ALTERATIONS AND RESOLUTION OF SINGULARITIES

Course material for

"Mountains and Singularities"

working week on resolution of singularities

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INFORMAL NOTES

1. Introduction

On July 26, 1995, at the University of California, Santa Cruz, a young Dutch mathematician by the name A.J. de Jong made a revolution in the study of the arithmetic, geometry and cohomology theory of varieties in positive or mixed characteristic (his results were already announced in the Arbeitstagung, Bonn, June 1995). The talk he delivered, first in a series of three entitled "Dominating Varieties by Smooth Varieties", had a central theme: a systematic application of fibrations by nodal curves. Among the hundreds of awe struck members of the audience, participants of the American Mathematical Society Summer Research Institute on Algebraic Geometry, many recognized the great potential of Johan de Jong's ideas even for *complex* algebraic varieties, and indeed soon more results along these lines began to form.

These pages include *informal* preliminary material for a course to be given by the authors during the Working Week on Resolution of Singularities, September 7-14, 1997 in Obergurgl, Tirol, Austria. In the course, we intend to explain de Jong's results in detail, with proofs as complete as possible, and give some applications, both arithmetic and in characteristic zero. Hopefully, the participants will come to share some of the excitement we felt on that beautiful July day in Santa Cruz.

2. Prerequisites

We are trying to keep the prerequisetes for the course at minimum, but some basics are necessary.

The participants are expected to have a good understanding of basic algebraic geometry. The standard text is Hartshorne's book [HAG]: Chapter II is essential, as well as most of Chapter III (though you can take some parts of it on faith). A good alternative to parts of [HAG], is Mumford's Red Book [M.RedBook]. This book does a great job at building one's intuition for the subject.

De Jong's results build on the theory of moduli of curves. We hope the participants will take some time before the course to read about the subject and build some intuition for it. The functor of points of a scheme is introduced in [M.RedBook], II§6. Lecture 21 in Harris's book [Harris] is good bed-time reading about parameter spaces and moduli spaces. Chapter IV in Eisenbud - Harris [E-H] is a bit more detailed, and will also give an understanding of the functor of points. Finally, Mumford's lectures [M.C+J] give an excellent introduction to the theory of moduli of curves, from a more analytic point of view. The standard reference for moduli spaces is [M.GIT].

For the applications in characteristic 0, we will use the theory of toroidal embeddings. The participants should be familiar with toric varieties, as in the first two chapters in Fulton's book [F], or in [KKMS], Chapter I, §1-2. This theory will also be discussed in the parallel course given by B. Teissier. If one wants to become familiar with toroidal embeddings in advance, one could look in [KKMS], Chapter II, §1-2.

Finally, if you would like to learn about the structure of de Jong's proofs, you may want to read the introduction of de Jong's paper [dJ], or Berthelot's Bourbaki seminar [BerBour]

2.1. **To summarize, please read and study:** here is the list of prerequisites we will take for granted in our course:

1. *Basic algebraic geometry*: [HAG] Chapters I and II (we will also use III). Alternatively, or in addition, read [M.RedBook], Chapters I and II;
2. *Moduli of curves:* in addition to the above, we suggest [M.GIT], pp. 96-104, and Th. 5.11 (p. 115); please have a look at [D-M], say pp. 75-96; please see [Knu], II;
3. *Toroidal geometry*: the first two chapters in Fulton's book [F], or [KKMS], Chapter I, §1-2.

2.2. **Supplementary reading:** if you want, and if you have time and energy it might be good to have a look at topics like:

- Please read [dJ], it is nice if you already have tasted the flavour of the subject,
- for ℓ-adic representations, see [S-T], see [SGA], pp.326-354, or see [O];
- for toroidal embeddings, [KKMS], Chapter II, §1-2.

2.3. **This pamphlet.** One important fact you can conclude from the discussion above is that *the material in this pamphlet is* **not** *a prerequisite for the course.* If you can understand all the material in this pamphlet, do all the exercises, and see how the proofs should go, the course will not be able to compete with the attractions of the Alps around. What we put in this pamphlet are three types of data:

1. a few definitions which we will use;
2. statements and discussion of the main results we wish to discuss, without proofs; and
3. a preliminary form for some sets of exercises.

We hope that you will be able to read the definitions and get some idea about the statement of results. You may also try your hands at some of the exercises. All of this may help you get an idea about what the course is about. It may also help you formulate for yourself some questions which will hopefully be answered during the course.

The exercises are certainly not in their final form. We intend to rewrite them and add many more, of various types - some will help you prepare, some will be done during the course, and some will be intended for further investigation.

Part 1. Results and tools

3. Some preliminaries and generalities on varieties

3.1. **Definition.** To fix notation, by a *variety* defined over k we mean a geometrically integral scheme of finite type over k. If $k \subset k_1$ we write X_{k_1} for $X \times_{\mathrm{Spec}\ k} \mathrm{Spec}\ k_1$.

In more down to earth terms this means: an affine variety defined over k is given as a closed subvariety of an affine space $\mathbb{A}^n_k$ defined by an ideal $I \subset k[T_1, \cdots, T_n] = k[T]$ such that $k_1 \cdot I \subset k_1[T]$ is a prime ideal for every (or for one) algebraically closed field k_1 containing k. A variety then is defined by glueing a finite number of affine varieties in a separated way. For more information, please see [M.RedBook], I.5 definition 1 (p. 35) and I.6 definition 2 (p. 52).

3.2. **Remark.** This definition differs slightly from that in [dJ]. De Jong requires the algebraic scheme to be integral, we require that the schemes stay integral after extending the field. For example for any finite field extension $k \subset K$, the scheme $\mathrm{Spec}(K)$ is called a k-variety by de Jong, but we only say it is a variety defined over k if $k = K$. For most geometric situations the differences will not be important.

We now introduce some important types of operations on varieties:

3.3. **Definition.** A morphism of varieties $Y \to X$ is called a *modification* if it is proper and birational.

3.4. **Definition (de Jong).** A morphism of varieties $Y \to X$ is called an *alteration* if it is proper, surjective and generically finite. This notion of alteration will also be used for integral schemes.

See [dJ], 2.20.

3.5. **Remark.** A modification is a birational alteration. We adopt the terminology: we say we have a *resolution of singularities* if we achieve a regular variety by a modification. If we can only do this by an alteration, we shall not use the word "resolution". We agree that this is the right way to avoid possible confusion.

3.6. **Exercise.** Show that an alteration $\varphi : Y \to X$ can be factored as

$$Y \xrightarrow{\pi} Z \xrightarrow{f} X,$$

where π is a modification, and f is a finite morphism.

3.7. **Exercise.** Suppose moreover that a finite group G acts on Y by automorphisms such that the field of invariants $K(Y)^G$ contains the function field $K(X)$. Formulate other factorizations of φ.

3.8. **Remark.** Given a variety X and a nonzero coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, the blowing up $Bl_{\mathcal{I}}X = Proj_X(\oplus_{j \geq 0} \mathcal{I}^j) \to X$ gives naturally a modification, such that the inverse image of $\mathcal{I}$ becomes invertible. If $Z \subset X$ is a subscheme with ideal sheaf $\mathcal{I}$, the blowing up $Bl_Z(X)$ of X along Z is defined to be the blowing up $Bl_{\mathcal{I}}X$.

See [HAG], II.7, p. 163.

To pick up the "important part" of the pullback by an alteration, we do the following:

3.9. **Definition (de Jong).** Let $Y \to Z$ be an alteration, $f : X \to Z$ a morphism of varieties. The *strict transform* $f' : \widetilde{X_Y} \to Y$ is obtained by taking the closed subscheme of $X \times_Z Y$ defined by dividing out the $\mathcal{O}_Z$ torsion. In other words, if η is the generic point of Y, then $\widetilde{X_Y}$ is the closure of $X \times_Z \eta$ in $X \times_Z Y$.

See [dJ], 2.18.

To compare varieties with projective varieties we have:

3.10. **Chow's Lemma.** Given a variety X, there is a modification $Y \to X$ such that Y is quasi-projective.

See [M.RedBook] , I.10, p. 85, or [HAG], Exc. II.4.10 p. 107.

One way projectivity comes in handy is the following result, which uses the theory of Hilbert schemes:

3.11. **Flattening Lemma.** Given a variety Z (more generally, an integral scheme of finite presentation) and a projective morphism $X \to Z$, there is a modification $Y \to Z$ such that the strict transform $f' : \widetilde{X_Y} \to Y$ is flat.

3.12. **Definition.** Let X be a variety. A *resolution of singularities* in the weak sense is a modification $Y \to X$ such that Y is nonsingular.

3.13. **Definition.** Let X be a variety. A *resolution of singularities* in the strong sense is a modification $Y \to X$, which is an isomorphism over the nonsingular locus X_{reg}, such that Y is nonsingular.

