Resolution and Reduction in characteristic 0

My preprints available/linked in my home page:
`http://math.bu.edu/INDIVIDUAL/abrmovic`

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I. THE PROBLEMS

1. What is a variety?

In $\mathbb{C}^n$ or in $\mathbb{CP}^n$ take the common zero set of polynomials
$V(f_1, ..., f_n)$ resp. $V(F_1, ..., F_n)$
(F_i homogeneous).

This is what we study.

2. Nice ones: manifolds = nonsingular varieties

These can be studied using: *Topology, differential geometry, complex analysis, algebraic geometry...*

3. What to do with the one which are not manifolds?
Why, of course, compare them with ones which are!
a. examples:

$$xy = 0$$

$$y^2 = x^3$$

$$z^2 = x^2 + y^2$$

b. Terminology for what we are doing:
Polynomial maps are called "morphisms"
e.g. $\mathbb{C}^3 \to \mathbb{C}^2$ defined by

$$(x, y, z) \mapsto (xy, x + z + y^2).$$

(i) **Modification** $X' \to X$: a proper birational morphism.

(ii) **Alteration** $Y \to X$: proper, surjective, generically finite morphism.

$$Y \xrightarrow{\text{mod.}} Y_1 \xrightarrow{\text{fin.}} X.$$

4. **Hironaka's Theorem on Resolution of Singularities (stated in weak form)**. Given an algebraic variety X, there is a modification $X' \to X$ such that X' is nonsingular.

Hironaka actually tells you more (we'll come back to that).

5. That's about varieties. How about morphisms (families of varieties)?
Examples of nice ones:

a. $(x, y) \mapsto y$

b. $(x, y) \mapsto t = xy$

Examples of bad ones:

a. $t = y^2 - x^3$

b. $t = x^2 y$

6. **Semistable reduction problem:** given a family $X \to B$ find a diagram

$$\begin{array}{ccccc} X_1 & \xrightarrow{r} & X' & \xrightarrow{f} & X \\ & \searrow & \downarrow & & \downarrow \pi \\ & & B_1 & \xrightarrow{a} & B \end{array}$$

where a is an alteration, r a modification, such that $X_1 \to B_1$ is "nice".

7. **Motivation (one of many):** Calculate the genus of a quartic plane curve!

II. RESOLUTION vs. REDUCTION THROUGH THE AGES

1. case of curves:

nonsingular algebraic curve
= nonsingular Riemann surface

a. What distinguishes these?

Riemann's theorem: given a bounded holomorphic function f on the punctured disc $\Delta^* \Longrightarrow$ the function f extends over the puncture.

e.g. on $\{xy = 0\} - (0, 0)$ take $u = 1$ on x axis and $u = 0$ on y axis.

b. **Normalization:** take the curve obtained by "adding the rational functions which are bounded" e.g.:

for $y^2 = x^3$ take the set in (x, y, t) space where $t = y/x$, namely $V(t - x^2, t - y^3)$. In fact this is just an incarnation of the t-line

for $xy = 0$ take the set in (x, y, u) space

$$V(xy, uy, (1-u)x).$$

c. Ancient theorem: the normalization of an algebraic curve is nonsingular. (Kronecker?)

2. Embedded curves: want to resolve $C \subset S$, keeping it embedded in a surface.

a. Blowing up: replace point P by all the tangent lines through P

In equations: one coordinate patch: $t = y/x$ namely $y = tx$

e.g.: $y^2 = x^3 \mapsto (xt)^2 = x^3$. Now $x = 0$ is the exceptional curve, factor out: $t^2 = x$.

Theorem ("exercise") given $C \subset S$, where S is a smooth surface, can resolve it by successive blowups at the singular points, so that the inverse image of C has **normal crossings**, namely, in local coordinates: $V(x^k y^l)$.

3. Surfaces.

a. **Theorem. (Zariski)** A surface S can be resolved $S' \to S$ by a succession of blowups and normalizations

b. Alternative: Jung's method.

Exercise: project $S \to \mathbb{P}^2$. The branch locus is a curve $C \subset \mathbb{P}^2$ use embedded resolution to make $C' \subset P'$ normal crossings. Pullback S and get $S' \to P'$. The local equations are (essentially) $z^n = x^k y^l$. Now you can resolve explicitly!!

4. **Semistable reduction - the first generation**.

Given: a 1-parameter family of curves:

$$\begin{array}{c} S \\ \pi \downarrow \\ C \end{array}$$

Want only "$xy = t$"

Let F be the union of the singular fibers. Let their images in C be Δ.

i: desingularize S.
ii: Blow up to make F normal crossings.

iii: Analyse the fibers: $x^{k_i}y^{l_i} = t$.
iv: Take a finite $C_1 \to C$ branched along Δ with multiplicities n_i divisible by these k_i, l_i (so locally $s^{n_i} = t$). Then normalize.

v: Exercise: new equations of the form $uv = s^m$
vi: Exercise: this can be resolved by exactly $[m/2]$ blowups.

5. Semistable reduction for curves in general: Using (4) and moduli theory, can prove:

Theorem (de Jong): Given $X \to B$ a family of curves, and $Z \in X$ a proper closed subset, there is a diagram

$$\begin{array}{ccccc} X_1 & \to & X & \supset & Z \\ \downarrow & & \downarrow & & \\ B_1 & \to & B & & \end{array}$$

6. Resolution in general.

a. Hironaka: resolve by blowing up nonsingular loci over $\operatorname{Sing} X$.

Notoriousely dificult

b. E. Bierstone - P. Milman, O. Villamayor: an explicit procedure. Simplification.

c. Mumford: the special case of monomial equations is easy. This is called **toroidal geometry**.

d. $\aleph$ - A. J. de Jong, also F. Bogomolov - T. Pantev: can use de Jong's ideas plus the toroidal case to resolve relatively easily, using induction.

7. Sketch of proof of $\aleph$ - de Jong
i. project $X \to \mathbb{P}^{n-1}$.

ii. use de Jong's theorem

iii. replace X by X_1/G and P by P_1/G. Special subset $\Delta \subset P$.
iv. Inductive hypothesis: resolve (P, Δ).
v. Now $P_1 \to P$ toroidal. Also $X_1 \to P_1$ toroidal.

vi. **Theorem** ($\aleph$ - de Jong). There is a canonical G-equivariant blowup of X_1 so that X_1/G inherits a toroidal structure.
vii. **Mumford $\Rightarrow$ resolution.**
8. Can now generalize this to get semistable reduction in general.