Contemporary Mathematics

# Algebraic orbifold quantum products

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## 1. Introduction

The purpose of this note is to give an overview of our work on defining algebraic counterparts for W. Chen and Y. Ruan's Gromov-Witten theory of orbifolds. This work will be described in detail in a subsequent paper.

The presentation here is generally based on lectures given by two of us at the Orbifold Workshop in Madison, Wisconsin. Following the spirit of the workshop, our presentation is intended to be understandable not only to algebraic geometers, but also practitioners of the other disciplines represented there, including differential geometers and mathematical physicists.

We make a special effort to make our constructions as canonical as we can, systematically using the language of algebraic stacks. Our constructions are based on the theory of twisted stable maps developed in [ℵ-**V3**], but require making explicit some details which were not studied in the paper [ℵ-**V3**]. Apart from the pleasure we take in understanding these details, our efforts bear some concrete fruits; in particular, we are able to define the Chen-Ruan stringy product in degree 0 (the so called stringy cohomology) with integer coefficients.

We work over the field $\mathbb{C}$ of complex numbers (although our discussion works just as well over any field of characteristic 0).

## 2. Stacks and their moduli spaces

There are two, quite different, ways in which "orbifolds" or "stacks" arise.

**2.1. Groupoids in schemes.** This is the way most differential geometers, as well as many algebraic geometers, are introduced to the subject, since it is in some sense concrete and geometric: one thinks about an object which is locally modeled on a quotient of a variety by the action of an algebraic group. Then one needs to define a good notion of maps between such objects - this is the difficult part of the picture.

Concretely, one is given a "relation", namely a morphism $R \to U \times U$, where each of the projections $R \to U$ is étale (or, more generally, smooth), with the extra data of an "inverse" $R \to R$ and "composition" $R \times_U R \to R$ satisfying natural assumptions which the reader may guess. This is what one calls a groupoid. One then defines morphisms of groupoids, Morita equivalences, and finally one defines the category of orbifolds using groupoids up to Morita equivalence.

In the theory of stacks, there is a natural algebraic stack $[U/R]$ associated to a groupoid. Two groupoids are Morita equivalent if and only if the associated stacks are isomorphic; therefore the groupoid appears as a *presentation* of the stack. The stack is a more intrinsic object.

For instance, the quotient stack $[U/G]$ of a scheme $U$ by the action of a finite group $G$ is the stack associated to the groupoid given by $R = G \times U$. A particularly simple and useful case is the *classifying stack* $\mathcal{B}G = [pt/G]$ of the group $G$ (where $pt$ stands for the point $\operatorname{Spec}\mathbb{C}$).

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**2.2. Moduli problems.** This is the way stacks arise in a more abstract, categorical incarnation, but still extremely natural.

One can encode the data of a "moduli problem" in the *category* of families of objects which we want to parametrize. For instance, the data of "moduli of curves of genus $g$" is encoded in the category $\mathcal{M}_g$ whose objects are

$$\mathcal{O}b(\mathcal{M}_g) = \left\{ \begin{array}{c} C \\ \downarrow \\ S \end{array} \;\middle|\; \begin{array}{c} S \text{ is a scheme,} \\ \text{and} \\ C \to S \text{ is a smooth family of curves of genus } g \end{array} \right\},$$

and the arrows from one object $C \to S$ to another $C' \to S'$ are the *fiber diagrams*.

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C' \arrow[d] \\ S \arrow[r] & S' \end{tikzcd}$$

There is also a *structural functor* $\mathcal{M}_g \to Sch$ which sends a family of curves $C \to S$ to the underlying scheme $S$.

Formally, an *algebraic stack* is a category $\mathcal{M}$ with a functor $\mathcal{M} \to Sch$, satisfying certain assumptions which guarantee that it is "somewhat close to being representable by a scheme". These assumptions imply, in particular, that an algebraic stack is always the stack associated to a groupoid in schemes $[U/R]$. If the automorphism group of every object in the category is *finite*, then in fact $\mathcal{M}$ is *locally* given as the quotient $[U/G]$ of a scheme by a *finite group*. Such a stack is called a *Deligne–Mumford stack*.

If $\mathcal{M}$ is a stack of a moduli problem, then exactly as in the example, the objects are families $C \to S$, and the arrows are fiber diagrams.

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C' \arrow[d] \\ S \arrow[r] & S' \end{tikzcd}$$

The functor $\mathcal{M} \to Sch$ simply sends a family $C \to S$ to its base scheme $S$. Simple examples include: $\mathcal{M}_g$, the moduli stack of smooth curves of genus $g$, and $\mathcal{B}G$, the moduli stack of principal homogeneous $G$-spaces, say for a finite group $G$.

**2.3. Coarse moduli spaces.** Every Deligne–Mumford stack $\mathcal{M}$ has an associated algebraic space called the *coarse moduli space* $\mathbf{M}$ (for a general proof see [**K-M**]):

- In case $\mathcal{M}$ is given via a groupoid as $[U/R]$, the coarse moduli space is the orbit space $U/R$.
- In case $\mathcal{M}$ is the moduli stack of a moduli problem, then $\mathbf{M}$ is the usual coarse moduli space parametrizing objects modulo isomorphisms.

## 3. Twisted stable maps

**3.1. Stable maps into a projective variety.** Recall that, when $X$ is a projective variety, one has a *projective* moduli space $\overline{\mathbf{M}}_{g,n}(X,\beta)$ of *$n$-pointed stable maps of genus $g$ and image class $\beta$.* This is always the coarse moduli space of a Deligne–Mumford stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$, whose objects over a scheme $T$ are triples:

$$(C \to T, \ \ \{s_1, \dots s_n : T \to C\}, \ \ f : C \to X)\,.$$

Here

(1) $C \to T$ is a family of prestable curves of arithmetic genus $g$,
(2) $s_i : T \to C$ are sections of $C \to T$ with disjoint images lying in the smooth locus, and
(3) $f : C \to X$ is a morphism with $f_*([C_t]) = \beta$ for all points $t \in T$, such that the group of automorphisms of fibers of $C \to T$ commuting with $f$ and fixing all the sections $s_i$ is finite.

It is no secret in this workshop that the spaces (and stacks) of stable maps have been useful in symplectic geometry, enumerative geometry and mathematical physics through Gromov–Witten invariants. They have also served as a stepping stone for constructing objects of interest in algebraic geometry (such as other moduli spaces). For a recent application in higher dimensional geometry see [**G-H-S**].

**3.2. Stable maps into a stack.** Since here we are in the business of enlarging our vocabulary of geometric objects from "varieties" to "orbifolds" (or "stacks"), one may wonder if it is worthwhile to replace the projective variety $X$ in the definition of a stable map by a stack $\mathcal{X}$.

Let us consider two examples of stacks which were mentioned above:

(1) Fix an integer $h > 1$, and say $\mathcal{M} = \overline{\mathcal{M}}_h$. Then a map of a curve $C$ to $\mathcal{M}$ is the same as an object of $\mathcal{M}$ over $C$, in other words a *family $X \to C$ of stable curves of genus $h$ over $C$*. Thus the moduli space of all such things is in particular a moduli space of a certain type of surfaces with extra structure (given by a map to a curve). For an algebraic geometer, it is evident that moduli of surfaces are of interest. So having a space of stable maps into $\overline{\mathcal{M}}_g$ is useful at least as a construction tool. See [ℵ-**V1**].
(2) Fix a finite group $G$, and say $\mathcal{M} = \mathcal{B}G$. Then a map of a curve $C$ to $\mathcal{M}$ is a family $P \to C$ of principal homogeneous $G$-spaces, in other words, a principal $G$-bundle. The moduli of these objects are closely related to Hurwitz schemes, level structures and admissible covers, which have been a subject of interest in algebraic geometry for years. See [ℵ-**C-V**].

It is from this point of view, of using stable maps as a basic construction tool, that two of us approached the problem of constructing a suitable space of stable maps into a stack, see [ℵ-**V3**]. Admittedly, Gromov–Witten theory for stacks seemed to us a distant possibility until the work of W. Chen and Y. Ruan on the subject was made public.

**3.3. The non-compactness problem.** One still needs to give a definition of what one means by a "stable map into a stack". As a first attempt, one can define stable maps into a stack $\mathcal{X}$ just as stable maps into a projective variety were defined. The only problem with this is that the stack of usual stable maps into $\mathcal{X}$ is not necessarily compact.

This phenomenon is already manifest with the simplest series of examples where $\mathcal{X} = \mathcal{B}G$, so that maps to $\mathcal{X}$ correspond to principal $G$-bundles, see [ℵ-**V3**]. Say $G = (\mathbb{Z}/m\mathbb{Z})^4, m \geq 2$, and $g = 2$. A smooth curve $C$ of genus 2 carries a *connected* principal $G$-bundle $P \to C$, since its first homology group is $\mathbb{Z}^4$. But if one takes a one-parameter family $P_t \to C_t, t \neq 0$ of such bundles where the underlying curve $C_t$ degenerates, as $t$ approaches 0, to a nodal curve $C_0$ of geometric genus 1, then there is nothing to which the one-parameter family of bundles can degenerate. This is simply because the first homology of the limiting curve $C_0$ is $\mathbb{Z}^3$, and thus $C_0$ carries no connected principal $G$-bundle.

An example of a similar phenomenon with $\mathcal{X} = \overline{\mathcal{M}}_{1,1}$ is described in [ℵ-**V1**].

**3.4. Adding orbispace structure on curves.** A simple analysis, detailed in [ℵ-**V2**], [ℵ-**V3**], of what goes wrong in the example above, reveals that the problem is centered above the node of the degenerate curve $C_0$, and moreover, at the node there is a natural orbispace structure $\mathcal{C}_0$ on $C_0$ which does afford a limiting connected principal bundle $P_0 \to \mathcal{C}_0$.

One then realizes that, in order to have a *compact* moduli space (or stack) of maps into a stack $\mathcal{X}$, *one needs to allow the source curves to acquire an orbispace structure as well.* This may seem as a surprise at first, but we have come to believe that this is completely natural and beautiful. In a way, nature imposes this solution upon us, so why fight the elements?

**3.5. Twisted curves.** Let us describe in some detail the orbispace curves - which we call *twisted curves*, underlying these new maps. We restrict ourselves here to *balanced* twisted curves, the only type that is relevant to our discussion in this paper.