The entire workshop revolves around finding resolutions of singularities. To put things in prespective, we have:

3.14. **Theorem (Hironaka).** Suppose X is a variety over a fied k of characteristic 0. There is a sequence of blowings up $X_n \to \cdots \to X_0 = X$, each along a regular variety lying over $\mathrm{Sing}X$, such that X_n is regular. Thus $X_n \to X$ is a resolution of singularities in the strong sense.

See [H]. This type of results will be discussed in a parallel course in the workshop, taught by O. Villamayor.

We define a type of nice subvarieties:

3.15. **Definition.** Let X be a variety. A subscheme $Z \subset X$ is called a *strict normal crossings divisor* if for each point $x \in Z$, there is a regular system of parameters $y_1, \ldots, y_k$ for x in X (in particular the point $x \in X$ is supposed to be a regular point on X), such that Z is given on a Zariski neighborhood of x by the equation $y_1 \cdots y_l = 0$.

Suppose furthermore we have a finite group acting on Z and X equivariantly: $G \subset \mathrm{Aut}(Z \subset X)$. We say that Z is a *G-strict normal crossings divisor* if it has normal crossings, and for any irreducible component $Z' \subset Z$, the orbit $\cup_{g \in G}\, g(Z')$ is normal.

See [dJ], 7.1.

4. **Results**

Our main goal is to prove the following result, due to A. J. de Jong:

4.1. **Theorem.** Let X be a variety over an algebraically closed field. There is a separable alteration $Y \to X$ such that Y is quasi projective and regular.

4.2. **Corollary.** Let X/k be a variety. There is a finite extension $k \subset k_1$ and a separable alteration $Y \to X_{k_1}$ such that Y is quasi projective and regular.

This result is completely new in positive characteristic. Notice that in characteristic 0, the result is an easy corollary of Hironaka's theorem. Nevertheless, the mothods bring about new and important results in characteristic 0 as well.

In order for the induction in the proof to work, de Jong's theorem gives more:

4.3. **Theorem (de Jong).** Let X be a variety over an algebraically closed field, $Z \subset X$ a proper closed subset. There is a separable alteration $f : Y \to X$, and an open immersion $j : Y \subset \bar{Y}$, such that $\bar{Y}$ is projective and regular, and the subset $j(f^{-1}Z) \cup \left(\bar{Y} \setminus Y\right)$ is a strict normal crossings divisor.

See [dJ], 4.1.

De Jong's theorem has a few important variants. First,

4.4. **Theorem of semistable reduction up to alteration (de Jong).** Let R be a discrete valuation ring, with fraction field K and residue field k. Let $X \to \operatorname{Spec} R$ be an integral scheme of finite type such that X_K is a variety. There exists a finite extension $R \subset R_1$, where R_1 is a discrete valuation ring with residue field k_1, and an alteration $Y \to X_{R_1}$, such that Y is nonsingular, the special fiber Y_{k_1} is a reduced, strict normal crossings divisor.

See [dJ], 6.5.

4.5. **Theorem of equivariant resolution up to alteration (de Jong).** Let X be a variety over an algebraically closed field, $Z \subset X$ a proper closed subset, $G \subset \operatorname{Aut}(Z \subset X)$. There is an alteration $f : Y \to X$, and a finite subgroup $G_1 \subset \operatorname{Aut} Y$, satisfying:

1. there is a surjection $G_1 \to G$ such that f is G_1 equivariant, and the field extension $K(X)^G \subset K(Y)^{G_1}$ is purely inseparable;
2. Y is quasi projective and nonsingular; and
3. $f^{-1}Z$ is a G-strict normal crossings divisor.

See [dJ], 7.3.

Note that, taking $G = \{id\}$, this implies:

4.6. **Corollary.** Let X be a variety over an algebraically closed field. There is a purely inseparable alteration $Y \to X$ such that Y is a quotient of a nonsingular variety by the action of a finite group.

In characteristic 0, any purely inseparable alteration is birational, and the quotient singularities can be improved:

4.7. **Weak Resolution Theorem.** Let X be a variety over an algebraically closed field of characteristic 0. Then there is a projective resolution of singularities (weak sense) $Y \to X$.

4.8. **Remark.** This is a rather weak version of Hironaka's theorem. The point is, that new proofs, by Abramovich and de Jong [ℵ-dJ], and by Bogomolov and Pantev [B-P], were given based on de Jong's ideas. The proof by Bogomolov and Pantev is extremely simple, drawing only on toric geometry. The proof by Abramovich and de Jong lends itself to various generalizations in the flavor of de Jong's semistable reduction theorem.

4.9. **Question.** Can we improve the methods and obtain a weak resolution of singularities in all characteristics? At least weak resolution up to purely inseparable alterations?

The next few sections discuss a few of the main toold involved in the proofs of the theorems.

5. Moduli of curves

The reader is referred to exercise set I for illuminations of the following definitions:

5.1. **Definition.** An S scheme $C \to S$ is called a *nodal curve* over S if it is proper and flat, such that all geometric fibers are connected curves with at most ordinary double points (locally $xy = 0$). (If S is allowed to be non-noetherian, we may also require $C \to S$ to be of finite presentation).

5.2. **Definition.** The *discriminant locus* $\Delta \subset S$ is the closed subset over which $C \to S$ is not smooth.

5.3. **Definition (Deligne and Mumford).** A nodal curve $f : C \to S$, together with sections $s_i : S \to C, \quad i = 1, \ldots, n$ with image schemes $S_i = s_i(S)$, is called a *stable n-pointed curve of genus g* if

1. The schemes S_i are mutually disjoint.
2. The schemes S_i are disjoint from the locus $\mathrm{Sing}(f)$.
3. All the geometric fibers have arithmetic genus g.
4. The sheaf $\omega_{C/S}(\sum S_i)$ is f-ample.

The definition is made so that a stable pointed curve has a finite automorphism group (relative over S). These definitions can be given in more down to earth terms - this will be done in the exercise sessions.

5.4. **The functor of points and representability.** We use the notion of *reperesentable functors* (as introduced by Grothendieck) in order to come to the definition of a fine moduli scheme with a universal family:

To any scheme M one naturally associates a contravariant functor

$$\mathcal{F}_M : \{\text{Schemes}\} \to \{\text{Sets}\}$$

via

$$X \mapsto \mathrm{Mor}(X, M).$$

This is known as *the functor of points of M*, see [M.RedBook], II§6.

We say that a contravariant functor $\mathcal{F} : \{\text{Schemes}\} \to \{\text{Sets}\}$ is *representable* by M, if it is isomorphic to $\mathcal{F}_M$, i.e. there is a functorial isomorphism

$$\mathcal{F}(-) \cong \mathrm{Mor}_S(-, M).$$

5.5. **Moduli spaces - what are they?** Suppose such a contravariant functor $\mathcal{F}$ is a *moduli functor*, i.e., it assigns to a scheme S the set of isomorphism classes of certain objects over S; e.g.

$$\mathcal{M}_g(S) = \{\text{isom. classes of families of curves of genus } g \text{ over } \quad S\}$$

Assume $\mathcal{F}$ is representable by a scheme M. Then we call M a *fine moduli scheme* for this functor $\mathcal{F}$, and the object

$$\mathcal{C} \to M \quad \text{corresponding to the identity} \quad id \in \mathcal{F}(M) = \mathrm{Mor}_S(M, M)$$

is called a *universal family.*

It is a fact of life that several natural moduli functors in algebraic geometry are not representable. For this reason Mumford introduced the notion of a *coarse moduli scheme*, see [M.GIT], 5.2, requiring that the scheme gives geometrically what you want, and does it in the best possible way:

5.6. **Definition.** A scheme M and a functorial morphism

$$\varphi : F \to \mathrm{Mor}_S(-, M)$$

is called a *coarse moduli scheme* for F if:

1. for every algebraically closed field k the map

$$\varphi(k) : F(\mathrm{Spec}(k)) \to \mathrm{Mor}_S(\mathrm{Spec}(k), M) = M(k)$$

   is bijective, and
2. for any scheme N and any morphism $\psi : F(-) \to \mathrm{Mor}_S(-, N)$ there is a uniqe $\chi : N \to M$ factoring ψ.

It can be seen that for any $g \geq 2$ the functor $\mathcal{M}_g$ "of curves of genus g" is not representable, but it does admit a coarse moduli scheme. The same happens in the case of elliptic curves, where $g = 1$ with one marked point. We shall study some of these examples in the exercises.

5.7. **Remark.** Even if over (some open part of) a coarse moduli scheme there exsits a family giving the morphism $\varphi : F \to \mathrm{Mor}_S(-, M)$ required in the definition of a cuarse moduli scheme, but the functor is not representable, we shall not call this family a universal family. Examples will be discussed.

