v = v_{C,P}$ defined by the discrete valuation ring $\mathcal{O}_{C,P}$ computed on the function on C given by F,

see [HAG], Exercise (5.4) on page 36, and Remark (7.8.1) on page 54.

4.2. Consider two varieties $V, W \subset \mathbb{P}^n$ which have an isolated point of intersection at $P \in V \cap W$. One could try to define the intersection of V and W at P as the length of

$$\mathcal{O}_{V,P} \quad \otimes_{\mathcal{O}_{\mathbb{P}^n,P}} \quad \mathcal{O}_{W,P}.$$

Analogous situations of intersections of arbitrary schemes in some regular ambient scheme can be considered.

4.3. **Exercise** (See Gröbner [10], 144.10/11, also see [29], [28], [3], see [HAG], I.7):

a) Let $C \subset \mathbb{P}^3$ be the space curve with parametrization

$$(x_1 \ : \ x_2 \ : \ x_3 \ : \ x_4) = (t^4 \ : \ t^3{\cdot}s \ : \ t{\cdot}s^3 \ : \ t{\cdot}s)$$

(we work over some field K). Show that the prime ideal given by this curve equals

$$j := (T_1^2T_3 - T_2^3 \ , \ T_1T_4 - T_2T_3 \ , \ T_1T_3^2 - T_2^2T_4 \ , \ T_2T_4^2 - T_3^3) \subset K[T_1, T_2, T_3, T_4].$$

b) Consider C as a curve embedded in $\mathbb{P}^4$: choose the hyper plane $\mathbb{P}^3 \cong \mathcal{Z}(T_0) = H$, and we get $C \subset H \subset \mathbb{P}^4$. Let $P := (x_0 = 1 : 0 : 0 : 0 : 0) \in \mathbb{P}^4$. Define $V \subset \mathbb{P}^4$ as the cone with vertex P over the curve $C \subset \mathbb{P}^4$, i.e. V is defined by the ideal

$$J := K[T_0, T_1, T_2, T_3, T_4] \cdot j, \quad V = \mathcal{Z}(J).$$

Note that the dimension of V equals two, that the degree of $V \subset \mathbb{P}^4$ equals four.

c) Let W be the 2-plane given by

$$I := (T_1, T_4) \subset K[T_0, T_1, T_2, T_3, T_4], \quad W := \mathcal{Z}(I).$$

Note that $P \in W$. Remark that (set-theoretically):

$$W \cap V = \{P\}$$

(use the geometric situation, or give an algebraic computation).
We like to have a Bézout type of theorem for the situation $W \cap V \subset \mathbb{P}^4$, however:

d) Define

$$M := \mathcal{O}_{W,P}, \quad A := \mathcal{O}_{\mathbb{P}^4,P}, \quad N := \mathcal{O}_{V,P},$$

and compute

$$\dim_K (M \otimes_A N)$$

(surprise: this is not equal to four).

e) Compute

$$\dim_K \left(\mathrm{Tor}_i^A(M, N)\right), \quad \forall i$$

(either using, or reproving $\chi_A(M, N) = 4$, for notation see below).

4.4. Hence we see that just the length of the appropriate tensor product does not define necessarily the correct concept. Serre proposed in 1957/58 to define the intersection multiplicity as the alternating sum of the lengths of the Tor_i (note that $\mathrm{Tor}_0 = \otimes$), i.e. by

$$\chi_A(M, N) := \sum_{i \geq 0} \ \mathrm{length}_A \mathrm{Tor}_i^A(M, N)$$

(we follow notation of [29], also see [3], 6.1, see [28]), here A is a regular local ring, and M and N are A-modules such that $M \otimes_A N$ has finite length. In equal characteristic this is the right geometric concept (i.e. satisfies Bézout's theorem, coincides with previously defined intersection mutiplicities etc.).

4.5. **Theorem** (Conjectured by Serre, proved by Gabber, see [3], 6.1): *Let the characteristic of A be equal to zero. Suppose $p \in \mathfrak{m}^2$, hence its residue field $A/\mathfrak{m}$ has characteristic $p > 0$. Then:*

$$\chi_A(M, N) \geq 0.$$

5. Extending families of curves.

Here is a basic question: given a scheme S, an open dense subset $U \subset S$, a family of curves $\mathcal{C}_U \to U$, can this be extended to a family $\mathcal{C} \to S$ (having certain reasonable properties). An analogous question can be asked for families of abelian varieties, for other types of families. These seem difficult questions.