3.5.1. *Nodes.* First, consider the structure at a node. The orbispace curve here is locally the quotient $[U/\boldsymbol{\mu}_r]$ of a nodal curve $U = \{xy = f(t)\}$ by the following action of the cyclic group $\boldsymbol{\mu}_r$ of $r$-th roots of 1:

$$(x, y) \mapsto (\zeta x, \zeta^{-1} y).$$

The *coarse curve* $C$ underlying the orbispace curve $\mathcal{C}$ is, locally, the schematic quotient $U/\boldsymbol{\mu}_r$ defined by $uv = f(t)^r$, where $u = x^r$ and $v = y^r$.

This kind of orbispace structure by itself would suffice for solving the compactification problem we discussed above. But we also know that it is useful to be able to describe the "boundary" of moduli by gluing *marked* curves into nodal ones. Reversing this line of thought, it is useful to understand what kind of structure one obtains when *separating* a node into two marked points on a maked curve.

3.5.2. *Markings.* We are led to consider an orbispace structure along a marking. Here the orbispace curve $C \to T$ is locally the quotient $[U/\boldsymbol{\mu}_r]$ of a smooth curve $U$, with local coordinate $z$ defining the marking, by the following action of $\boldsymbol{\mu}_r$:

$$z \mapsto \zeta z.$$

The coarse curve $C$ is locally a smooth curve with coordinate $w = z^r$.

The integer $r$ in these local descriptions is called the *index* of the orbifold curve at the point in question.

3.5.3. *Global structure of a marking.* Before imposing on the reader the formal definition of a twisted curve, we need to give a warning. Having given the *local* description of a family of twisted curves $\mathcal{C} \to T$ at a marking, one might be misled to believe that there is automatically *globally* a section $T \to \mathcal{C}$. *This is not the case in general.* This fact is going to be important later when we consider evaluation maps.

Take the coarse family of curves $C \to T$, with its section $T \to C$, having image $\Sigma^C \subset C$. The reduction $\Sigma^{\mathcal{C}} \subset \mathcal{C}$ of the inverse image of $\Sigma^C$ in the twisted curve $\mathcal{C}$ is canonically a *gerbe* over $T$ banded by the group $\boldsymbol{\mu}_r$. This means, in particular, that *locally* $\Sigma^{\mathcal{C}}$ is isomorphic to the stack-theoretic quotient of $\Sigma^C$ by the trivial action of $\boldsymbol{\mu}_r$, but globally there is an obstruction for such a quotient description. The obstruction, which naturally lies in $H^2(T, \boldsymbol{\mu}_r)$, comes in our case from $H^1(T, \mathbb{G}_m) = \mathbf{Pic}(T)$ in the following manner:

Let $L = s^*N$ be the pullback to $T$ of the normal bundle of $\Sigma^C \subset C$. Then $\Sigma^{\mathcal{C}}$ is *the moduli stack of $r$-th roots of $L$,* whose objects over a scheme $S$ are triples

$$(g : S \to T, \quad M, \quad \sigma : M^{\otimes r} \xrightarrow{\sim} g^*L),$$

where

- $g : S \to T$ is a morphism,
- $M$ is a line bundle on $S$, and
- $\sigma$ is an isomorphism of line bundles.

The group $\boldsymbol{\mu}_r$ acts on every object here by fiberwise multiplication on $M$. All automorphisms arise this way. It is clear from this description that $\Sigma^{\mathcal{C}} \to T$ has a section if and only if the line bundle $L$ has an $r$-th root over $T$.

Incidentally, there is also a simple prescription for obtaining $\mathcal{C}$ from the coarse curve $C$, at least away from the nodes. We will not need that description in the present exposition.

3.5.4. Finally, here is a formal definition of twisted curves, as in [ℵ-**V3**]:

DEFINITION 3.5.5. A *twisted nodal $n$-pointed curve over a scheme $T$* is a diagram

$$\begin{tikzcd}
\Sigma^{\mathcal{C}} \arrow[r, hook] \arrow[ddr] & \mathcal{C} \arrow[d] \\
 & C \arrow[d] \\
 & T
\end{tikzcd}$$

where

(1) $\mathcal{C}$ is a Deligne-Mumford stack, proper over $T$, which étale locally is a nodal curve over $T$;
(2) $\Sigma^{\mathcal{C}} = \cup_{i=1}^n \Sigma_i^{\mathcal{C}}$, where $\Sigma_i^{\mathcal{C}} \subset \mathcal{C}$ are disjoint closed substacks in the smooth locus of $\mathcal{C} \to S$;
(3) $\Sigma_i^{\mathcal{C}} \to T$ are étale gerbes;
(4) the morphism $\mathcal{C} \to C$ exhibits $C$ as the coarse moduli scheme of $\mathcal{C}$; and
(5) $\mathcal{C} \to C$ is an isomorphism over $C_{\text{gen}}$.

The notation $C_{\text{gen}}$ in the definition above stands for the *generic locus*, namely the complement of the nodes and markings on $C$.

One categorical issue we have to mention is the following: the collection of all stacks naturally forms a 2-category, and therefore families of twisted curves also form a 2-category. However, the fact that each twisted curve has a dense open subset which is a scheme, can be shown to imply that the 2-category of twisted curves is equivalent to a category (so it has a chance of being a stack).

**3.6. Twisted stable maps.** We can now define a *twisted stable map* as follows (see [ℵ-**V3**]):

DEFINITION 3.6.1. Let $\mathcal{X}$ be a Deligne–Mumford stack, $\mathcal{X} \to X \subset \mathbb{P}^N$ a projectively embedded coarse moduli scheme.

An $n$-pointed **twisted stable map** $f : \mathcal{C} \to \mathcal{X}$ to $\mathcal{X}$ of genus $g$, degree $d$, is a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d] & \mathcal{X} \arrow[d] \\ C \arrow[r, "\overline{f}"] & X \end{tikzcd}$$

where

(1) $\mathcal{C}$ is a twisted marked curve, with coarse moduli space $\mathcal{C} \to C$,
(2) $C \to X$ a stable map of genus $g$ and degree $d$, and
(3) $\mathcal{C} \to \mathcal{X}$ representable

We recall that a morphism of algebraic stacks is representable if and only if for any geometric object $\xi \in \mathcal{X}(\operatorname{Spec} \mathbb{C})$, the map $f : \operatorname{Aut}\xi \to \operatorname{Aut} f(\xi)$ is a monomorphism.

With this definition, we have the following theorem:

THEOREM 3.6.2. *The category of $n$-pointed twisted stable maps of genus $g$ and degree $d$ to $\mathcal{X}$ is a proper Deligne–Mumford stack $\mathcal{K}_{g,n}(\mathcal{X}, d)$ admitting a projective coarse moduli space $\mathbf{K}_{g,n}(\mathcal{X}, d)$.*

Of course, the stacks $\mathcal{K}_{g,n}(\mathcal{X}, d)$ can be further decomposed using a numerical equivalence class, or a homology class $\beta \in H_2(X, \mathbb{Z})$ instead of the degree.

The notation using the letter $\mathcal{K}$ rather than $\overline{\mathcal{M}}$ was chosen originally to avoid confusion when inserting something like $\overline{\mathcal{M}}_\gamma$ as an argument in place of $\mathcal{X}$. In this note we will introduce the notation $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)$ for a certain variant of $\mathcal{K}_{g,n}(\mathcal{X}, d)$ (namely the stack of twisted stable maps with trivialized markings).

The stacks $\mathcal{K}_{g,n}(\mathcal{X}, d)$ can also be decomposed in terms of the type of the markings. To avoid cumbersome notation, we will not do this explicitly. Note, however, that the index $r_i$ of the $i$-th marking is a locally constant function $\mathcal{K}_{g,n}(\mathcal{X}, d) \to \mathbb{Z}$.

Also, as in the case of usual stable maps, there is a stabilization morphism $\mathcal{K}_{g,n}(\mathcal{X}, *) \to \mathcal{K}_{g,n}(\mathcal{Y}, *)$ associated to a morphism of stacks $\mathcal{X} \to \mathcal{Y}$. This is going to be useful in this paper in the special case where $\mathcal{Y}$ is a point.

Theorem 3.6.2 is closely related to Proposition 2.3.8 in Chen and Ruan's paper [**C-R2**]. There are several differences: Chen and Ruan work with pseudo-holomorphic maps into smooth symplectic orbifolds in general. In [**ℵ-V3**] we work only with algebraic Deligne–Mumford stacks. On the other hand we allow the target stack $\mathcal{X}$ to be singular (and allow tame stacks in positive characteristics). Chen and Ruan's moduli space is a differential orbifold, whereas we work systematically with stacks.

Some words on the proof of the theorem in [**ℵ-V3**]: often, the construction of a moduli stack (or space) is a standard process based on the existence of Hilbert and Quot schemes. Unfortunately, there is no well-developed machinery of universal constructions analogous to Hilbert and Quot schemes for stacks. The proof in [**ℵ-V3**] constructs the stack $\mathcal{K}_{g,n}(\mathcal{X}, d)$ "with bare hands" using deformation theory and algebraization, following the list of axioms in M. Artin's paper [**Ar**]. Some of the methods developed in [**ℵ-V3**] might serve as building blocks for a future theory of universal constructions for stacks.

## 4. Gromov-Witten theory of varieties and stacks

**4.1. Quantum intersection theory.** In standard treatments of Gromov-Witten theory in symplectic geometry, as well as [**C-R2**], one defines the quantum product in terms of "$n$-point Gromov-Witten numbers", using the inverse matrix of the intersection pairing on cohomology, both for defining the product and proving associativity. In the context of algebraic cycles on varieties, let alone stacks, this cannot be done in general, since there is no analogue of Poincaré duality. A formalism which circumvents this issue was described in the paper [**G-P**]:

Let $X$ be a smooth projective variety. Consider the correspondence

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n+1}(X, \beta) \arrow[r, "e_{n+1}"] \arrow[d, "e_{1,\dots,n}"'] & X \\ X^n & \end{tikzcd}$$

where $e_{1,\dots,n}:\overline{\mathcal{M}}_{g,n+1}(X,\beta)\to X^n$ is the evaluation map at the first $n$ points, and $e_{n+1}$ is the last evaluation map.