5.8. **Theorem/Notation.** The functor $\mathcal{M}_g$ of smooth curves of genus g admits a coarse moduli scheme. We will denote it by $M_g \to \mathrm{Spec}\mathbb{Z}$. The functor of smooth stable n-pointed curves of genus g is coarsely representable; notation: $M_{g,n}$. If $n + 2g - 2 > 0$, there is a coarse moduli scheme $\overline{M_{g,n}}$, projective over Spec $\mathbf{Z}$, for stable n-pointed curves of genus g. The scheme $M_{g,n}$ is open and dense in $\overline{M_{g,n}}$.

See [M.GIT], [D-M], [Knu].

Note that we write $M_{g,0} = M_g$.

The fact that $\overline{M_{g,n}} \to \mathrm{Spec}\mathbb{Z}$ is proper follows from what we call the "the weak valuative criterion", which is based on the following extension result:

5.9. **Theorem.** Let S be the spectrum of a discrete valuation ring, $\eta \in S$ the generic point. Let $C_\eta \to \{\eta\}$ be a smooth curve of genus $g > 1$. There exists a finite extension of discrete valuation rings $S \hookrightarrow S_1$, with generic point η_1, and an extension

$$\begin{array}{ccc} C_{\eta_1} & \hookrightarrow & C \\ \downarrow & & \downarrow \\ \{\eta_1\} & \hookrightarrow & S_1 \end{array}$$

such that $C \to S_1$ is a family of stable curves.

5.10. **Moduli schemes for curves with a level structure.** We see that $M_g \to \mathrm{Spec}\mathbb{Z}$ is (only !) a coarse moduli scheme, which sometimes causes difficulties. Also $M_{g,n}$ is coarse for small n, and $\overline{M_{g,n}}$ is most often only a coarse moduli space. This may seem to be a serious drawback. To overcome that, Mumford has provided us with the following:

5.11. **Definition.** Let k be a field, and let m be a positive integer, not divisible by the characterisitc of k. A level-m-structure on a regular curve C over k of genus g consists of an isomorphism

$$\alpha : \mathrm{Jac}(C)[m] \to (\mu_m)^g \times (\mathbb{Z}/m)^g$$

which is symplectic with respect to the Weil pairing; here $\mathrm{Jac}(C)$ is the Jacobian of C, and $\mathrm{Jac}(C)[m]$ denotes the m- torsion points. In case all m-th root of unity are in k we can write α as a symplectic $\mathbb{Z}/m$-base $\{x_1, \cdots, x_{2g}\}$ for $\mathrm{Jac}(C)[m]$.

5.12. **Theorem.** Suppose $m \in \mathbb{Z}_{\geq 3}$. There is a fine moduli scheme

$$M_g^{(m)} \to \mathrm{Spec}\ \mathbf{Z}[1/m]$$

for smooth curves over schemes on which m is invertible. The scheme

$$M_{g,n}^{(m)} \to \mathrm{Spec}\ \mathbf{Z}[1/m]$$

is a fine moduli scheme for smooth n-pointed curves over schemes on which m is invertible.

5.13. **Definition.** Define $\overline{M_g^{(m)}} \to \mathbf{Z}[1/m]$ to be the normalization of $\overline{M_{g,n}}$ in the function field of $M_g^{(m)}$. Let $\overline{M_{g,n}^{(m)}}$ be the normalization of $\overline{M_{g,n}}$ in the function field of $M_{g,n}^{(m)}$. Let $\pi^{(m)} : \overline{M_{g,n}^{(m)}} \to \overline{M_{g,n}} \times \mathrm{Spec}\ \mathbf{Z}[1/m]$ be the natural finite morphism.

5.14. **Remark.** Note the somewhat strange approach: indeed we define schemes like $\overline{M_g}$ by first defining a functor, and then showing it admits a coarse moduli scheme. However the schemes $\overline{M_{g,n}^{(m)}}$ are defined by the somewhat mysterious process of first defining moduli schemes of stable curves via a coarsely representable functor, and then normalizing these in the function fields of moduli schemes of smooth curves with level structure. There is a good reason for this. We do not know an a priori obvious moduli-functor which defines the schemes like $\overline{M_{g,n}^{(m)}}$, this is semetimes expressed by saying: “we do not know a good moduli interpretation” of the points of $\overline{M_{g,n}^{(m)}}$. This will be discussed in the course.

5.15. **Theorem.** Suppose $m \geq 3$. There exists a family of stable n-pointed curves

$$\pi^{(m)} : C^{(m)} \to \overline{M_{g,n}^{(m)}}, \quad \text{with sections } s_i : \overline{M_{g,n}^{(m)}} \to C^{(m)},$$

such that $\pi^{(m)}$ coincides with the moduli map of this family.

From this, de Jong deduces the following theorem, which in a sense resolves the singularities of the fibers of a family of curves:

5.16. **Theorem.** Let $f : X \to B$ be a projective morphism of finite type of integral schemes, $Z \subset X$ a proper closed subset. Assume the generic fiber of this morphism is a smooth absolutely irreducible curve. Then there exists a diagram

$$\begin{array}{cccc} & X_1 & \xrightarrow{f} & X \\ & \pi_1 \downarrow & & \downarrow \\ \Sigma \subset & B_1 & \to & B \end{array}$$

and sections $s_i : B_1 \to X_1$, satisfying:

1. $B_1 \to B$ is an alteration;
2. $X_1 \to \widetilde{X_{B_1}}$ is a modification;
3. $X_1 \to B_1$ is flat, projective and nodal;
4. s_i are disjoint sections, not meeting $\mathrm{Sing}(\pi_1)$;

5. $\Sigma \subset B_1$ is a proper closed subset.
6. $f^{-1}Z = \cup s_i(B_1) \cup \pi_1^{-1}\Sigma$.

6. **Toroidal geometry**

6.1. **Definition.** A variety X together with an open dense embedding $T \subset X$ is called a *toric variety* if X is normal, T is a torus (geometrically isomorphic to $\mathbb{G}_m^k$), and the action of T on itself by translations extends to an action on X.

To get an intuitive idea about the singularities of a toric variety, it is worth noting that a normal, affine variety, defined by equations between monomials is toric, and every toric variety is locally of this type.

6.2. **Definition.** A variety X together with an open embedding $U \subset X$ is called a *toroidal embedding* if any point $x \in X$ has an étale neighborhood X' such that $X' \times_X U \subset U$ is isomorphic to an étale neighborhood of a point on a toric variety.

Thus a toroidal embedding looks locally like a toric variety, and the big open set U is a device which ties together these local pictures.

A toroidal embedding is said to be *strict* (or a *toroidal embedding without self intersections*) if every irreducible component of $X \setminus U$ is normal. For instance, if X is a nonsingular variety and D is a strict normal crossings divisor, and $U = X \setminus D$, then $U \subset X$ is a strict toroidal embedding.

We will only work with strict toroidal embeddings.

If $G \subset \mathrm{Aut}(U \subset X)$ is a finite group, we say that G *acts toroidally* if for any point $x \in X$, the stabilizer G_x of x can be identified with a subgroup of the torus in an appropriate étale neighborhood of x.

A morphism between toric varieties is called a *toric morphism*, if it is surjective and torus-equivariant. A morphism of toroidal embeddings $(U_X \subset X) \to (U_Y \subset Y)$ is called a *toroidal morphism* if locally on X it looks like a toric morphism.

6.3. **The polyhedral complex.** Recall that there is a correspondence between

$$\{\text{affine toric varieties } T \subset X\}$$
$$\text{and}$$
$$\{\text{strictly convex rational polyhedral cones } \sigma \subset N\}$$

Here N may be canonically identified with the group of 1-parameter subgroups on T.

Since every toroidal embedding $U \subset X$ looks locally like a toric variety, we can associate a cone σ_x to (the neighborhood of) a point $x \in X$. However, since we don't have a fixed torus around, there is a slightly different definition of a linear embedding of this cone - it is embedded in N_σ, the dual of the group of Cartier divisors on a neighborhood of x, supported on $X \setminus U$. When $U \subset X$ is actually toric, then $N_\sigma \subset N$ is the subgroup spanned by σ.

Recall also, that given a general toric variety $T \subset X$, we can glue together the cones σ attached to toric affine open sets $V_\sigma \subset X$ and form a *fan* $\Delta_X \subset N$. This is in fact a one-to-one correspondence: we can build $T \subset X$ back from the data $\Delta_X \subset N$.

On the toroidal side we can do the same - we can glue together the cones of open sets in X. However, since they do not all lie canonically in one space N, what we get is not a fan - it is a rational conical polyhedral complex, also denoted Δ_X. Here we have no chance of getting a one-to-one correspondence.

Recall that there is a one-to-one correspondence between (proper) toric modifications $T \subset X' \to X$ and (complete) subdivisions $\Delta'_X \to \Delta_X$. The proof of this goes through word-for-word in the toroidal case - so proper "allowable" toroidal modifications correspond with subdivisions. Also, it is not difficult to describe when such a modification is projective.

Finally, recall that an affine toric variety is nonsingular if and only if the corresponding cone is generated by a subset of an integral basis for N. The same works in the toroidal case, if we replace N by the appropriate group as mentioned above.