5.1. Lemma: *Let S be a* normal *locally Noetherian scheme, $U \subset S$ a dense open subset, and $\mathcal{C}_U \to U$ a stable curve. Then there exists at most one stable curve $\mathcal{C} \to S$ which extends $\mathcal{C}_U$.*
See [14], 2.3 for a more general statement; for a statement concerning semi-abelian schemes, see [7], page 350, Lemma 1, see [8], page 9, Prop. 2.7.

Using the theory of stable curves we can show an extension theorem for families of curves, to be found in [14]. To this end we define:

5.2. **Definition**: Let C be an n-pointed stable curve over an algebraically closed field k. The *topological type* of this pointed curve is a graph (the "dual graph") where edges are labeled with pairs of non-negative integers, given by:

- Every irreducible component of C corresponds with a vertex,
- the genus of that component appears as first label at that vertex, the number of marked points on that component as second label.
- Every singularity of C is a node, and it corresponds with an edge of the graph; the edge has as endpoints the component(s) on which the singularity is situated (NB if the node is a singularity of one component, the edge starts and and ends in the same vertex).

There is a slight difference with the definition as on page 86 of [6], here every singularity gives an edge, in [6] only singularities situated on two different irreducible components define an edge. For example, an irreducible curve with one singular point (a node) defines a graph with one edge and one vertex, while in [6] it corresponds with a graph with one vertex, and no edges.

5.3. **Definition**: Let $\mathcal{C}_U \to U$ be a family of stable n-pointed curves. We say that this family is *locally of constant topological type* if for every two geometric points $u_1, u_2 \in U$ the fibres have equal topological types (i.e. the dual graph with labelling is constant on geometric fibres).

5.4. **Theorem** (see [14], Th. 5.1): *Let S be a* regular *scheme, $D \subset S$ a normal crossings divisor, and $U := S - D$. Let $\mathcal{C}_U \to U$ be a stable curve which is locally of constant topological type, such that $\mathcal{C}_U$ extends to a stable curve over the generic point of every component of D. Then $\mathcal{C}_U \to U$ extends to a stable curve over S.*

5.5. **Remarks**: It is easy to give a surface S, a closed point $P \in S$, such that $S - \{P\}$ is regular (but S singular at P), and a smooth curve $\mathcal{C}^0 \to S - \{P\}$ which cannot be extended to any curve over S, i.e. the condition "S is regular" is needed. There are examples which show that without the condition "D is a normal crossing divisor" the conclusion of the theorem need not be true.

5.6. **Key Lemma** (see [14], 4.1): *Let k be an algebraically closed field, let $\mathcal{C} \to \mathbb{P}^1_k$, with $s_i : \mathbb{P}^1_k \to \mathcal{C}$ be a family of stable n-pointed curves which is locally of constant topological type when restricted to $U := \mathbb{P}^1_k - \{0, \infty\}$. Then this family is* constant *i.e. take the pointed stable curve $(C_1, \{s_i(1)\}$ over $1 \in U$, then*

$$(\mathcal{C}, \{s_i\}) \cong (C_1, \{s_i(1)\}) \times_{\mathrm{Spec}(k)} \mathbb{P}^1_k.$$

5.7. **Remarks**: There exist smooth pointed curves over $U := \mathbb{P}^1_k - \{0, \infty\}$ which are not constant, i.e. without the condition "$(\mathcal{C}, \{s_i\})$ is stable over $\mathbb{P}^1_k$" the conclusion of the lemma does not hold in general.

For families of stable curves over $\mathbb{P}^1_k$ with at most two singular fibres: see the result by L. Szpiro, [33], Th. 4 on page 61. For extension problems in codimension 2, and for certain families over $\mathbb{P}^1_k$, see [18].

For a discussion and an example (by Raynaud-Ogus-Gabber) on the extension problem for families of abelian varieties, see [14], Section 6.

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