Define a map

$$\begin{array}{rcl} A^*(X)^n & \longrightarrow & A^*(X) \\ \gamma_1\times\cdots\times\gamma_n & \mapsto & \langle\gamma_1,\dots,\gamma_n,*\rangle_{g,\beta}\,, \end{array}$$

where

$$\langle\gamma_1,\dots,\gamma_n,*\rangle_{g,\beta}=(e_{n+1})_*\ \left(e^*_{1,\dots,n}(\gamma_1\times\cdots\times\gamma_n)\cap[\overline{\mathcal{M}}_{g,n+1}(X,\beta)]^v\right)$$

Here we used the notation $[\overline{\mathcal{M}}_{g,n+1}(X,\beta)]^v$ for the *virtual fundamental class* of $\overline{\mathcal{M}}_{g,n+1}(X,\beta)$, defined using the cotangent complex and deformation theory, see, e.g., [**B-F**], [**L-T**].

Gromov–Witten invariants of smooth projective varieties satisfy a number of relations which are coded by a number of well known axioms. In particular, the genus 0 invariants can be combined to give the multiplication law of the associative *quantum cohomology ring.* For instance, the so-called "small" quantum product is defined by

$$\gamma_1*\gamma_2=\sum_{\beta\in A^1(X)}\langle\gamma_1,\gamma_2,*\rangle_{0,\beta}\,q^\beta.$$

To prove the axioms of Gromov–Witten invariants and, in particular, to prove associativity of the quantum product, one needs to compare the fiber product

$$\overline{\mathcal{M}}_{g_1,n_1+1}(X,\beta_1)\ \times_X\ \overline{\mathcal{M}}_{g_2,n_2+1}(X,\beta_2),$$

along with its virtual fundamental class, with a boundary divisor in

$$\overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}(X,\beta_1+\beta_2),$$

as we will illustrate later.

In order to generalize this picture, we will need

(1) an analogue of $A^*(X)$ for a smooth Deligne–Mumford stack,
(2) an analogue of the evaluation maps $e_{1,\dots,n}$ and $e_{n+1}$,
(3) an analogue of the virtual fundamental class, and
(4) an analogue of the fiber product description of the boundary divisors.

**4.2. Intersection theory of a stack.** The first ingredient in the "non-orbifold" case which needs a revision in the stack situation is the intersection theory $A^*(X)$.

Two important ingredients in intersection theory of smooth varieties are

(1) a ring $A^*(X)$, functorial under pull-back along a morphism $f:X\to Y$, and
(2) push-forward maps $f:X\to Y$ for $f$ proper, satisfying the projection formula.

It has been observed long ago that such a theory *with integer coefficients* cannot exist for Deligne–Mumford stacks. The problem lies with the push-forward. Let $G$ be a finite group of $r$ elements, and let $p=\operatorname{Spec} k$ be a point. Consider the fiber diagram.

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\phi"'] & p \arrow[d, "\psi"] \\ p \arrow[r] & \mathcal{B}G \end{tikzcd}$$

Since the degree of $G\to p$ is the order $r$ of $G$ (i.e. $\phi_*[G]=r[p]$), and since $p\to\mathcal{B}G$ is proper and flat, it follows by the projection formula that the degree of $\psi:p\to\mathcal{B}G$ is $r$ as well. But we also have the map $\pi:\mathcal{B}G\to p$, since $p$ is the coarse moduli space of $\mathcal{B}G$, and of course $\pi\circ\psi=id:p\to p$. So the degree of $\pi$ must be $1/r$, so $r\pi_*[\mathcal{B}G]=[p]$, which is impossible since the generator $[p]$ is not divisible by $r$.

In [**Gi**] and [**V**], Gillet and Vistoli defined an analogous intersection theory *with rational coefficients* for smooth Deligne-Mumford stacks. Basically

(1) as a group, $A^*(\mathcal{X})_{\mathbb{Q}}$ is isomorphic to $A_*(X)_{\mathbb{Q}}$, where $X$ is the coarse moduli space, and it has the structure of a commutative ring, but
(2) push forward is a bit more subtle, and takes into account the degrees of stabilizers.

This construction can be generalized in the sense of [**F**], Chapter 17 (see also [**B-F**]) to a bivariant intersection theory of Deligne–Mumford stacks - we will use this when proving associativity of the quantum product.

In [**K**], A. Kresch defined an intersection theory $A^*(\mathcal{X})$ with *integer coefficients* but where push-forward maps are defined only for *representable morphisms* $f : \mathcal{X} \to \mathcal{Y}$. We will use Kresch's theory when we define an integer-coefficients version of Chen and Ruan's stringy cohomology ring of $\mathcal{X}$.

The constructions of Kresch and Vistoli are related by the fact that $A^*(\mathcal{X}) \otimes \mathbb{Q} = A^*(\mathcal{X})_{\mathbb{Q}}$.

**4.3.** $A^*(\mathcal{X})$ **is too small.** Both constructions discussed above are not quite sufficient for what we need. Even for global quotients, it has long been observed that the $K$ theory of $\mathcal{X} = [V/G]$, namely the equivariant $K$-theory of $V$, is bigger than the $K$ theory of $V/G$, even with rational coefficients. Indeed, the decomposition theorem says that the equivariant $K$-ring $K_0(V//G)$ is a product of $K_0(V^g/C(g))$, suitably twisted by cyclotomic rings. Thus any reasonable Riemann-Roch formula must take into account contributions from all the fixed-point sets. These contributions are lost in $A_*(X)_{\mathbb{Q}}$. A detailed theory of Riemann-Roch type theorems for stacks, in which such contributions are introduced, was developed in Toen's work [**T**].

In enumerative geometry, T. Graber studied in [**Gr**] the number of hyperelliptic curves of genus $g$ and degree $d$ in $\mathbb{P}^2$ through $3d+1$ general points. It is explained there that the cohomology of $Sym^2\mathbb{P}^2$ does not carry enough information to encode all these invariants, and therefore the Hilbert scheme $Hilb_2(\mathbb{P}^2)$ was used instead. The extra information in $Hilb_2(\mathbb{P}^2)$ is completely accounted for by contributions of fixed-point sets. It is now known (see [**B-K-R**]) that the derived categories of $Hilb_2(\mathbb{P}^2)$ and the *stack* symmetric square $[(\mathbb{P}^2)^2/(\mathbb{Z}/2\mathbb{Z})]$ are equivalent, and therefore their $K$ groups are isomorphic.

In [**C-R2**], Chen and Ruan were motivated by string theory to introduce similar contributions of fixed-point sets. We follow their approach, with a slight modification which avoids any choice of roots of 1.

**4.4. The inertia stack and variants.** The stringy cohomology of $\mathcal{X}$ according to Chen-Ruan is, as a vector space, the cohomology of another smooth stack, namely the inertia stack

$$\begin{aligned}\mathcal{X}_1^{\text{Chen-Ruan}} = I_{\mathcal{X}} &= \{\ (x,g)\ \mid x \in \mathcal{O}b(\mathcal{X}),\ g \in \operatorname{Aut} x\} \\ &= \{(x,H,g) \mid x \in \mathcal{O}b(\mathcal{X}),\ H \subset \operatorname{Aut} x,\ g \text{ a generator of } H\}\end{aligned}$$

A slightly more canonical Gromov-Witten formalism arises if instead one takes a Galois twist of this:

$$\begin{aligned}\mathcal{X}_1 &= \{(x,H,\chi) \mid x \in \mathcal{O}b(\mathcal{X}),\ H \subset \operatorname{Aut} x,\ \chi : H \tilde{\to} \boldsymbol{\mu}_r \text{ for some } r\} \\ &= \bigcup_r HomRep(\mathcal{B}\boldsymbol{\mu}_r, \mathcal{X})\end{aligned}$$

i.e., the stack of representable morphism from a constant cyclotomic gerbe to $\mathcal{X}$.

Of course, over $\mathbb{C}$ there is a natural isomorphism of $\boldsymbol{\mu}_r$ with $\mathbb{Z}/r\mathbb{Z}$, which gives an isomorphism

$$\mathcal{X}_1^{\text{Chen-Ruan}} \simeq \mathcal{X}_1.$$

But over other fields the group-schemes are different, in which case one has to use $\boldsymbol{\mu}_r$: with $\mathbb{Z}/r\mathbb{Z}$ one cannot even define the Gromov-Witten invariants of $\mathcal{X}$.

There is an interesting variant of this stack, which we denote by $\overline{\mathcal{X}}_1$. It arises as a *rigidification* of $\mathcal{X}_1$. The point is that the group $\boldsymbol{\mu}_r$ acts on every object of $\mathcal{X}_1$ in the component corresponding to $\boldsymbol{\mu}_r$, and the rigidification process (see [$\aleph$**-C-V**]) allows one to remove this action from the picture (just as the Picard functor is obtained by removing the $\mathbb{C}^*$-action from the stack of line bundles).

$$\begin{aligned}\overline{\mathcal{X}}_1 &= \{(x,H,\chi) \text{ up to } \boldsymbol{\mu}_r\} \\ &= \text{stack of cyclotomic gerbes in } \mathcal{X}\end{aligned}$$

The geometric objects of $\mathcal{X}_1$ and $\overline{\mathcal{X}}_1$ are the same $(x,H,\chi)$, but the automorphisms are different: in the first case an automorphism is an element of the centralizer $C(H)$, in the second it is an element of $C(H)/H$.

We note that

- there are obvious inclusions $\mathcal{X} \subset \mathcal{X}_1$ and $\mathcal{X} \subset \overline{\mathcal{X}}_1$.
- There is also a "forgetful map" $\mathcal{X}_1 \to \mathcal{X}$, but there is in general no map from $\overline{\mathcal{X}}_1$ to $\mathcal{X}$.
- Finally the rigidification map $\pi_1 : \mathcal{X}_1 \to \overline{\mathcal{X}}_1$ exhibits $\mathcal{X}_1$ as the universal gerbe over $\overline{\mathcal{X}}_1$.
  This implies that the coarse moduli space $X_1$ of $\mathcal{X}_1$ is also the moduli space of $\overline{\mathcal{X}}_1$.

The total stringy Chow group of $\mathcal{X}$ is defined as the total Chow group $A^*(\mathcal{X}_1)_{\mathbb{Q}}$. We can also consider the integral version $A^*(\mathcal{X}_1)$ defined by Kresch.

There is a useful endomorphism

$$r : A^*(\mathcal{X}_1) \to A^*(\mathcal{X}_1),$$

induced by the locally constant function

$$r : \mathcal{X}_1 \to \mathbb{Z}$$

with value $r$ on the piece corresponding to $HomRep(\mathcal{B}\boldsymbol{\mu}_r, \mathcal{X})$.