The following theorem is thus relatively easy to prove:

6.4. **Theorem.** For any toroidal embedding $U \subset X$ there is a projective toroidal modification $U \subset X' \to X$ such that X' is nonsingular.

See [KKMS], theorem 11*.

Part 2. Exercises

0. Methods of proof: a guide to the exercises

One striking feature of the proofs of de Jong's theorem and its derivatives is, that all the ingredients, with the exception of one beutiful subtle result (the Three Point Lemma), were known and understood nearly two decades before. The way they are put together is quite ingenious. The purpose of this section is to help you understand how they are put together, once you put the effort into solving the exercises.

0.1. **First steps.** The proof of de Jong's theorem starts with a series of simple reduction steps. These are discussed in the exercises in section I. Thus one may assume that the variety X is projective and normal, and the subset Z is the support of a Cartier divisor. Moreover, one may replace Z by a larger divisor. Using a fairly standard projection argument one may assume that we have a morphism $X \to P$ which makes X into a generically smooth family of curves, satisfying some nice properties, in particular $Z \to P$ is finite and unramified. Taking a separable alteration of the base P, we may assume that Z is the union of sections $s_i : P \to X$.

0.2. **Simplifying the fibers.** De Jong's idea is to simplify the fibers of the morphism $X \to P$. Then by induction on dimension one can simplify the base P, and finally put these simplifications together.

The method of simplifying the fibers uses the deepest ingredient in the program: the theory of moduli of curves. See section 5 for discussion. Many exercises are available in sections II, III and IV. Using this theory, it is easy to see that one can find a diagram as follows:

$$\begin{array}{ccc} X_1 & \overset{\beta}{\dashrightarrow} & X \\ \downarrow & & \downarrow \\ P_1 & \to & P \end{array}$$

where $X_1 \to P_1$ is a family of stable pointed curves, where the "points" arise from the sections coming from Z. But in order for the inductive argument to work one needs $\beta : X_1 \dashrightarrow X$ to be an alteration, not a rational map! Here comes the following result:

0.3. **Three Point Lemma (de Jong).** Suppose Z meets the smooth locus of every irreducible component of every fiber in at least three points. Then $\beta : X_1 \to X$ extends to a morphism.

See exercises in section V.

Once this lemma is proven, one simply needs to use induction on the dimension in order to resolve the base P and make the degeneracy locus into a divisor of normal crossings. A simple set of blowups (see exercises in section VI) finish the proof of theorem 4.3.

0.4. **The variant theorems.** Proving the variant theorems 4.4 and 4.5 goes by relatively easy modifications of the proof of 4.3. These modifications are discussed in VII. Basically, some of the good properties of the projection $X \to P$ used in the proof of 4.3 do not necessarily hold, therefore a little more work needs to be done. An important ingredient in the proof of 4.5 is the *uniqueness* of stable reduction of pointed curves.

0.5. **Weak resolution in characteristic 0.** Both new proofs of the weak resolution theorem start with reduction steps as in de Jong's theorem. Once the projection is produced, the methods diverge.

The proof by Bogomolov and Pantev, discussed in exercises in section IX relies on a variation of a lemma of Belyi. One arrives at a situation where X is a finite cover of a smooth variety P branched over a divisor of normal crossings. It follows that X carries a toroidal structure, and therefore can be resolved easily. In a way, this is a generalization of the old method of Jung (also known as “Abhyankar's dream” for resolutin in positive characterisic).

The proof by Abramovich and de Jong follows de Jong's theorem more closely. One uses the stable reduction method to cover $X \to P$ by a family of nodal curves $X_1 \to P_1$. Some easy steps allow to make this into a finite Galois cover. After modifying the base P one may assume that X_1 carries a toroidal structure. A simple argument provides an equivariant modification of X_1 such that X_1/G is toroidal, and therefore can be resolved.

Both methods of proof seem to come close to resolving singularities in positive characteristic *up to a purely inseparable alteration*, but both lack crucial steps to produce such a result!

I. Simple reduction steps for de Jong's theorem

I.1. Suppose you want to prove de Jong's theorem 4.3. First you want to perform some elementary reductions:

1. Show that you may replace X by a modification $X' \to X$ while replacing Z by its inverse image in X'.
2. By blowing up, show that you may assume that Z is the support of a Cartier divisor.
3. If $Z_i \subset X$ are the supports of a Cartier divisors and $Z_1 \subset Z_2$, show that to prove the theorem for (X, Z_1) it suffices to prove it for (X, Z_2).
4. Show, using Chow's Lemma, that we may assume X is projective.
5. Show, using the first exercise, that we may assume X is normal.

I.2. The next step is to produce a projection with some nice properties. Let $X \subset \mathbb{P}^N$ be a projective variety of dimension n.

1. Assume $\dim X < N - 1$. Show that, if p is a general point in $\mathbb{P}^N$, then the projection of X through p to $\mathbb{P}^{N-1}$ is finite and birational onto its image.
2. Assume $\dim X = N - 1$. Show that if p is a general point in $\mathbb{P}^N$, then the projection of X through p to $\mathbb{P}^{N-1}$ is finite and generically étale onto $\mathbb{P}^{N-1}$.
3. In the situation $Z \subset X$ in the theorem, show, using a projection through a general plane of dimension $N - n$, that there is a finite set S of regular closed points in $X \setminus Z$, such that if $\tilde{X}$ is the blow up of X at S, then there is a morphism $f : \tilde{X} \to \mathbb{P}^{n-1}$, satisfying the following conditions:

(a) f is equidimensional of relative dimension 1
(b) The smooth locus of f is dense in all fibers
(c) Let $Z' = \phi^{-1}Z$. Then $f|_{Z'}$ is finite and generically étale
(d) At least one fiber of f is smooth
(e) The morphism f has connected fibers.

I.3. For the sake of the Three Point Lemma, it will be useful to enlarge Z so it meets every fiber sufficiently. This is done as follows:

1. Let X be a projective variety, $C \subset X$ an irreducible curve, L a very ample line bundle. Let $\mathbb{P} = \mathbb{P}(H^0(X, L^{\otimes n}))$ and let $\phi_n : X \to \mathbb{P}$ be the associated morphism. Then the image of C does not lie in any linear projective subspace of dimension $< n$.
2. Now let X be as in the theorem, with a projection $X \to P$ as produced above. Let $C \subset X$ be an irreducible component of a geometric fiber of $X \to P$. Show that there is a divisor $H \subset X$ such that
   (a) $f|_H : H \to P$ is finite and generically étale, and
   (b)
   $$\#sm(X/P) \cap C \cap H \geq 3.$$
   (Here we count the points without multiplicities.)
3. apply Noetherian induction to obtain:
   Let $X \to P$ be as above. There exists a divisor $H \subset X$ such that
   (a) $f|_H : H \to P$ is finite and generically étale, and
   (b) for any irreducible component C of a geometric fiber of f, we have
   $$\#sm(X/P) \cap C \cap H \geq 3.$$
   Again, we count the points without multiplicities.

II. **Exercises on moduli of curves**

II.1. **Definition (Deligne - Mumford).** Let k be an algebraically closed field, and let C be an algebraic curve defined over k. Suppose that C is complete and connected. Let $g \in \mathbb{Z}_{\geq 2}$. We say that C is a *stable* curve of genus g if:

1. $\dim_k(H^1(C, \mathcal{O}_C)) = g$ (i.e. C has arithmetic genus g),
2. the singularities of C are normal crossings of two regular branches (i.e. a point on C is either regular, or the completion of its local ring is on two parameters with equation $x{\cdot}y = 0$),
3. if there is a component of C which is regular and rational (i.e. isomorphic to $\mathbb{P}^1$), then it has at least three points in common with other components.

II.2. **Definition.** Let K be a field, let k be an algebraically closed field containing K. Let C be an algebraic curve defined over K. We say that C is a stable curve if $C \otimes k$ is stable in the sense just defined.

II.3. **Definition.** A nodal curve $C \to S$ such that all fibres are stable (as defined above) is called a stable curve.

II.4. **Exercise.** Assume all conditions in the definition up to (2). Show that then (3) is equivalent by requiring:

3'. $\mathrm{Aut}(C)$ is a finite group.

II.5. **Exercise.** Show these definitions coincides with the one previously given.

II.6. **Exercise.** Give examples of stable curves, e.g. describe all "types" of stable curves of genus 3.

II.7. **Definition (Deligne and Mumford, Knudsen).** Let k be an algebraically closed field, and let C be an algebraic curve defined over k. Suppose that C is complete and connected. Let $P_1, \cdots, P_n \in C(k)$ be points on C. Let $\dim_k(H^1(C, \mathcal{O}_C)) = g$ (i.e. C has arithmetic genus g). Suppose:

$$2g - 2 + n > 0.$$

We write $S = \mathrm{Sing}(C)$ for the set of singularities. We say that $(C, P_1, \cdots, P_n)$ is a *stable n-pointed curve of genus g* if:

1. every singular point of C is a normal crossing of two regular branches,
2. for $1 \leq i < j \leq n$ we have $P_i \neq P_j$,
3. for every $1 \leq i \leq n$ we have $P_i \notin S$,
4. if there is a component of C which is regular and rational, then then number of points either in S or among $\{P_1, \cdots, P_n\}$ on that component is at least three.