**4.5. Evaluation maps.** Given a family of $n$-pointed twisted stable maps parametrized by a scheme $S$:

$$(\mathcal{C} \to S, f : \mathcal{C} \to \mathcal{X})$$

we have, for each $i$ in the range $1 \leq i \leq n$, a map of the marking $(\Sigma_i)$ into $\mathcal{X}$. This means that $S$ parametrizes a family of cyclotomic gerbes in $\mathcal{X}$. By definition this gives a morphism $(\bar{e}_i)_S : S \to \overline{\mathcal{X}}_1$, and since this is true for every twisted stable map, we get a map of moduli stacks

$$\bar{e}_i : \mathcal{K}_{g,n}(\mathcal{X}, \beta) \to \overline{\mathcal{X}}_1.$$

Passing to moduli spaces we get

$$e_i^{\rm coarse} : \mathbf{K}_{g,n}(\mathcal{X}, \beta) \to X_1.$$

Chen and Ruan [**C-R2**] use this latter map $e_i^{\rm coarse}$ of coarse moduli spaces. It so happens that the formalism of Gromov-Witten theory basically goes through (with an important twist to be introduced soon) if one works with this map *as if it came from a map* $e_i^{\rm orb} : \mathcal{K}_{g,n}(\mathcal{X}, \beta) \to \mathcal{X}_1$. It is a fact of life that such a "virtual map" does not exist in general. However, as long as one works with cohomology or intersection theory *with rational coefficients,* then one can define a "cohomological pull-back":

$$(e_i^{\rm orb})^* = r^{-1} \cdot \bar{e}_i^* \ (\pi_1)_*.$$

(here again $r$ denotes the operator which multiplies a class on a component of index $r$ by the integer $r$). A similar trick works also for pushforward

$$(e_i^{\rm orb})_* = r \cdot \pi_1^* \ (\bar{e}_i)_*.$$

In order to avoid the confusion of working with such "weighted maps" we can simply work with the map $\bar{e}_i : \mathcal{K}_{g,n}(\mathcal{X}, \beta) \to \overline{\mathcal{X}}_1$, changing the products by a suitable factor. For the sake of exposition, it may be a bit more transparent to use $\mathcal{X}_1$ and work with the following formalism:

Define $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta)$ to be the stack of twisted stable maps *with sections of all the gerbes.* This is simply the fibered product of the $n$ universal gerbes over $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$. Now we have evaluation maps

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta) \arrow[r, "e_i"] \arrow[d] & \mathcal{X}_1 \arrow[d] \\
\mathcal{K}_{g,n}(\mathcal{X}, \beta) \arrow[r, "\bar{e}_i"] & \overline{\mathcal{X}}_1
\end{tikzcd}$$

There is an obvious involution $\iota$ on $\overline{\mathcal{X}}_1$ and $\mathcal{X}_1$ via $\chi \mapsto \chi^{-1}$. We define the *twisted evaluation map* $\check{e}_i = \iota \circ e_i$. The twisted evaluation map comes about when one glues two twisted stable maps along two markings to form a *balanced* twisted stable map.

We can work with $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta)$, as long as we remember to account for the degree $1/(r_1 \cdots r_n)$ of the map $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta) \to \mathcal{K}_{g,n}(\mathcal{X}, \beta)$. Note that this degree is only locally constant, and varies from one connected component to another.

**4.6. Deformation and obstruction.** We want to define an appropriate virtual fundamental class for $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta)$, generalizing the situation of the space of maps into a smooth projective variety.

The first obstacle one needs to overcome is mostly psychological - does deformation theory as we know it work for stacks?

The answer is yes - and already in the literature. The point is that Illusie's work on deformation theory [**I**] works by deforming a ringed topos (essentially the category of sheaves on a site), as soon as it is reasonable enough to allow the definition of a cotangent complex. The fact that a stack has such a topos is discussed in [**L-MB**], Chapter 12, and the existence of a cotangent complex is discussed in [**L-MB**], Chapter 17 (in fact the case of Deligne–Mumford stacks can be more easily deduced from the case of schemes).

In particular, as in the case of schemes, the infinitesimal deformations of a twisted curve $\mathcal{C}$ are measured by $\mathrm{Ext}^1(\Omega^1_{\mathcal{C}}, \mathcal{O}_{\mathcal{C}})$, and if one wants to deform some additional non-twisted marking, then infinitesimal deformations are in $\mathrm{Ext}^1(\Omega^1_{\mathcal{C}}(D), \mathcal{O}_{\mathcal{C}})$, where $D$ is the divisor given by these markings. Obstructions would lie in the corresponding $\mathrm{Ext}^2$ group, which turns out to be zero for twisted pointed curves (see [**ℵ-C-V**]).

Similarly, infinitesimal deformations of a map $f : \mathcal{C} \to \mathcal{X}$ fixing the structure of $\mathcal{C}$ (as well as of $\mathcal{X}$) are measured by $\mathrm{Hom}(f^*\Omega^1_{\mathcal{X}}, \mathcal{O}_{\mathcal{C}}) = H^0(\mathcal{C}, f^*T\mathcal{X})$. Obstructions lie in $H^1$. More precisely, $R^\bullet\pi_*(f^*T\mathcal{X})$ gives a relative obstruction theory for this moduli problem over the stack of marked twisted prestable curves, where the symbols refer to the diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{X} \\ {\scriptstyle\pi}\downarrow & & \\ \overline{\mathcal{M}}_{g,n}(\mathcal{X},\beta) & & \end{array}$$

Now, the formalism of the virtual fundamental classes (see [**L-T**], [**B-F**], [**K**]) automatically allows one to define a virtual fundamental class $[\mathcal{K}_{g,n}(\mathcal{X},\beta)]^v$. There is similarly a class $[\overline{\mathcal{M}}_{g,n}(\mathcal{X},\beta)]^v$, which is just the pullback of $[\mathcal{K}_{g,n}(\mathcal{X},\beta)]^v$. As we indicated before, to get the degrees right we need to replace the latter by a multiple: $[\overline{\mathcal{M}}_{g,n}(\mathcal{X},\beta)]^w = r_1 \cdots r_n[\overline{\mathcal{M}}_{g,n}(\mathcal{X},\beta)]^v$.

We remark that in the special case where the moduli space is smooth and the morphism from the deformation space of the map surjects onto the deformation space of the underlying twisted curve for all points in the moduli space, there is a simple expression for the virtual fundamental class. The formula in this case is

$$[\overline{\mathcal{M}}_{g,n}(\mathcal{X},\beta)]^v = c_{top}(E)$$

where $E = R^1\pi_*(f^*T\mathcal{X})$ is the obstruction bundle. (The hypotheses above ensure that this sheaf is locally free.) There are two important cases where these conditions are satisfied. One is when $g = 0$ and the target space is a homogeneous variety. In this case the obstruction bundle is trivial, and the virtual class can be identified with the ordinary fundamental class. The other case is of special interest in this paper. It is when $d = 0$ – the case of constant maps. When the map is constant, it is clear that the underlying curve can be deformed freely, which explains why we get the surjectivity on deformation spaces. It is by studying the spaces of constant maps that we define the product structure on the stringy Chow rings in Section 6. As a result, we see that this product can be understood without recourse to the somewhat technical theory of virtual classes. In order to make this exposition more accessible, we will assume that we are in one of these nice cases and identify the virtual class with this chern class in some of our arguments below. Those familiar with the arguments of [**B**] and [**B-F**] should have no trouble adapting them in general. Those who are not can still take this presentation as a complete proof in the case of constant maps which is in some sense the most interesting from our point of view.

**4.7. The product.** We can now define the Gromov-Witten invariants and the quantum product just as it is done for smooth projective varieties. Consider the diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n+1}(\mathcal{X},\cdot) & \xrightarrow{\check{e}_{n+1}} & \mathcal{X}_1 \\ {\scriptstyle e_{1,\dots,n}}\downarrow & & \\ \mathcal{X}_1^n & & \end{array}$$

and define

$$\begin{array}{ccc} A^*(\mathcal{X}_1)^n_{\mathbb{Q}} & \longrightarrow & A^*(\mathcal{X}_1)_{\mathbb{Q}} \\ \gamma_1 \times \cdots \times \gamma_n & \mapsto & \langle \gamma_1, \dots, \gamma_n, * \rangle_{g,\beta}, \end{array}$$

where

$$\langle \gamma_1, \dots, \gamma_n, * \rangle_{g,\beta} = \check{e}_{n+1\,*}\,\left( e^*_{1,\dots,n}(\gamma_1 \times \cdots \times \gamma_n) \cap [\overline{\mathcal{M}}_{g,n+1}(\mathcal{X},\beta)]^w \right).$$

Again - the reason we are using the twisted evaluation map $\check{e}_{n+1}$ has to do with the fact that the twisted curves are *balanced*, and will become more explicit below.

Having defined the Gromov-Witten invariants, we immediately get a new product structure on $A_*(\mathcal{X}_1)_{\mathbb{Q}}[[q]]$ defined by taking

$$\gamma_1 * \gamma_2 = \sum_\beta \langle \gamma_1, \gamma_2, * \rangle_{0,\beta} q^\beta .$$

## 5. Associativity

As in the now "classical" case of smooth varieties, the most subtle property of the quantum product is its associativity. Here we go through the main steps in the proof. We revisit these steps in the next section when discussing the stringy cohomology ring of $\mathcal{X}$.

**5.1. Associativity: the product diagram.** For simplicity of the discussion, we restrict from now on to the case where $g = 0$ and $n = 3$.

We want to prove that the "3-point" quantum product is associative. Since it is commutative, it suffices to show

THEOREM 5.1.1.

$$\sum_{\beta_1+\beta_2=\beta} \langle \langle \gamma_1, \gamma_2, * \rangle_{0,\beta_1}, \gamma_3, * \rangle_{0,\beta_2} = \sum_{\beta_1+\beta_2=\beta} \langle \langle \gamma_1, \gamma_3, * \rangle_{0,\beta_1}, \gamma_2, * \rangle_{0,\beta_2} .$$

Fix two homology classes $\beta_i \in H_1(X, \mathbb{Z})$, and for simplicity write $\mathcal{M}_i = \overline{\mathcal{M}}_{0,3}(\mathcal{X}, \beta_i)$ for the corresponding stacks of twisted stable maps. The corresponding evaluation maps are denoted $e_i^1 : \mathcal{M}_1 \to \mathcal{X}_1$ and $e_i^2 : \mathcal{M}_2 \to \mathcal{X}_1$.

The relevant diagram is the following.

$$\begin{tikzcd}
 & & \mathcal{M}_2 \arrow[r, "\check{e}_3^2"] \arrow[dl, "e_1^2"'] \arrow[d, "e_2^2"] & \mathcal{X}_1 \\
\mathcal{M}_1 \arrow[r, "\check{e}_3^1"] \arrow[d, bend right, "e_1^1"'] \arrow[d, bend left, "e_2^1"] & \mathcal{X}_1 & \mathcal{X}_1 & \\
\mathcal{X}_1 & & &
\end{tikzcd}$$

As a first step we complete the top left corner of the diagram.

$$\begin{tikzcd}
\mathcal{M}_1 \times_{\mathcal{X}_1} \mathcal{M}_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathcal{M}_2 \arrow[d, "e_1^2"] \\
\mathcal{M}_1 \arrow[r, "\check{e}_3^1"] & \mathcal{X}_1
\end{tikzcd}
\quad \text{equivalently} \quad
\begin{tikzcd}
\mathcal{M}_1 \times_{\mathcal{X}_1} \mathcal{M}_2 \arrow[r, "p_1 \times p_2"] \arrow[d] & \mathcal{M}_1 \times \mathcal{M}_2 \arrow[d, "\check{e}_3^1 \times e_1^2"] \\
\mathcal{X}_1 \arrow[r, "\delta"] & \mathcal{X}_1^2
\end{tikzcd}$$

Consider the following morphisms from $\mathcal{M}_1 \times_{\mathcal{X}_1} \mathcal{M}_2$ to $\mathcal{X}_1$ :

$$\begin{aligned}
e_1^\times &= e_1^1 \circ p_1 , \\
e_2^\times &= e_2^1 \circ p_1 , \\
e_3^\times &= e_1^2 \circ p_2 , \\
e_4^\times &= e_1^2 \circ p_2 .
\end{aligned}$$

Starting with the definition of the Gromov-Witten invariants, a straightforward push-pull argument detailed in [**G-P**] gives

$$\langle \langle \gamma_1, \gamma_2, * \rangle_{0,\beta_1}, \gamma_3, * \rangle_{0,\beta_2} = \check{e}_{4*}^\times \left( \left( e_1^{\times *} \gamma_1 \cdot e_2^{\times *} \gamma_2 \cdot e_3^{\times *} \gamma_3 \right) \cap \delta^! \left( [\mathcal{M}_1]^w \times [\mathcal{M}_2]^w \right) \right) . \tag{1}$$

As we will see, the class $\delta^!( [\mathcal{M}_1]^w \times [\mathcal{M}_2]^w )$ has an interpretation as a weighted virtual fundamental class of $\mathcal{M}_1 \times_{\mathcal{X}_1} \mathcal{M}_2$ with respect to a natural obstruction theory.

**5.2. Associativity: the divisor diagram.** The stack $\mathcal{M}_1 \times_{\mathcal{X}_1} \mathcal{M}_2$ can be viewed as the stack of pairs of 3-pointed genus 0 twisted stable maps with *balanced gluing data* along $\check{e}_3^1$ and $e_1^2$, and trivialization of all the gerbes. Writing $\beta = \beta_1 + \beta_2$, this means that there is a gluing morphism

$$gl : \bigcup_{\beta_1+\beta_2=\beta} \mathcal{M}_1 \times_{\mathcal{X}_1} \mathcal{M}_2 \to \overline{\mathcal{M}}_{0,4}(X,\beta),$$

just as in the case of usual stable maps. It is essential here, however, that the fiber product be taken with respect to the twisted evaluation on the second map. Otherwise the curve that we would obtain by gluing would not be balanced.

Consider the stabilization morphism $st : \overline{\mathcal{M}}_{0,4}(X,\beta) \to \overline{\mathbf{M}}_{0,4}$ and the divisor $D := (12|34) \subset \overline{\mathbf{M}}_{0,4}$ (corresponding to the reducible stable 4-pointed curve of genus 0 where the two first points are on the same component). We have the top and bottom of the following fiber diagram.

(2)
$$\begin{tikzcd}
\bigcup_{\beta_1+\beta_2=\beta} \mathcal{M}_1 \times_{\mathcal{X}_1} \mathcal{M}_2 \arrow[d, "g"] & & \\
D(\mathcal{X}) \arrow[r] \arrow[d] & D \times_{\overline{\mathbf{M}}_{0,4}} \overline{\mathcal{M}}_{0,4}(\mathcal{X}) \arrow[r, hook] \arrow[d] & \overline{\mathcal{M}}_{0,4}(\mathcal{X},\beta) \arrow[d] \\
D^{orb} \arrow[r, "j"] & D \times_{\overline{\mathbf{M}}_{0,4}} \overline{\mathcal{M}}_{0,4}^{orb} \arrow[r, hook] \arrow[d] & \overline{\mathcal{M}}_{0,4}^{orb} \arrow[d] \\
& D \arrow[r, hook] & \overline{\mathbf{M}}_{0,4}
\end{tikzcd}$$

The notation $D(\mathcal{X})$ stands for the substack of reducible *twisted* stable maps to $\mathcal{X}$ with a specified node separating the first two markings from the last two. We also denote by $e_i : \overline{\mathcal{M}}_{0,4}(\mathcal{X},\beta) \to \mathcal{X}_1$ the evaluation maps. Note that by construction, we have

(3)
$$e_i^{\times} = e_i \circ gl.$$

On the second row we inserted the Artin stack $\overline{\mathcal{M}}_{0,4}^{orb}$ of *quasistable twisted curves,* i.e. connected proper twisted curves without a stability requirement. This is a smooth stack which is neither separated nor of finite type, yet it comes in handy just as the usual stack of quasistable curves was useful in [**B**]. We let $D^{orb}$ denote the smooth stack of quasistable curves with a specified node separating the first two markings from the last two, and let

$$j : D^{orb} \to \overline{\mathcal{M}}_{0,4}^{orb}$$

denote the natural morphism from $D^{orb}$ to the boundary of $\overline{\mathcal{M}}_{0,4}^{orb}$. This morphism is analogous to the standard gluing maps between moduli spaces of ordinary stable curves. In particular, it is birational onto its image, although not an embedding. We will use $[D^{orb}]$ to denote the class of this boundary divisor in $\overline{\mathcal{M}}_{0,4}^{orb}$.

A local analysis establishes the following two equations.

PROPOSITION 5.2.1.

1. *The map* $g : \bigcup \mathcal{M}_1 \times_{\mathcal{X}_1} \mathcal{M}_2 \to D(\mathcal{X})$ *has locally constant degree* $1/r_{\times}$. *In fact* $\bigcup \mathcal{M}_1 \times_{\mathcal{X}_1} \mathcal{M}_2$ *is the universal gerbe over the distinguished node of the universal curve of* $D(\mathcal{X})$.
2. *We have an equality of divisors on* $\overline{\mathcal{M}}_{0,4}^{orb}$:

$$[D \times_{\overline{\mathbf{M}}_{0,4}} \overline{\mathcal{M}}_{0,4}^{orb}] = r_{\times}[D^{orb}].$$

Here we are using $r_{\times}$ to denote the locally constant function of $D^{orb}$ giving the index of the distinguished node. By abuse of notation, we will sometimes use the same symbol to refer to the pullback of this function to $\mathcal{M}_1 \times_{\mathcal{X}_1} \mathcal{M}_2$ where it is also equal to $p_1^* r_3$ or $p_2^* r_1$.

**5.3. Associativity: compatibility of virtual fundamental classes.** A more subtle issue is the following:

PROPOSITION 5.3.1.
$$\delta^!(\ [\mathcal{M}_1]^w \times [\mathcal{M}_2]^w\ ) = r_\times^2 \cdot g^*(j^![\overline{\mathcal{M}}_{0,4}(X)]^w)$$

PROOF. It is for the proof of this statement that we will make the simplifying assumptions mentioned at the end of Section 4.6. It is true in complete generality, however. By definition of $[\mathcal{M}_1]^w, [\mathcal{M}_2]^w, [\overline{\mathcal{M}}_{0,4}(X)]^w$, and $r_\times$ we need to show

$$\delta^!(c_{top}(E_1) \times c_{top}(E_2)) = g^*(j^! c_{top}(E)) \tag{4}$$

where $E_1, E_2$, and $E$ are the relevant obstruction bundles.

The right hand side of this equation is easy to understand. Since we have assumed that the morphism from $\overline{\mathcal{M}}_{0,4}(X)$ to $\overline{\mathcal{M}}_{0,4}^{orb}$ is smooth, the $j^!$ just means restriction to $D(\mathcal{X})$, and the right hand side of Equation (4) is equal to $c_{top}(gl^*E)$. Denote
$$\mathcal{Y} = \mathcal{M}_1 \times_{\mathcal{X}_1} \mathcal{M}_2.$$
By the theorem on cohomology and base change, we can compute $gl^*E$ as a higher direct image sheaf on $\mathcal{Y}$. If we take $\mathcal{C}_\mathcal{Y}$ to be the pull-back under $gl$ of the universal family of twisted curves over $\overline{\mathcal{M}}_{0,4}(X)$, then $gl^*E = R^1\pi_*(f^*T\mathcal{X})$ where by abuse of notation we let $\pi$ be the map from $\mathcal{C}_\mathcal{Y}$ to $\mathcal{Y}$. Essentially by the definition of $\mathcal{Y}$, $\mathcal{C}_\mathcal{Y}$ is naturally a union of two families of curves, which we will denote $\mathcal{C}_1$ and $\mathcal{C}_2$, which are pulled back from the projections from $\mathcal{Y}$ to $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. If we let $\mathcal{C}'_\mathcal{Y}$ be the disjoint union of these two families, then there is a normalization map $\nu : \mathcal{C}'_\mathcal{Y} \to \mathcal{C}_\mathcal{Y}$ which glues the two curves together along the gerbe $\Sigma$ associated to the distinguished node of $\mathcal{C}_\mathcal{Y}$. We consider the normalization sequence
$$0 \to f^*T\mathcal{X} \to \nu_* f'^* T\mathcal{X} \to (f^*T\mathcal{X})|_\Sigma \to 0.$$
Pushing forward to $\mathcal{Y}$ gives
$$0 \to \pi_* f^*T\mathcal{X} \to \pi'_* f'^* T\mathcal{X} \to \pi_*(f^*T\mathcal{X})|_\Sigma \to gl^*E \to p_1^*E_1 \oplus p_2^*E_2 \to 0.$$
We break this into two shorter exact sequences.
$$0 \to \pi_* f^*T\mathcal{X} \to \pi'_* f'^* T\mathcal{X} \to \pi_*(f^*T\mathcal{X})|_\Sigma \to Q \to 0 \tag{5}$$
and
$$0 \to Q \to gl^*E \to p_1^*E_1 \oplus p_2^*E_2 \to 0.$$
At this point we conclude that
$$c_{top}(gl^*E) = c_{top}(Q) \cdot p_1^*\, c_{top}(E_1) \cdot p_2^*\, c_{top}(E_2).$$
Comparing to the equation we are trying to prove, and considering the diagram