II.8. **Exercise.** Assume all conditions in the definition up to (3). Show that then (4) is equivalent by requiring:

4'. $\mathrm{Aut}((C, P_1, \cdots, P_n))$ is a finite group.

II.9. **Exercise.** Consider all stable n-pointed curves of genus g. Suppose that

$$2g - 2 + n \geq 3.$$

1. Show that there exists such a curve which has no nontrivial automorphisms.
2. (variant:) Show that such a generic curve has no nontrivial automorphisms.
3. (variant:) Show that such a general curve has no nontrivial automorphisms.

II.10. **Definition.** Let K be a field, let k be an algebraically closed field containing K. Let C be an algebraic curve defined over K, and let $P_1, \cdots, P_n \in C(K)$. We say that $(C, P_1, \cdots, P_n)$ is a stable n-pointed curve if $(C, P_1, \cdots, P_n) \otimes k$ is stable n-pointed in the sense just defined.

II.11. **Exercise.** Let $n > 2g+2$ and let $(C, P_1, \cdots, P_n)$ be any stable n-pointed curve of genus g. Suppose that C is *regular* (and hence irreducible). Show that

$$\mathrm{Aut}((C, P_1, \cdots, P_n)) = \{1\}$$

(if you want, assume that $\mathrm{char}(k) = 0$).

II.12. **Exercise.** Choose some g (e.g. $g = 3$), choose a very large integer n (e.g. $n = 1997$), and construct a stable n-pointed curve of genus g which has a non-trivial group of automorphisms.

Variant: Let $2g - 2 + n \geq 2$; show that there exist stable n-pointed curves of genus $g \geq 3$ in codimension two in the moduli space with non-trivial groups of automorphisms.

III. **The j-line**

In this exercise we analyse "moduli of elliptic curves" The motivation is: in history people looked for a long time for the right concept, it seems that Grothendieck with his notion of "representable functor" pinnned down the precise concept, but people soon found out that it did not give what you want in all cases[1].

Strictly speaking this set of exercises will not be used, but it can be used as a blueprint for al cases of moduli space of curves (without markings, without level structures).

Advice: if you know enough about ℓ-adic representations we suggest you substitute the exercise sets III, IV by more advanced problems and methods! For example see how to do the exercises mentioned, and also provide exercises and solutions for curves of arbitrary genus.

Let us do an easy case. Suppose $\operatorname{char}(K) \neq 2, \neq 3$, suppose E is is given over K by the equation $Y^2 = X^3 + AX + B$, with $4A^3 + 27B^2 \neq 0$. Then define

$$j(E) := 1728{\cdot}4{\cdot}A^3/(4A^3 + 27B^2).$$

Note: for any elliptic curve, there is a general definition of the j-invariant (see [Silv]).

Suppose that E_t is given by the equation

$$Y^2 = X^3 + aX + a, \quad \text{with} \quad a = \frac{27{\cdot}t}{4(1728 - t)}.$$

III.1. **Exercise.** For $t \neq 0, \neq 1728$ compute $j(E_t)$. Conclude: there exists a familiy of elliptic curves

$$\mathcal{E} \to \operatorname{Spec}(K\Big[j, \frac{1}{j}, \frac{1}{j - 1728}\Big]) \subset \mathbb{A}^1_K$$

where each fibre E_t has the j-value equal to t.

However this family we do not call a "universal family", it does not represent the functor of elliptic curves (see below).

III.2. * **Exercise.** Show that any family

$$\mathcal{E} \to \operatorname{Spec}(K\Big[j, \frac{1}{j}, \frac{1}{j - 1728}\Big]) \subset \mathbb{A}^1_K$$

having the property $j(E_t) = t$ cannot be extended to a family over the point $t = 0$ or $t = 1728$.

If you want, take $K = \mathbb{C}$, and use analytic methods, or do smething else... In the course we shall discuss methods which prove such statements "in general".

III.3. * **Exercise.** Let $K = \mathbb{C}$. How many families (up to isomorphisms) have the properties mentioned in (2) ?

Define a moduli functor for elliptic curves which do not have $j = 0$ or $j = 1728$, show it admits a coarse moduli space, show it does not admit a fine moduli space.

[1]See the introductory lines in the paper P. Samuel - invariants arithmétiques des courbes de de genre 2 [d'après Jun Ichi Igusa]. Sém. Bourbaki **14** (1961/62) N. 228: "Signalons ausitôt que le travail d' IGUSA ne résoud pas pour les courbes de genre 2, le "problème des modules tel qu'il a été posé par GROTHENDIECK à diverses reprises dans ce Séminaire."

III.4. **Exercise.** Let $g \in \mathbb{Z}_{\geq 3}$. Show there exists a non-empty Zariski-open set $U \subset M_g$ and a smooth curve $\mathcal{C} \to U$ which gives the moduli map, i.e. for every geometric point u in U, the fibre $\mathcal{C}_u$ defines the moduli point $[\mathcal{C}_u] = u \in M_g$. Show that once U is choosen, this family is unique.

IV. Stable reduction for elliptic curves.

In this exercise we illustrate the concept of stable reduction by studying the case of elliptic curves (i.e. curves of genus one with one marked point). In this case examples are easy to give because in many cases we can choose plane models (Weierstrass equations). These exercises can be used at motivation for more abstract methods which apply for higher genus.

IV.1. **Notation.** Let $R = R_v$ be a discrete valuation ring, with $K = \text{fract}(R)$ its field of fractions, and $k = R_v/m_v$ the residue class field. An elliptic curve E defined over K is said to have *good reduction* (at the given valuation) if there exists a smooth proper morphism $\mathcal{E} \to \text{Spec}(R)$ with generic fibre isomorphic to $E \to \text{Spec}(K)$.

IV.2. **Exercise.** Suppose $R = k[T]$, with $\text{char}(k) \neq 2, \neq 3$, and let E over $K = k(T)$ be given by the equation $Y^2 = X^3 + T^6$. Show that E has good reduction at the valuation given by $v(T) = 1$.

We say that E has *bad reduction* at v if it does not have good reduction at v.

IV.3. **Exercise.** Suppose $R = k[S]$, with $\text{char}(k) \neq 2, \neq 3$, and let E over $K = k(S)$ be given by the equation $Y^2 = X^3 + S$. Show that E has bad reduction at the valuation v given by $v(S) = 1$.

IV.4. **Definition.** We say that E has *potentially good reduction*, if there exists a finite extension $K \subset L$, where B is the integral closure of R in L, and w a valuation over v, such that $E \otimes L$ has good reduction at w.

IV.5. **Exercise.** Suppose $R = k[S]$, with $\text{char}(k) \neq 2, \neq 3$, and let E overt $K = k(S)$ be given by the equation $Y^2 = X^3 + S$. Show that E has potentially good reduction at the valuation v given by $v(S) = 1$.

IV.6. **The case of Weierstrass equations.** Suppose that E is given over K by a Weierstrass equation with coefficients in R (see [Silv]). Such an equation defines a plane curve $\mathcal{E} \subset \mathbb{A}^1_B$ over $\text{Spec}(R) = B$, and it is easy to see that the curve $E_0 := \mathcal{E} \otimes_R k$ is irreducible and has at most one singular point. If this point is a cusp, we say that this reduction is of additive type, if it is a node we say that this reduction is of multiplicative type, or we say in this case the reduction is stable. We say "potentially stable" if we can find a model which has stable reduction over a finite extension.

IV.7. **Exercise.** Suppose $R = k[T]$, with $\text{char}(k) \neq 2$, and let E over $K = k(T)$ be given by the equation $Y^2 = X^2 \cdot (X - 1)$. Show that any model of this curve given by a Weierstrass equation has stable reduction at the valuation given by $v(T) = 1$ (and conclude that this curve has potentially bad reduction).

IV.8. **Exercise.** Let R be a DVR, with residue characteristic $\neq 2$. Let E be an elliptic curve over K. Show this curve has potentially stable reduction !

IV.9. * **Exercise.** Let R_v be a DVR, with residue characteristic $\neq 3$. Suppose E is an elliptic curve over K given by a Weierstrass equation $E = \mathcal{Z}(F) \subset \mathbb{P}^2_K$ such that all flex points of E have coordinates in $L \supset K$, with w a discrete valuation of L over v. Show that $E \otimes_K L$ has stable reduction at w.

IV.10. **Conclusion.** Every elliptic curve over a field K with a discrete valuation has potentially stable reduction at that valuation.

In the course we show this to be true for regular curves of arbitrary genus.

IV.11. * **Exercise.** Let R_v be a DVR, and E an elliptic curve of the fraction field K. Show that E has potentially good reduction at R_v if and only if $j(E) \in R_v$. - Formulate (and prove?) the same result for curves of arbitrary genus. - If E is an elliptic curve over a field K, and $\mathrm{End}_K(E) \neq \mathbb{Z}$, then E has potentially good reductions at every place of K.

V. Exercises on the Three Point Lemma

V.1. **Performing stable reduction.** Let $X \to P$ be a family of curves satisfying the conditions in I.2. Suppose $Z \subset X$ satisfies the conditions in I.3. We first perform stable reduction to this situation

1. Show that, by replacing P by a separable alteration and X by the strict transform, we may assume that there are sections $s_i : P \to X$ such that Z is the union of their images.
2. Use theorem 5.15 to show that there exists an alteration $P_1 \to P$, a family of pointed stable curves $C \to P_1$, $\quad \tau_i : P_1 \to C$ and an open set $U_1 \subset P_1$ with an *isomorphism* $C|_{U_1} \to X \times_P U_1$ carrying τ_i to σ_i.

We may as well replace P by P_1 and X by its strict transform. We thus have a birational map $C \dashrightarrow X$ carrying τ_i to σ_i. We want it to extend to a morphism, at least after a modification of P!

V.2. **The graph.** Let $T \subset X \times_P C$ be the closure of the graph of this rational map. We have two projection maps $pr_1 : T \to C$ and $pr_2 : T \to X$.

1. Use the flattening lemma to show that there exists a modification P' of P such that the strict transform of X, and the closure of the graph of $C \dashrightarrow X$ are both flat. Thus we might as well assume $X \to P$ and $T \to P$ are flat.
2. Let $p \in P$ be a geometric point, X_p, T_p, C_p the fibers over p. Show that there exists a finite set $W \subset X_p$ such $T_p \to X_p$ is finite away from W.
3. If $x \in X_p \setminus W$, conclude that there is an open set $V \subset X$ such that $pr_2^{-1}V \to V$ is finite and birational.
4. If $x \in sm(X_p) \setminus W$ show that we may choose V to be normal; thus $pr_2^{-1}V \to V$ has to be an isomorphism in this case.
5. Show that if $X' \subset X_p$ is an irreducible component, then there is a *unique* irreducible component T' of T_p mapping finitely and birationally onto X' via $pr_2 : T \to X$.
6. By repeating the argument for $pr_1 : T \to C$, sho that if $C'' \subset C_p$ is an irreducible component, then there is a unique irreducible component T'' of T_p mapping finitely and birationally onto C'' via $pr_1 : T \to C$.
7. Let $x \in sm(X_p)$ be a closed point. By considering the Stein factorization $T \to \tilde{X} \to X$, show that the fiber $pr_2^{-1}(x)$ is connected.

V.3. **Using the three point assumption.** Let $X' \subset X_p$ be an irreducible component, and $T' \subset T_p$ the unique component mapping finitely onto it, as in exercise 5 above. We will prove that $pr_1 : T' \to C$ is non-constant. Assume by contradiction that $pr_1(T') = c$ is a point.

We will now use the three point assumption. Let $\sigma_i : P \to X, i = 1, \dots, 3$ be three of the given sections such that $\sigma_i(p) = x_i$ are three distinct points on $sm(X')$. Let $T_i = pr_2^{-1} x_i$. Let $\tau_i(p) = c_i \in C_p$.

1. Show that the point $t_i = (c_i, x_i) \in C \times X$ is in T.
2. Assume $c \notin \{c_i, i = 1, \dots, 3\}$. Show that each of $T_i, i = 1, \dots, 3$ contains an irreducible component T_i' whose image in C is again a curve passing through c. Show that these image components are *distinct*. This contradicts the fact that C_p is nodal.
3. Assume, without loss of generality, $c = c_1$. Repeat the argument of the previous exercise to conclude that there are two distinct components of C_p passing through c, contradicting the fact that C_p has a marked point at $c = c_1$.
4. Conclude that $pr_1 : T \to C$ is finite and birational.
5. Show, using Serre's criterion, that C is normal (condition S_2 follows since $C \to P$ has reduced 1-dimensional fibers and P is normal).
6. Conclude that $T \to C$ is an isomorphism, hence β extends as a morphism.

VI. **Exercises on elementary resolution of nice nodal families**

An important idea in de Jong's work, is that once one has a variety X mapping onto a variety S as a nice family of curves over a nice base, it is easy to resolve its singularities. The purpose of these exercises is to go through the resolution in such a case.

Let $f : X \to S$ be a flat morphism of varieties over an algebraically closed field k, with $n = \dim X = \dim S + 1$. Let $D \subset S$ be a reduced divisor. We make the following assumptions.

- N1 The base S is nonsingular.
- N2 The divisor D has strict normal crossings.
- N3 The morphism f is smooth over $S \setminus D$.
- N4 The morphism $f : X \to S$ is a nodal curve.

Let $x \in X$ be a closed point and let $s = f(x) \in S$. By assumption we may choose a regular system of parameters $t_1, ..., t_{n-1}$ at s such that D coincides on a neighborhood with the zero locus of $t_1 \cdots t_r$ for some $r \leq n - 1$. It can be seen that if x is a singular point of X, then the completed local ring of X at x can be described as

$$(*) \quad k[[u, v]]/(uv - t_1^{n_1} \cdot \dots \cdot t_r^{n_r})$$

.

Step 1: Assume $\operatorname{codim}_X \operatorname{Sing}(X) = 2$.

1. Show that there is an irreducible component $D_1 \subset D$ and $\Sigma_1 \subset \operatorname{Sing}(X)$ such that $f(\Sigma_1) = D_1$.
2. Fix a point $x \in \Sigma_1$, and use formal coordinates as in $(*)$, such that $D_1 = V(t_1)$. Show that the power n_1 of t_1 is > 1.
3. Show that the ideal of Σ_1 in the formal completion is (u, v, t_1).
4. Conclude that $\Sigma_1 \to D_1$ is étale, in particular Σ_1 is nonsingular.
5. Let $X_1 = Bl_{\Sigma_1} X$. Show that $X_1 \to S$ satisfies conditions N1-N4, there is at most one component of $\operatorname{Sing}(X_1)$ over Σ_1, with the exponent n_1 replaced by $n_1 - 2$
6. Conclude by induction that there is a blowup $X' \to X$ centered above $\operatorname{Sing}(X)$, such that X' satisfies N1-N4, and $\operatorname{codim}_X \operatorname{Sing}(X) > 2$.

7. Show that now each component of $\mathrm{Sing}(X')$ is defined by $u = v = t_i = t_j$ in equation $(*)$, in particular it is nonsingular.

Step 2: Assume $\mathrm{codim}_X \mathrm{Sing}(X) > 2$. Define $Z = f^{-1}D$. Unfortunately here we need to abandon the structure $X \to B$ of a family of nodal curves. Instead we look at X itself. The situation is as follows:

T1 whenever x is a nonsingular point of X, Z has normal crossings at x.
T2 whenever $x \in \mathrm{Sing}(X)$, we have formal description

$$(**) \quad k[[u,v]]/(uv - t_1 \cdots t_s), \quad 2 \le s \le r \le n-1$$

and $Z = V(t_1 \cdots t_r)$.
T3 All components of $\mathrm{Sing}(X)$ are nonsingular.

1. Let $E \subset \mathrm{Sing}(X)$ be an irreducible component. Show that the blowup $Bl_E X$ satisfies T1-T3, and its singular locus has one fewer irreducible component.
2. Conclude by induction that there is a resolution of singularities $X' \to X$.

VII. **Modifications of the proof for the variant theorems**

VII.1. First we consider some possible variants of the elementary reduction steps, and their consequences. Thus, consider the situation $X \to P$ we have obtained in the elementary reduction steps, see exercises in section I. We will try to see what happens if we drop some of the nice properties it had.

1. Using an alteration, show that the condition that $Z \to P$ be finite is unnecessary.
2. Show that, if one is willing to accept inseparable alterations in the theorem, the condition that $Z \to P$ be generically étale is unnecessary.
3. * By reviewing the arguments, show that the condition that every fiber of $X \to P$ be generically smooth is unnecessary. (Here a modification of the three point lemma may be necessary!)
4. Using the flattening lemma and the previous exercise, show that the condition that $X \to P$ be equidimensional is unnecessary.
5. Show that, if one is willing to accept inseparable alterations in the theorem, the condition that the generic fiber of $X \to P$ be smooth is unnecessary.

VII.2. Let us address the Theorem on semistable reduction up to alteration 4.4. Suppose $S = \mathrm{Spec}\ R$ where R is a discrete valuation ring, and $X \to S$ is a morphism as in the theorem, and Z a proper closed subset.