$$\begin{tikzcd}
\mathcal{M}_1 \times_{\mathcal{X}_1} \mathcal{M}_2 \arrow[r, "p_1 \times p_2"] \arrow[d] & \mathcal{M}_1 \times \mathcal{M}_2 \arrow[d, "\check{e}_3^1 \times e_1^2"] \\
\mathcal{X}_1 \arrow[r, "\delta"] & \mathcal{X}_1^2
\end{tikzcd}$$

we see that what we need to prove is that $Q$ is the excess bundle for the intersection of $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{X}_1$. The result would then follow immediately from the excess intersection formula. Using our identification of the relative tangent space to the space of stable maps as $H^0(\mathcal{C}, f^*T\mathcal{X})$, and of the curve $\mathcal{C}'_\mathcal{Y}$ as the union of $\mathcal{C}_1$ and $\mathcal{C}_2$, we can rewrite sequence (5) as
$$0 \to T\mathcal{Y} \to p_1^*T\mathcal{M}_1 \oplus p_2^*T\mathcal{M}_2 \to \pi_*(f^*T\mathcal{X})|_\Sigma \to Q \to 0.$$
Now it is evident that the result will follow from the following lemma.

LEMMA 5.3.2. $\pi_*\left((f^*T\mathcal{X})|_\Sigma\right) \ \simeq\ p_1^*\,(\check{e}_3^1)^*\ T\mathcal{X}_1$

PROOF OF THE LEMMA. Over a geometric point $y$ of $\mathcal{Y}$, we can identify the fiber of $\Sigma$ with $\mathcal{B}\boldsymbol{\mu}_r$. The point $y$ maps to a geometric point $x$ of $\mathcal{X}$, with stabilizer $G$, and we can locally describe $\mathcal{X}$ around $x$ as $[U/G]$. The pulback $T$ of the tangent space of $\mathcal{X}$ to $\mathcal{Y}$ has a natural action of $\boldsymbol{\mu}_r$, and the fiber of $\pi_*(f^*T\mathcal{X})|_\Sigma$ is naturally the space of invariants $T^{\boldsymbol{\mu}_r}$. This is naturally isomorphic to the tangent space of $\mathcal{X}_1$ at the point $\check{e}_3^1(p_1(y))$, which is what we needed. ♣

**5.4. Associativity: end of proof.** The associativity now follows formally from our calculations. Consider the following commutative diagram.

$$
\begin{tikzcd}
\bigcup \mathcal{M}_1 \times_{\mathcal{X}_1} \mathcal{M}_2 \arrow[d, "g"] \arrow[dr, "gl"] & \\
D(\mathcal{X}) \arrow[r, "\iota"] \arrow[d] & \overline{\mathcal{M}}_{0,4}(\mathcal{X},\beta) \arrow[d, "s"] \arrow[dd, bend left, "st"] \\
D^{orb} \arrow[r, "j"] \arrow[d] & \overline{\mathcal{M}}^{orb}_{0,4} \arrow[d] \\
D \arrow[r, hook] & \overline{\mathbf{M}}_{0,4}
\end{tikzcd}
$$

Note that this is a simplified version of diagram (2). In particular, the bottom square of this diagram is not cartesian, although the middle square is. Denote

$$c = (e_1^{\times})^*\gamma_1 \cdot (e_2^{\times})^*\gamma_2 \cdot (e_3^{\times})^*\gamma_3$$

and

$$[\overline{\mathcal{M}}]^w = [\overline{\mathcal{M}}_{0,4}(\mathcal{X},\beta)]^w.$$

Combining Equation (1) with Proposition 5.3.1 we obtain

$$S := \sum \langle\langle \gamma_1, \gamma_2, *\rangle_{0,\beta_1}, \gamma_3, *\rangle_{0,\beta_2} = \check{e}^{\times}_{4*}(gl^*(c) \cap (r_{\times}^2 \cdot g^* j^! [\overline{\mathcal{M}}]^w)).$$

Now we use the fact that $\check{e}_4^{\times} = \check{e}_4 \circ gl$ to simplify to

$$S = \check{e}_{4*}\ gl_*\ (gl^* c \cap (r_{\times}^2 \cdot g^* j^! [\overline{\mathcal{M}}]^w)).$$

The projection formula gives

$$S = \check{e}_{4*}\ (c \cap gl_*(r_{\times}^2 \cdot g^* j^! [\overline{\mathcal{M}}]^w)).$$

Factoring $gl = \iota \circ g$ gives

$$S = \check{e}_{4*}\ (c \cap \iota_* g_*(g^* r_{\times}^2 \cdot j^! [\overline{\mathcal{M}}]^w))$$

and since $g$ has degree $1/r^{\times}$ we get

$$S = \check{e}_{4*}\ (c \cap \iota_*(r_{\times} \cdot j^! [\overline{\mathcal{M}}]^w)).$$

Now, standard intersection theory gives us

$$S = \check{e}_{4*}\ (c \cdot r_{\times} \cdot s^*[D^{orb}] \cap [\overline{\mathcal{M}}]^w).$$

Finally, by the second part of Proposition 5.2.1, we can conclude

$$\sum \langle\langle \gamma_1, \gamma_2, *\rangle_{0,\beta_1}, \gamma_3, *\rangle_{0,\beta_2} = \check{e}_{4*}\ (c \cdot st^*[D] \cap [\overline{\mathcal{M}}]^w).$$

The final expression is clearly independent of the way the markings are grouped (since $\check{e}_4$ is invariant under the $S_3$ action permuting the first three points), which is what we need for the associativity theorem:

$$\sum \langle\langle \gamma_1, \gamma_2, *\rangle_{0,\beta_1}, \gamma_3, *\rangle_{0,\beta_2} = \sum \langle\langle \gamma_1, \gamma_3, *\rangle_{0,\beta_1}, \gamma_2, *\rangle_{0,\beta_2}.$$

♣

## 6. Stringy Chow rings with integral coefficients

We now concentrate on the part of quantum Chow ring of $\mathcal{X}$ involving only $g = 0, n = 3$ and $\beta = 0$. Following a suggestion of Y. Ruan during a discussion at the conference, we call it the *stringy Chow ring* of $\mathcal{X}$.

**6.1. The main result.** A-priori, one gets a new ring structure on the group $A^*_{st}(\mathcal{X})_{\mathbb{Q}} := A^*(\mathcal{X}_1)_{\mathbb{Q}}$. It is easy to see that this is well defined whenever $\mathcal{X}$ is smooth and separated, without any properness assumptions. Our main claim is the following:

THEOREM 6.1.1. *Let $\mathcal{X}$ be a smooth and separated Deligne–Mumford stack, not necessarily proper. There is a commutative, associative ring structure on Kresch's Chow group $A^*_{st}(\mathcal{X}) := A^*(\mathcal{X}_1)$ lifting the stringy chow ring structure (with rational coefficients) on $A^*(X_1)_{\mathbb{Q}}$.*

*Moreover, let $j : \mathcal{X} \hookrightarrow \mathcal{X}_1$ be the embedding. Then the group homomorphism $j_* : A^*(\mathcal{X}) \to A^*(\mathcal{X}_1) = A^*_{st}(\mathcal{X})$ is a ring homomorphism, in particular $j_*1$ is the identity in $A^*_{st}(\mathcal{X})$.*

The proof of this theorem is more subtle than the rational case.

**6.2. Refined evaluation and the product law.** First, we need to define the product. The crucial point is that in the case $g = 0, n = 3$ and $\beta = 0$ we have refined evaluation maps:

LEMMA 6.2.1. *Each evaluation map $\bar{e}_i : \mathcal{K}_{0,3}(\mathcal{X}, 0) \to \bar{\mathcal{X}}_1$ admits a lifting $\mathbf{e}_i : \mathcal{K}_{0,3}(\mathcal{X}, 0) \to \mathcal{X}_1$. The morphism $\mathbf{e}_i$ is representable and finite.*

PROOF. The existence of $\mathbf{e}_i$ stems from the following: given a family of twisted stable maps $(\mathcal{C} \to S, \mathcal{C} \to \mathcal{X})$ in $\mathcal{K}_{0,3}(\mathcal{X}, 0)$ $(S)$, the coarse curve is just $\mathbb{P}^1 \times S$, and the markings can be identified as $\{0\} \times S, \{1\} \times S$ and $\{\infty\} \times S$. Constructing a lifting $e_i$ is equivalent to constructing an $r_i$-th root of the normal bundle of the $i$-th marking, functorially in $S$, which is the same as constructing an $r_i$-th root of the tangent space of $\mathbb{P}^1$ restricted to the $i$-th marking. But a line bundle over a point obviously has an $r_i$-th root.

The morphism $\mathbf{e}_i$ is easily seen to be proper and with finite fibers. Representability is a straightforward calculation of stabilizers. ♣

We note that $\mathcal{K}_{0,3}(\mathcal{X}, 0)$ is always smooth. The obstruction class $O_{0,3}$ defining $[\mathcal{K}_{0,3}(\mathcal{X}, 0)]^v$ is given by the top Chern class of the bundle

$$E_{0,3} = R^1(\pi_{0,3})_* f^*_{0,3} T\mathcal{X}.$$

We can now define the product as follows:

DEFINITION 6.2.2. Let $\gamma_1, \gamma_2 \in A^*(\mathcal{X}_1)$. Define

$$\gamma_1 \smile \gamma_2 = \check{\mathbf{e}}_{3*} \ \left( \mathbf{e}_1^*(\gamma_1) \cdot \mathbf{e}_2^*(\gamma_2) \cdot O_{0,3} \right)$$

As things stand, this definition depends on the choice of the liftings $\mathbf{e}_i$. However it is not difficult to show that the group homomorphisms $\mathbf{e}_{i*}$ and $\mathbf{e}_i^*$ are independent of the choices, and therefore the product $\gamma_1 \smile \gamma_2$ is independent of the choices. Note that after tensoring with $\mathbb{Q}$ this definition agrees with the definition of the quantum product in degree 0.