1. Show that we may assume X projective over S, that Z is the support of a Cartier divisor.
2. Show that we may assume that the generic fiber is a normal variety, and that X is a normal scheme.
3. $*$ Show that we may assume that the special fiber X_s is generically reduced.
4. Let d be the dimension of X_η. Produce a projection $X \to \mathbb{P}^{d-1}_S$ with connected fibers.
5. Use the semistable reduction argument, with the Three Point Lemma, to replace X by a nice family of curves $X \to P \to S$.
6. Use induction on the dimension to conclude the proof of the theorem.
7. Can you think of other situations where a similar theorem can be proven, where S is not necessarily the spectrum of a discrete valuation ring? (This is interesting even in characteristic 0!)

VII.3. We now address the equivariant version of the theorem. Suppose X is a variety, Z a proper closed subset and G acts on X stabilizing Z. We wish to prove theorem 4.5.

1. Produce an equivariant version of Chow's lemma, so that we may assume X is projective.
2. Show that we may assume Z is a divisor.
3. Show that we may replace Z by a bigger *equivariant* divisor; in particular we may assume Z contains the fixed point loci of elements in G.
4. Using a projection of X/G, show that we may assume we have an equivariant projection $X \to P$ making X into a nice family of curves.
5. Consider the case $X = \mathbb{A}^2_k$ where $\mathrm{char} k = p$, and $G = \mathbb{Z}/p$ acting via $(x, y) \mapsto (x, x + y)$. Show that the fixed point set maps inseparably to the image. In particular, the map $Z \to P$ in the previous exercise might be inseparable!
6. making an alteration "Galois": Given a variety W, a finite group action $H \subset \mathrm{Aut}\, W$ and an alteration $V_0 \to W$, show that there exists an alteration $V \to V_0$, and a finite group H' with a surjection $H' \to H$, and a lifting of the H action $H' \subset \mathrm{Aut}(V \to W)$ such that the extension of fixed fields $K(W)^H \subset K(V)^{H'}$ is purely inseparable.
7. $*$ Use the uniqueness in the stable reduction theorem to show that there is an alteration $P' \to P$, a family of stable pointed curves $C \to P'$ and a finite group G' with a quotient $G' \to G$ and a diagram
$$\begin{array}{ccc} C & \to & X \\ \downarrow & & \downarrow \\ P' & \to & P \end{array}$$
on which G' acts equivariantly, such that C is birational to $\widetilde{X_{P'}}$, and the extensions $K(X)^G \subset K(C)^{G'}$ and $K(P)^G \subset K(P')^{G'}$ are purely inseparable.
8. Use induction on the dimension and a suitable modification of the elementary blowups argument to conclude the theorem.

VIII. **Exercises on toric varieties and toroidal embeddings**

1. Show that $\mathbb{G}_m^n \subset \mathbb{A}^n$ is a toric variety. Describe its cone.
2. Show that $\mathbb{G}_m^n \subset \mathbb{P}^n$ is a toric variety. Describe its fan.
3. Let $X \subset \mathbb{A}^n$ be a normal variety defined by monic monomial equations of type
$$\prod x_j^{n_j} = \prod x_j^{m_j}.$$
Show that X is toric. (Identify the torus!)
4. Now do the same if the monomial equations are not necessarily with coefficients $= 1$.
5. Describe the cone associated to the affine toric variety defined by
$$xy = t_1^{k_1} \cdots t_r^{k_r}.$$
6. Look at the affine 3-fold $xy = zw$. Let $X' \to X$ be the blowup of X at the ideal (x, z). Describe this blowup, show that it is toric, and describe the cone subdivision associated to it.
7. Let $X = \mathbb{A}^2$, $D = \{xy(x + y - 1) = 0\}$, $U = X \setminus D$. Show that $U \subset X$ is a toroidal embedding. Describe its conical polyhedrral complex. (Compare with the fan of $\mathbb{P}^2$!)
8. Now do the same for $D = \{y(x^2 + y^2 - 1) = 0\}$. Show that the resulting complex can not be linearly embedded in a vector space.
9. Consider the surface $X = \{z^2 = xy\}$, $U = \{z \neq 0\}$. Show that $U \subset X$ is toric and describe its cone.

10. Consider the surface X above, let $D_1 = \{x = 0\}$, $D_2 = \{y = x(x-1)^2$ and $z = x(x-1)\}$. Let $U = X \setminus (D_1 \cup D_2)$. Show that $U \subset X$ is toroidal. Describe its conical polyhedral complex. Make sure to describe the integral structure!

IX. **Weak Resolution according to Bogomolov and Pantev**

Granted toroidal resolution, the proof of weak resolution in characteristic 0 by Bogomolov and Pantev is arguably the simplest available. It does not even require surface resolution, or moduli spaces.

We will now go through this proof. The steps of proof here include some simplifications on the arguments in [B-P], which came up in discussions with T. Pantev. These and additional simplifications were discovered independently by K. Paranjape, and we have used his beautiful exposition in some of the following exercises.

IX.1. Let X be a variety over an algebraically closed field of characteristic 0, and $Z \subset X$ a proper closed subset. Let $n = \dim X$, and again assume we know the weak resolution theorem for varieties of dimension $n-1$.

First a few reduction steps:

1. Show that we may assume X projective and normal, and Z the support of a Cartier divisor.
2. Show that there is a finite projection $X \to \mathbb{P}^n$.
3. Let $P \to \mathbb{P}^n$ be the blowup at a closed point. Show that $P \simeq \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1))$. Denote by E the exceptional divisor of $P \to \mathbb{P}^n$.
4. By blowing up a general point on $\mathbb{P}^n$, and blowing up X at the points above, show that we may assume we have a finite morphism $f : X \to P$, which is étale along E, such that the image of Z is disjoint from E, and maps finitely to $\mathbb{P}^{n-1}$.
5. By the Nagata - Zariski purity theorem, note that the branch locus of $X \to P$ is a divisor B mapping finitely to $\mathbb{P}^{n-1}$.

We now replace Z by $Z \cup f^{-1}B$.

IX.2. The next steps are aimed at replacing P by another $\mathbb{P}^1$-bundle over $\mathbb{P}^{n-1}$, such that the branch locus of $X \to P$ becomes simpler. Let Y be any variety, F a rank-2 vector bundle on Y, $P = \mathbb{P}_Y(E)$. Let $E \subset P$ be a divisor which is a section of $\pi : P \to Y$ and let $D \subset P$ be another effective divisor disjoint from E. Denote by $\mathcal{O}_P(1)$ the tautological bundle, and by d the relative degree of D over Y.

1. Consider the exact sequence
$$0 \to \mathcal{I}_D(d) \to \mathcal{O}_P(d) \to \mathcal{O}_P(d)|_D \to 0.$$
Show that there is an invertible sheaf $\mathcal{L}_D$ on Y such that $\mathcal{I}_D(d) \simeq \pi^*\mathcal{L}_D$.
2. If $D_1, D_2 \subset P$ are any two *disjoint* divisors finite of degree d over Y, show that there is an embedding of vector bundles $\mathcal{L}_{D_1} \oplus \mathcal{L}_{D_2} \subset sym^d F$ inducing a *surjection* $\pi^*(\mathcal{L}_{D_1} \oplus \mathcal{L}_{D_2}) \to \mathcal{O}_P(1)$.
3. Assume the characteristic is 0. Consider the case $D_1 = dE, D_2 = D$. show that the resulting morphism $P \to \mathbb{P}_Y(\mathcal{L}_{D_1} \oplus \mathcal{L}_{D_2}) = P'$ maps E to a section E' and D to another section; and its branch locus is of the form $(d-1)E + D'$ where D' has degree $d-1$ over Y and is disjoint from E'.

4. Continue by induction to show that there is a $\mathbb{P}^1$ bundle $Q \to Y$ and a morphism $g : P \to Q$ over Y, such that the image of D and the branch locus of g form a union of sections of $\pi_Q : Q \to Y$.

IX.3. Back to our theorem, where $Y = \mathbb{P}^{n-1}$. Composing with the morphism $f : X \to P$, we obtain that the image $g(f(Z)) \subset Q$ is the image D_1 of a section $s_1 : \mathbb{P}^{n-1} \to Q$ of $Q \to \mathbb{P}^{n-1}$ and the banch locus of $g \circ f$ is the union of images D_i sections $s_i : \mathbb{P}^{n-1} \to Q$ as well. Denote $\Delta = \pi_Q(\cup_{i \neq j} D_i \cap D_j)$.

The following steps use moduli theory; however it has been shown (in the preprint version of [B-P], and in Paranjape's exposition) that this can be circumvented.