**6.3. Associativity.** Next we need to prove associativity. The discussion in section 5.1 goes through word for word and one obtains (with notation analogous to that in section 5.1)

$$(\gamma_1 \smile \gamma_2) \smile \gamma_3 = (\check{\mathbf{e}}_4^{\times})_* \left( \left( (\mathbf{e}_1^{\times})^* \gamma_1 \, (\mathbf{e}_2^{\times})^* \gamma_2 \, (\mathbf{e}_3^{\times})^* \gamma_3 \right) \delta^!( [\mathcal{K}_1]^v \times [\mathcal{K}_2]^v ) \right).$$

We can now follow the arguments of section 5.2. As before, we have a gluing morphism

$$gl : \mathcal{K}_1 \times_{\mathcal{X}_1} \mathcal{K}_2 \xrightarrow{g} D(\mathcal{X}) \overset{\iota}{\hookrightarrow} \mathcal{K}_{0,4}(\mathcal{X}, 0).$$

This time, the morphism $g$ is a $\boldsymbol{\mu}_{r_\times}$-bundle.

There is a little problem with Equation (3) in section 5.2: the morphism $\bar{e}_i : \mathcal{K}_{0,4}(\mathcal{X}, 0) \to \bar{\mathcal{X}}_1$ does not lift to $\mathcal{X}_1$, since the normal bundle to the $i$-th section of the universal curve over $\overline{\mathbf{M}}_{0,4}$ has degree 1, and thus has no $r_i$-th root unless $r_i = 1$. However, if we restrict to the inverse image $\tilde{\mathcal{K}}_{0,4}(\mathcal{X}, 0)$ of the open set $\mathbb{A}^1 = \overline{\mathbf{M}}_{0,4} \smallsetminus (14|23) \subset \overline{\mathbf{M}}_{0,4}$, then a section does exist. We denote by $gl : \mathcal{K}_1 \times_{\mathcal{X}_1} \mathcal{K}_2 \to \tilde{\mathcal{K}}_{0,4}(\mathcal{X}, 0)$ the gluing map. So we can define

$$\mathbf{e}_i : \tilde{\mathcal{K}}_{0,4}(\mathcal{X}, 0) \to \mathcal{X}_1 \ , \quad i = 1, \ldots, 4$$

and if we do so carefully then

$$\mathbf{e}_i \circ gl = \mathbf{e}_i^{\times}.$$

Continuing with the arguments, the proof of Proposition 5.3.1 goes through and gives

$$\delta^!( [\mathcal{K}_1]^v \times [\mathcal{K}_2]^v ) = g^*(j^![\mathcal{K}_{0,4}(\mathcal{X}, 0)]^v).$$

We also have that $\mathcal{K}_{0,4}(\mathcal{X},0) \to \mathcal{K}_{0,4}^{orb}$ is flat, therefore

$$g^*(j^![\mathcal{K}_{0,4}(\mathcal{X},0)]^v) = gl^*[\mathcal{K}_{0,4}(\mathcal{X},0)]^v.$$

This implies that

$$(\gamma_1 \smile \gamma_2) \smile \gamma_3 = \check{\mathbf{e}}_{4*}^{\times}\, gl^* \Big( \mathbf{e}_1^*\gamma_1\, \mathbf{e}_2^*\gamma_2\, \mathbf{e}_3^*\gamma_3\ [\tilde{\mathcal{K}}_{0,4}(\mathcal{X},0)]^v \Big).$$

Unfortunately, $\mathbf{e}_4$ is neither proper nor representable, so we cannot replace $\check{\mathbf{e}}_{4*}^{\times}$ by $\check{\mathbf{e}}_{4*}gl_*$. We remark that Chen and Ruan utilize a certain identification of the *coarse* moduli space $\mathbf{K}_{0,4}(\mathcal{X},0) = \overline{\mathbf{M}}_{0,4} \times \mathbf{X}_3$, which does not lift to the level of stacks.

Let $\tau : \tilde{\mathcal{K}}_{0,4}(\mathcal{X},0) \to \tilde{\mathcal{K}}_{0,4}(\mathcal{X},0)$ be the automorphism induced by the involution $(2,3)$ on the markings, and denote $gl_\tau = \tau \circ gl$. One immediately sees that associativity follows from the following:

Claim.
$$\check{\mathbf{e}}_{4*}^{\times}\ gl^* = \check{\mathbf{e}}_{4*}^{\times}\ (\tau \circ gl)^* : A^*(\tilde{\mathcal{K}}_{0,4}(\mathcal{X},0)) \to A^*\mathcal{X}_1.$$

To show this equality it is useful to refine the divisor diagram. We deal with one connected component of $\tilde{\mathcal{K}}_{0,4}(\mathcal{X},0)$ at a time, so let $\tilde{\mathcal{K}}$ be such a component. We denote $\mathcal{Z} := gl^{-1}\tilde{\mathcal{K}}$ and similarly $\mathcal{Z}_\tau := gl_\tau^{-1}\tilde{\mathcal{K}}$. One can show that the locally constant function $r_\times$ is constant on $\mathcal{Z}$ and $\mathcal{Z}_\tau$, and we denote these constants by $r_\mathcal{Z}$ and $r_{\mathcal{Z}_\tau}$ respectively.

We have a fiber diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r] \arrow[d] & \tilde{D}(\mathcal{X}) \arrow[r, hook] \arrow[d] & \tilde{\mathcal{K}} \arrow[d] \\ \operatorname{Spec}\mathbb{C} \arrow[r] & \mathcal{B}\boldsymbol{\mu}_{r_\mathcal{Z}} \arrow[r, hook] & \tilde{\mathcal{M}}_{0,4} \end{tikzcd}$$

and a similar diagram for the $\tau$ version.

Here $\tilde{\mathcal{M}}_{0,4}$ denotes the open stack of twisted 4-pointed curves over $\mathbb{A}^1$ with *stable* 4-pointed coarse curve, with markings of indices $r_1, \ldots, r_4$, and nodes of index $r_\mathcal{Z}$ over $\{0\} \subset \mathbb{A}^1$ and index $r_{\mathcal{Z}_\tau}$ over $\{1\} \subset \mathbb{A}^1$. It is simply obtained by endowing $\mathbb{A}^1$ with orbifold structure of index $r_\mathcal{Z}$ at $\{0\} \subset \mathbb{A}^1$ and index $r_{\mathcal{Z}_\tau}$ at $\{1\} \subset \mathbb{A}^1$. It thus contains two divisors $\mathcal{G}_0$ and $\mathcal{G}_1$ over $\{0\}$ and $\{1\}$ in $\mathbb{A}^1$, isomorphic to $\mathcal{B}\boldsymbol{\mu}_{r_\mathcal{Z}}$ and $\mathcal{B}\boldsymbol{\mu}_{r_{\mathcal{Z}_\tau}}$, respectively.

The square on the left is cartesian since $\operatorname{Spec}\mathbb{C} \to \mathcal{B}\boldsymbol{\mu}_{r_\mathcal{Z}}$ is the universal principal bundle. Ignoring the middle column, we can extend the fiber diagram as follows:

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "gl"] \arrow[d, "\check{\mathbf{e}}_4^{\times}"'] & \tilde{\mathcal{K}} \arrow[d, "s\times\check{\mathbf{e}}_4"] \\ \mathcal{X}_1 \arrow[r, "p\times id"] \arrow[d] & \tilde{\mathcal{M}}_{0,4} \times \mathcal{X}_1 \arrow[d] \\ \operatorname{Spec}\mathbb{C} \arrow[r, "p"] & \tilde{\mathcal{M}}_{0,4} \end{tikzcd}$$

(and a similar diagram with $\tau$).

Considering the projection formula for the upper square, associativity follows if we prove that

$$(p \times id)^* = (p_\tau \times id)^* : A^*(\tilde{\mathcal{M}}_{0,4} \times \mathcal{X}_1) \to A^*(\mathcal{X}_1).$$

We can apply the projection formula here, because $s \times \check{\mathbf{e}}_4$ is proper and representable. Given $\xi \in A^*(\tilde{\mathcal{M}}_{0,4} \times \mathcal{X}_1)$, we can restrict it to the open set over $\tilde{\mathcal{M}}_{0,4} \smallsetminus (\mathcal{G}_0 \cup \mathcal{G}_1) \ \simeq\ \mathbb{A}^1 \smallsetminus \{0,1\}$.

$$\begin{tikzcd} & \tilde{\mathcal{M}}_{0,4} \arrow[d] \\ \mathbb{A}^1 \smallsetminus \{0,1\} \arrow[ur] \arrow[r, hook] & \mathbb{A}^1 \end{tikzcd}$$

The usual exact sequence says that this restriction extends to a class on $\mathbb{A}^1 \times \mathcal{X}_1$, which can be pulled back to a class $\xi'$ on $\tilde{\mathcal{M}}_{0,4} \times \mathcal{X}_1$. On the one hand, $A^*(\mathbb{A}^1 \times \mathcal{X}_1) = A^*(\mathcal{X}_1)$, and therefore $(p \times id)^*\xi'$ is independent of the choice of $p$. On the other hand, $\xi$ and $\xi'$ differ by a class supported on $\mathcal{G}_0 \cup \mathcal{G}_1$. It thus suffices to show that

$$(p \times id)^*\xi = 0$$

whenever $\xi$ is supported on $(\mathcal{G}_0 \cup \mathcal{G}_1) \times \mathcal{X}_1$. It suffices to treat one of the two.