1. Use the stable reduction argument to show that there is a *modification* $Y' \to Y$, and a modification $Q' \to Q \times_Y Y'$ such that $Q' \to Y$ is a family of nodal curves of genus 0, and the sections lift to *disjoint* sections $s'_i : Y' \to Q'$. We now replace Y by Y', Δ by its inverse image, etc.
2. Use induction on the dimension to replace Y by a nonsingular variety such that Δ becomes a strict divisor of normal crossings.
3. Use either toroidal geometry, or exercise set VI, to replace Q by a nonsingular variety, such that the inverse image of D is a strict divisor of normal crossings.
4. Let $\tilde{X}$ be the normalization of Q in the function field of X. Use Abhyankar's lemma to show that $\tilde{X}$ has a toroidal structure, such that the inverse image $\tilde{Z}$ of Z is a toroidal divisor.
5. Conclude that there is a weak resolution of singularities $r : X' \to X$ such that $r^{-1}Z$ is a strict divisor of normal crossings.

X. **The Torific Ideal**

The weak resolution argument according to Abramovich - de Jong starts very much like de Jong's theorem: a projection $X \to P$ is produced, and a Galois alteration $P_1 \to P$ over which one has stable reduction $X_1 \to P_1$, equivariant under the Galois group G, is produced. Induction on the dimension for P allows one to assume that X_1 and P_1 are toroidal, and the Galois action on P_1 is toroidal as well. The only point left is to make the group action on X_1 toroidal, so that the quotient should be toroidal, and therefore admit toroidal resolution. The question boils down to the following situation:

Let $T_0 \subset X_0$ be an affine torus embedding, $X_0 = \mathrm{Spec}\ \ R$. Let $G \subset T_0$ be a finite subgroup of T_0, let $p_0 \in X_0$ be a fixed point of the action of G, and let ψ_u be a character of G. Consider the torus embedding of $T = T_0 \times \mathrm{Spec}\ \ k[u, u^{-1}]$ into $X = X_0 \times \mathrm{Spec}\ \ k[u]$, and let G act on u via the character ψ_u. Write $p = (p_0, 0) \in X$ and write $D = (X_0 \setminus T_0) \times \mathrm{Spec}\ \ k[u]$. We wish to find a canonical blowup $X_1 \to X$, such that if $U \subset X_1$ is the inverse image of T_0, then it is a toroidal embedding, and the group G acts toroidally.

X.1. **The ideal.** Let $M \subset R[u]$ be the set of monomials. for each $t \in M$ let χ_t be the associated character of T, and let let $\psi_t : G \to k^*$ be the restriction of χ to G. Define $M_u = \{t | \psi_t = \psi_u\}$, the set of mononials on which G acts as it acts on u. Defing $I_G = \langle M_u \rangle$, the ideal generated by M_u.

X.2. **Exercise (canonicity).** Show that if X'_0, T'_0, G', p'_0 and ψ'_u is a second set of such data, and if we have an isomorphism of formal completions

$$\varphi : \hat{X}_p \cong \hat{X'}_{p'},$$

which induces isomorphisms $G \cong G'$ and $\hat{D}_p \cong \hat{D'}_{p'}$, then φ pulls back I_G to the ideal $I_{G'}$.

X.3. **Exercise (gluing property).** If q_0 is any point of X_0 and if $G_q \subset G$ is the stabilizer of $q = (q_0, 0)$ in G, then the stalk of I_G at q is the same as the stalk of I_{G_q} at q.

X.4. **Exercise.** Show that I_G is generated by u and a finite number of monomials $t_1, \ldots , t_m$ in $M_u \cap R$.

X.5. **Exercise.** Let $X' = B_{I_G}(X)$ be the blowup. Let X'_u be the chart with coordinates $u, t_j/u$. Show that the action of G on X'_u is toroidal.

X.6. **Exercise.** Ley X'_i be the chart on X' with coordinates $t_i, v = u/t_i, s_j = t_j/t_i$. Show that G acts trivially on v, and that $X'_i = \text{Spec } R'_i[v]$ where R'_i is generated over R by s_j.

X.7. **Exercise.** Let X_1 be the normalization of X'. Show that if $U \subset X_1$ is the inverse image of T_0, then it is a toroidal embedding, and the group G acts toroidally.

References

[ℵ-dJ] D. Abramovich and A. J. de Jong, *Smoothness, semistability and toroidal geometry*, J. Alg. Geom., to appear. `alg-geom/9603018`

[ℵ-W] D. Abramovich and J. Wang, *Equivariant resolution of singularities in characteristic 0*, Math. Res. Letters, toappear. `alg-geom/9609013`

[A-W] M. Artin and G. Winters, *Degenerate fibres and reduction of curves.* Topology **10** (1971), 373-383.

[BerBour] P. Berthelot, *Altérations de variétés algébriques [d'après A. J. de Jong].* Sém. Bourbaki 1995-96, Exp. 815.

[B-P] F. Bogomolov and T. Pantev, *Weak Hironaka theorem*, Math. Res. Lett. 3 (1996), no. 3, 299–307.

[Bry] J-L. Brylinski, *Propriétés de ramification à l'infini du groupe modulaire de Teichmüller.* Ann. Sci. Ecole Norm. Sup. (4) **12** (1979), 295-333.

[D] P. Deligne, *Le lemme de Gabber.* In: Sém. sur les pinceaux arithmétiques: la conjecture de Mordell (Ed. L.. Szpiro). Astérisque **127** (1985), 131-150.

[D-M] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus.* Publ. IHES, No. 36 (1969) (Volume dédié au Professor Oscar Zariski à l'occasion de son 70^e anniversaire), pp. 75 - 109.

[E-H] D. Eisenbud and J. Harris, *Schemes: the language of modern algebraic geometry.* Wadsworth & Brooks/Cole, 1992.

[F] W. Fulton, *Introduction to toric varieties*, Annals of Math. Studies 131.

[vG-O] B. van Geemen and F. Oort, *A compactification of a fine moduli spaces of curves.* University of Utrecht, Preprint 301, 1983.

[H-M] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves.* Invent. Math. **67** (1982), 43-70.

[Harris] J. Harris, *Algebraic Geometry - a first course.* Grad. texts in Math. **133**, Springer - Verlag, 1992.

[HAG] R. Hartshorne, *Algebraic geometry.* Grad. texts Math. **52**, Springer - verlag, 1977.

[H] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero: I, II,* Ann. of Math. (2) **79** (1964), 109-326.

[KKMS] G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat, *Toroidal Embeddings I*, Springer, LNM 339, 1973.

[Knu] F. F. Knudsen, *The projectivity of the moduli space of stable curves, II: the stacks $M_{g,n}$. III: The line bundles on $M_{g,n}$ and a proof of the projectivity of $\overline{M_{g,n}}$ in characteristic zero.* Math. Scand. **52** (1983). 161-199, 200-221.

[dJ] A. J. de Jong, *Smoothness, semistability, and alterations*, Publications Mathématiques I.H.E.S. **83**, 1996, pp. 51-93.

[dJ2] A. J. de Jong, *Families of curves and alterations*, Annales de l'Institute Fourier, to appear.

[dJ-O] A. J. de Jong and F. Oort, *On extending families of curves*, J. Alg. Geom. **6** no. 3 (1997), 545-562.

[Mos] M. Mostafa, *Die Singularitäten de Modulmannigfaltigkeiten $\overline{M}_g(n)$ der stabilen Kurven vom Geslecht $g \geq 2$ mit n-Teilungspunkstruktur.* Journ reine angew. Math. (Crelle), **343** (1983), 81–98.

[M.C+J] D. Mumford, *Lectures on curves and their jacobians.* University of Michgan Press.

[M.GIT] D. Mumford, J. Fogarty and F. Kirwan *Geometric invariant theory.* Erben. Math. N.Folge, Bd. 34, Springer - Verlag, 1994.

[M.RedBook] D. Mumford, *The red book of varieties and schemes.* Lect. Notes Math. 1358, Springer - Verlag 1988.

[O] F. Oort, *Good and stable reduction of abelian varieties.* Manuscr. Math. **11** (1974), 171 - 197.

[Po1] H. Popp, *Moduli theory and classification theory of algebraic varieties.* Lect. Notes Math. 620, Springer - Verlag 1965.

[Po2] H. Popp, *On the moduli of algebraic varieites III, Fine moduli spaces.* Compos. Math. **31** (1975), 237 - 258.

[Po3] H. Popp, *The singularities of moduli schemes of curves.* Journ. Number theory **1** (199), 90 - 107.

[Ray] M. Raynaud, *Compactification du module des courbes.* Sém. Bourbaki **23**, (1970/71), Exp. 385.

[S-T] J-P. Serre & J. Tate, *Good reduction of abelian varieties.* Ann. Math. **88** (1968), 492-517 [Serre ŒII, 79].

[SGA] A. Grothendieck (with M. Raunaud and D. S. Rim), *Groupes de monodromie en géométrie algébrique (SGA 7 I).* Séminaire de géométrie algébrique du Bois-Marie. LNM 288, Springer - Verlag 1972.

[Silv] J. Silverman, *The arithmetic of elliptic curves.* Grad. texts in Math. **106**, Springer - Verlag, 1986.