Say $i : \mathcal{G}_0 \times \mathcal{X}_1 \hookrightarrow \tilde{\mathcal{M}}_{0,4} \times \mathcal{X}_1$ is the embedding of the gerbe over 0, with normal sheaf $N_0$. The point $p : \operatorname{Spec}\mathbb{C} \to \tilde{\mathcal{M}}_{0,4}$ lifts to $\widetilde{p} : \operatorname{Spec}\mathbb{C} \to \mathcal{G}_0$. We may assume the class $\xi = i_*\widetilde{\xi}$. Then

$$i^*i_*\widetilde{\xi} = c_1(N_0)\widetilde{\xi}.$$

Pulling back to $\mathcal{X}_1 = \operatorname{Spec}\mathbb{C} \times \mathcal{X}_1$, we have

$$(p \times id)^* i_*\widetilde{\xi} = (\widetilde{p} \times id)^* c_1(N_0)\widetilde{\xi}.$$

The fact that $\widetilde{p}^* N(\,\mathcal{G} \subset \tilde{\mathcal{M}}_{0,4}\,)$ is trivial implies that

$$(\widetilde{p} \times id)^* c_1(N_0) = 0,$$

therefore $(p \times id)^*\xi = (p \times id)^* i_*\widetilde{\xi} = 0$. ♣

## 7. Remarks on grading, expected dimension, divisors

**7.1. Degree shifting.** When we read about virtual fundamental classes on moduli spaces of "usual" stable maps, we often find it stressed that the virtual fundamental class has the "expected dimension". In our discussion here we (somewhat perversely) avoided any discussion of the expected dimension, and evidently it is not at all necessary for proving the associativity of the quantum product. On the other hand, the existence of such a dimension and its properties are of interest.

An interesting feature of the expected dimension of twisted stable maps is that it depends on the component of the moduli space in a subtle way, determined by the type of the evaluation maps, using the so called "degree shifting number" or "age" of the image component in $X_1$.

The complex $R\pi_{g,n,\beta *} f^*T\mathcal{X}$ was used for defining the relative obstruction theory, and thus the class $[\mathcal{K}_{g,n}(\mathcal{X},\beta)]^v$. There is a naïve approximation for the degree of the top Chern class, obtained by using the usual Grothendieck–Riemann–Roch formula. However, it is well known that the usual Grothendieck–Riemann–Roch formula does not hold for Deligne–Mumford stacks; it is explained in Toen's work [**T**] that one needs to introduce terms corresponding to pullback of the sheaf to the twisted sectors to correct the formula.

The issue can be understood if one considers a smooth marked twisted curve $\mathcal{C}$ with a representable map $f : \mathcal{C} \to \mathcal{X}$. Let $V = f^*T\mathcal{X}$. If $\pi : \mathcal{C} \to C$ is the map to the coarse curve, then $\chi(\mathcal{C}, V) = \chi(C, \pi_*V)$, since $\pi_*$ is an exact functor. Also, $\deg_C \pi_*V = \deg_{\mathcal{C}} \pi^*\pi_*V$. Thus the failure of Riemann–Roch is exactly the degree of the torsion sheaf $V/\pi^*\pi_*V$ supported at the twisted markings.

A neighborhood of a twisted marking locally looks like $[\operatorname{Spec}\mathbb{C}[[t]]/\boldsymbol{\mu}_r]$. We describe the action as follows: we let $\boldsymbol{\mu}_r$ act on the tangent space of $\mathcal{C}$ via the fundamental character. We can choose the parameter $t$ to be an eigenvector, and since it generates the *cotangent* space at $x$, the group $\boldsymbol{\mu}_r$ acts on $t$ via the *inverse* of the fundamental character:

$$t \ \mapsto\ \zeta_r^{-1}\, t.$$

The group $\boldsymbol{\mu}_r$ is identified via $f$ with a subgroup $H$ of the stabilizer $G_x$ of a point of $\mathcal{X}$. Let the tangent space $V_x$ of $\mathcal{X}$ at $x$ have basis $(v_1, \dots, v_n)$ consisting of $\boldsymbol{\mu}_r$-eigenvectors. The degree of $V/\pi^*\pi_*V$ is then exactly $\frac{1}{r}\sum_i^n k_i$ where

$$k_i = \min\{l | v_i \cdot t^l \text{ is invariant }\}.$$

In other words, $\boldsymbol{\mu}_r$ acts on $v_i$ via

$$v_i \mapsto \zeta_r^{k_i} v_i,$$

with $0 \le k_i < r$. Thus the contribution of a marking to the degree depends only on the component of $X_1$ where it evaluates. This contribution $\frac{1}{r}\sum_i^n k_i$ is known as the *degree shifting number* ([**C-R2**]) or the *age* (Miles Reid's terminology). If the connected components of $\mathcal{X}_1$ are $\mathcal{X}_1^i$, we denote this value by $a(\mathcal{X}_1^i)$.

As in [**C-R1**], one defines a grading on the group $A^*_{st}(\mathcal{X})$ as follows:

$$A^m_{st}(\mathcal{X}) = \oplus_i A^{m - a(\mathcal{X}_1^i)}(\mathcal{X}_1^i).$$

An argument similar to that of Chen and Ruan shows that this makes the stringy Chow ring with integer coefficients $A^*_{st}(\mathcal{X})$ into a graded ring.

**7.2. Forgetting a marking.** The Gromov–Witten invariants of projective varieties satisfy the so-called "divisor axiom", which comes from the fact that $\overline{\mathcal{M}}_{g,n+1}(X,\beta)$ is the universal curve over $\overline{\mathcal{M}}_{g,n}(X,\beta)$. This still works for stacks when we restrict to the case where the $(n+1)$-st marking is *untwisted*. In our setup there is no "forgetful map" for a twisted marking. There is, however, a way to change the formalism which allows for such forgetful maps. One way to do this entails

(1) systematically breaking up $\overline{\mathcal{M}}_{g,n}(\mathcal{X},\beta)$ according to the types of the markings, with components $\overline{\mathcal{M}}_{g,n}(\mathcal{X},\beta,\tau)$, where $\tau\in\pi_0(X_1)^n$ is an $n$-tuple of connected components of $X_1$ indicating where each evaluation map $e_i$ lands;
(2) for each index set $I\subset\{1,\dots,n\}$ of *twisted* markings in $\overline{\mathcal{M}}_{g,n}(\mathcal{X},\beta,\tau)$ of size $|I|=l$ introducing a new moduli stack

$$\overline{\mathcal{M}}_{g,n-l}(\mathcal{X},\beta,\tau')=[\overline{\mathcal{M}}_{g,n}(\mathcal{X},\beta,\tau)/\mathcal{S}_l].$$

The right hand side is the quotient of $\overline{\mathcal{M}}_{g,n}(\mathcal{X},\beta,\tau)$ by the action of the symmetric group on $l$ letters $\mathcal{S}_l$ acting on the indices of $I$. The point is that the only thing we can "forget" about a collection of twisted markings is their order. The notation $\tau'$ on the left denotes the image of $\tau$ in $\pi_0(X_1)^n/\mathcal{S}_l$ obtained by forgetting the order of the indices in $I$.

At this point we do not know which formalism carries more computational or theoretical advantages.

## 8. Examples

We include a few examples of integral stringy Chow rings, both to give the reader a sense of what these rings look like, and to illustrate the fact that stacks tend to have a lot of torsion in their Chow rings. The most basic example is $\mathcal{B}\mu_r$. We recall that the non-stringy Chow ring, $A^*(\mathcal{B}\mu_r)$ is $\mathbb{Z}[t]/\langle rt\rangle$, where $t$ is the first Chern class of the standard line bundle over $\mathcal{B}\mu_r$, see [**To**]. The inertia stack of $\mathcal{B}\mu_r$ is simply $r$ disjoint copies of $\mathcal{B}\mu_r$, so additively we know exactly what the stringy Chow ring is. It is easy to compute the product structure in this case. If we let $s$ be the element of $A^*_{st}(\mathcal{B}\mu_r)$ corresponding to the fundamental class of the component of the inertia stack given by the identity map from $\mathcal{B}\mu_r$ to itself, then its powers give the fundamental classes of the other components of the inertia stack. We conclude that

$$A^*_{st}(\mathcal{B}\mu_r)=\mathbb{Z}[s,t]/\langle rt, s^r-1\rangle.$$

Another simple example is the stack quotient $\mathcal{X}=[\mathbb{A}^1/\mu_r]$. This stack is a vector bundle over $\mathcal{B}\mu_r$, so its ordinary Chow ring is isomorphic to the Chow ring of $\mathcal{B}\mu_r$. Under this isomorphism, the generator $t$ corresponds to the substack of $\mathcal{X}$ isomorphic to $\mathcal{B}\mu_r$ supported at the origin. The inertia stack $\mathcal{X}_1$ is a copy of $\mathcal{X}$ together with $r-1$ disjoint copies of $\mathcal{B}\mu_r$. In this case, if we let $s$ denote the fundamental class of the component of $\mathcal{X}_1$ corresponding to the natural inclusion of $\mathcal{B}\mu_r$ into $\mathcal{X}_1$, we again find that powers of $s$ correspond to the fundamental classes of the other components of $\mathcal{X}_1$, except for the $r^{th}$ power, which now gives the generator $t$ of the Chow ring of the identity component of $\mathcal{X}_1$. We conclude that

$$A^*_{st}(\mathcal{X})=\mathbb{Z}[s]/\langle rs^r\rangle.$$

Note that even after tensoring with $\mathbb{Q}$, this ring does not agree with $A^*_{st}(\mathcal{B}\mu_r)$, showing that the stringy Chow ring does not enjoy the functorial properties of the standard Chow ring.

We conclude with a description of the stringy Chow ring of $\overline{\mathcal{M}}_{1,1}$. The ordinary Chow ring of $\overline{\mathcal{M}}_{1,1}$ was computed in [**E-G**] to be $\mathbb{Z}[\lambda]/\langle 24\lambda^2\rangle$, where $\lambda$ is the first Chern class of the Hodge bundle. The inertia stack consists of two copies of $\overline{\mathcal{M}}_{1,1}$, two copies of $\mathcal{B}\mu_4$, and four copies of $\mathcal{B}\mu_6$. We let $u$ denote the fundamental class of the second copy of $\overline{\mathcal{M}}_{1,1}$, $a$ denote the fundamental class of the copy of $\mathcal{B}\mu_4$ corresponding to an elliptic curve with four automorphisms and an isomorphism of $\mathrm{Aut}(E)$ with $\mu_4$ such that $\mu_4$ acts on $H^1(\mathcal{O}_E)$ via the standard representation, and let $b$ denote the analogous class for the elliptic curve with six automorphisms. Then considerations similar to those above (although more involved) lead to

$$A^*_{st}(\overline{\mathcal{M}}_{1,1})=\mathbb{Z}[u,a,b,\lambda]/\langle u^2-1, 4a\lambda, 6b\lambda, a^2-6\lambda u, b^3-4\lambda u, ab\rangle.$$

As in the nonstringy case, we get the stringy Chow ring for the open stack $\mathcal{M}_{1,1}$ by adding the additional relation $12\lambda=0$.

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